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NUMERICAL INTEGRATION WITH WEIGHTS OF CONVEX FUNCTIONS OF ALGEBRAIC POLYNOMIALS

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1. Introduction. On the unit interval $I = [0,1]$ we consider a *weight* w, i.e. a Lebesgue integrable nonnegative and nontrivial function. The space of all functions integrable with exponent p, $1 \le p < \infty$, with respect to the weight w is denoted by $L^p_w(I)$. Also, let a continuous convex function Φ on the real line $\mathbf{R} = (-\infty, \infty)$ be given. For later convenience we introduce the functional

$$\Phi_w(f) = \int_I \Phi(f(t))w(t)\,dt. \tag{1.1}$$

Our first goal is to construct, for $f \in L^1_w(I)$ with (1.1) finite, a sequence $(\Phi_n(f))$ such that

$$\Phi_n(f) \le \Phi_{n+1}(f) \le \Phi_w(f) \quad \text{for } n = 0, 1, \ldots,$$

where the $\Phi_n(f)$ are easily computable for a reasonable class of weights w. Moreover, the construction should be such that $\Phi_n(f) \to \Phi_w(f)$ as $n \to \infty$. For later use the space of all algebraic polynomials of one variable of degree at most n is denoted by Π_n. The second goal can now be formulated as follows: for $g \in \Pi_m$ construct $(\Phi^*_n(g))$ such that

$$\Phi_w(g) \le \Phi^*_{n+1}(g) \le \Phi^*_n(g) \quad \text{for } n = m, m+1, \ldots,$$

where $\Phi^*_n(g)$ should again be easily computable and $\Phi^*_n(g)$ should approach $\Phi_w(g)$ as $n \to \infty$ for the same class of weights.

The construction of $\Phi_n(f)$ will be given in terms of the Bernstein basis in Π_n, and the construction of $\Phi^*_n(g)$ in terms of its dual basis.

The basic Bernstein polynomials are given by the formula

$$N_{i,n}(t) = \binom{n}{i} t^i (1-t)^{n-i}, \quad i = 0, \ldots, n. \tag{1.2}$$

These basic polynomials are nonnegative on I and

$$\sum_{i=0}^{n} N_{i,n}(t) = 1. \tag{1.3}$$

For each $n = 0, 1, \ldots$ we assign to the weight w the discrete weight

(1.4) $$w_{i,n} = \int_I N_{i,n}(t)w(t)\,dt, \qquad i = 0, \ldots, n.$$

Clearly, (1.3) implies

(1.5) $$\int_I w(t)\,dt = \sum_{i=0}^{n} w_{i,n}.$$

In the space $L^2_w(I)$ we have

(1.6) $$(f,g)_w = \int_I f(t)g(t)w(t)\,dt,$$

and in Π_n the corresponding scalar product

(1.7) $$(u,v)_{w,n} = \sum_{i=0}^{n} u(i)v(i)w_{i,n}.$$

The dual to the Bernstein basis in Π_n with respect to (1.6) is denoted by $(N^*_{i,n},\ i = 0, \ldots, n)$ and is uniquely determined by

(1.8) $$(N_{i,n}, N^*_{j,n})_w = \delta_{i,j}, \qquad i,j = 0, \ldots, n.$$

It is a consequence of (1.3) that

(1.9) $$\int_I N^*_{j,n}(t)w(t)\,dt = 1.$$

Moreover, we need the polynomials

(1.10) $$M_{j,n} = N_{j,n}/w_{j,n},$$

for which we have

(1.11) $$\int_I M_{j,n}(t)w(t)\,dt = 1.$$

Now, define

(1.12) $$\Phi_n(f) = \sum_{i=0}^{n} \Phi((f, M_{i,n})_w)w_{i,n},$$

(1.13) $$\Phi^*_n(g) = \sum_{i=0}^{n} \Phi((g, N^*_{i,n})_w)w_{i,n}.$$

The *Durrmeyer type* operators are important in our construction:

(1.14) $$D_n(f) = \sum_{i=0}^{n} (f, M_{i,n})_w N_{i,n}.$$

This definition implies that the operator $D_n : \Pi_n \to \Pi_n$ is a linear isomorphism and

$$D_n(N^*_{i,n}) = M_{i,n} \quad \text{for } i = 0, \ldots, n. \tag{1.15}$$

Moreover, the following known formula for the *artificial lifting* of degree of the basic Bernstein polynomials is also important. Namely, for $n = 1, \ldots,$ we have

$$N_{i,n-1} = \frac{i+1}{n} N_{i+1,n} + \frac{n-i}{n} N_{i,n} \quad \text{for } i = 0, \ldots, n-1. \tag{1.16}$$

2. Main inequalities. In this section some inequalities for the functionals Φ_w, Φ_n and Φ^*_n, without any restrictions on the weight w, are established.

LEMMA 2.1. *Let $\Phi(t)$, $-\infty < t < \infty$, be a continuous convex function and let $f \in L^1_w(I)$. Then*

$$\Phi_w(D_{n-1}(f)) \le \Phi_{n-1}(f) \le \Phi_n(f) \le \Phi_w(f) \quad \textit{for } n = 1, 2, \ldots \tag{2.2}$$

Proof. The first inequality in (2.2) is obtained by using (1.14), (1.3), (1.11) and Jensen's inequality. Now, (1.16) implies

$$M_{i,n-1} = A_i\, M_{i+1,n} + B_i\, M_{i,n},$$

where

$$A_i = \frac{i+1}{n}\frac{w_{i+1,n}}{w_{i,n-1}} \quad \text{and} \quad B_i = \frac{n-i}{n}\frac{w_{i,n}}{w_{i,n-1}}.$$

Since $A_i + B_i = 1$ it follows by Jensen's inequality that $\Phi_{n-1} \le \Phi_n$. The third inequality in (2.2) is obtained by using once more (1.11) and Jensen's inequality.

LEMMA 2.3. *Let $\Phi(t)$, $-\infty < t < \infty$, be a continuous convex function and let $g \in \Pi_m$. Then*

$$\Phi_w(g) \le \Phi^*_n(g) \le \Phi^*_{n-1}(g) \le \Phi_w(D^{-1}_{n-1}(g)) \quad \textit{for } n > m. \tag{2.4}$$

Proof. Application of Lemma 2.1 with $n-1$ replaced by n, $f = D^{-1}_n(g)$ and (1.15) give

$$\Phi_w(g) \le \Phi_n(D^{-1}_n(g)) = \Phi^*_n(g) \le \Phi_w(D^{-1}_n(g)).$$

It remains to prove the middle inequality in (2.4). Since

$$g = \sum_{i=0}^{n-1}(g, N^*_{i,n-1})_w\, N_{i,n-1} = \sum_{i=0}^{n}(g, N^*_{i,n})_w\, N_{i,n},$$

it follows by (1.16) that

$$(g, N^*_{i,n})_w = \frac{i}{n}(g, N^*_{i-1,n-1})_w + \frac{n-i}{n}(g, N^*_{i,n-1})_w. \tag{2.5}$$

Now, the definition of Φ_n^*, (2.5) and Jensen's inequality give

$$\Phi_n^*(g) \le \sum_{i=0}^{n}(A_{i-1}\Phi((g,N_{i-1,n-1}^*)_w) + B_i\Phi((g,N_{i,n-1}^*)_w))w_{i,n-1},$$

where the A_i and B_i are as in the previous proof. Since $A_i + B_i = 1$ the inequality in question follows.

Lemmas 2.1 and 2.3 now yield

THEOREM 2.6. *Let w be a weight as in Section 1. Let $\Phi(t)$, $-\infty < t < \infty$, be a continuous convex function and let $g \in \Pi_m$. Then for $n > m$ we have*

$$\Phi_w(D_{n-1}(g)) \le \Phi_{n-1}(g) \le \Phi_n(g) \le \Phi_w(g) \le \Phi_n^*(g) \le \Phi_{n-1}^*(g) \le \Phi_w(D_{n-1}^{-1}(g)).$$

3. The case of Jacobi weights. From now on it is assumed that w is a *Jacobi weight*, i.e. $w(t) = t^\alpha(1-t)^\beta$ with some $\alpha > -1$ and $\beta > -1$. The definition of the beta function gives

$$\int_I w(t)\,dt = B(\alpha+1,\beta+1), \tag{3.1}$$

$$w_{i,n} = \binom{i}{n} B(i+\alpha+1, n-i+\beta+1) \quad \text{for } i = 0,\dots,n. \tag{3.2}$$

Moreover, it implies

$$\int_I t^j M_{i,n}(t)w(t)\,dt = \left(\frac{i+j+\alpha}{n+j+\alpha+\beta+1}\right)^{(j)} \quad \text{for } i = 0,\dots,n,\ j \ge 0, \tag{3.3}$$

where $(m)^{(j)} = m(m-1)\dots(m-j+1)$ and $(i/m)^{(j)} = i^{(j)}/m^{(j)}$. In addition to this we have the known formula

$$t^j = \sum_{i=0}^{n}(i/n)^{(j)}N_{i,n}(t) \quad \text{for } j = 0,\dots,n. \tag{3.4}$$

(3.3) and (3.4) together imply

PROPOSITION 3.5. *Let w be a Jacobi weight with parameters $\alpha > -1$ and $\beta > -1$ and let*

$$g(t) = \sum_{j=0}^{m} g_j t^j. \tag{3.6}$$

Then

$$(3.7)\quad (M_{i,n}, g)_w = \sum_{j=0}^{m} g_j \left(\frac{i+j+\alpha}{n+j+\alpha+\beta+1}\right)^{(j)} \quad \text{for } i = 0,\ldots,n,$$

$$(3.8)\quad (N^*_{i,n}, g)_w = \sum_{j=0}^{m} g_j \left(\frac{i}{n}\right)^{(j)} \quad \text{for } i = 0,\ldots,n,\ n \geq m.$$

4. Jacobi and Hahn polynomials. In this section we are going to give formulas for

$$(4.1)\qquad (M_{i,n}, g)_w \quad \text{and} \quad (N^*_{i,n}, g)_w,$$

provided that we have the Jacobi representation for the polynomial $g \in \Pi_m$.

Given $\alpha > -1$ and $\beta > -1$ the Jacobi polynomials orthogonal with respect to (1.6) are denoted by $(P_j^{(\alpha,\beta)},\ j = 0, 1, \ldots)$ and they are normalized by the formula

$$(4.2)\qquad P_j^{(\alpha,\beta)}(0) = (-1)^j P_j^{(\beta,\alpha)}(1) = \binom{j+\alpha}{j}, \quad j = 0, 1, \ldots$$

It should be remembered that these are the standard Jacobi polynomials transformed from $[-1, 1]$ to $I = [0, 1]$ by the map $x = 1 - 2t$ with $t \in I$. With this normalization we have

$$(4.3)\qquad (P_j^{(\alpha,\beta)}, P_j^{(\alpha,\beta)})_w = \frac{1}{(2j+\alpha+\beta+1)} \frac{B(j+\alpha+1, j+\beta+1)}{B(j+1, j+\alpha+\beta+1)}.$$

Given $\alpha > -1$, $\beta > -1$ and natural n we define the polynomials $(H_{j,n}^{(\alpha,\beta)},\ j = 0, \ldots, n)$ by the formula

$$(4.4)\qquad P_j^{(\alpha,\beta)} = \sum_{i=0}^{n} H_{j,n}^{(\alpha,\beta)}(i) N_{i,n}, \quad j = 0, \ldots, n.$$

This, (4.2) and (3.8) imply that

$$(4.5)\qquad H_{j,n}^{(\alpha,\beta)}(0) = (-1)^j H_{j,n}^{(\beta,\alpha)}(n) = \binom{j+\alpha}{j}, \quad H_{j,n}^{(\alpha,\beta)} \in \Pi_j, \quad \text{for } j = 0, \ldots, n.$$

For later use we introduce

$$\widehat{P}_j^{(\alpha,\beta)} = \frac{P_j^{(\alpha,\beta)}}{\|P_j^{(\alpha,\beta)}\|_{2,w}}, \qquad \widehat{H}_{j,n}^{(\alpha,\beta)} = \frac{H_{j,n}^{(\alpha,\beta)}}{\|H_{j,n}^{(\alpha,\beta)}\|_{2,n}},$$

where

$$\|f\|_{p,w} = \Big(\int_I |f(t)|^p w(t)\,dt\Big)^{1/p}, \qquad \|f\|_{p,n} = \Big(\sum_{i=0}^{n} |f(i)|^p w_{i,n}\Big)^{1/p}.$$

THEOREM 4.6. *The polynomials* ($H_{j,n}^{(\alpha,\beta)}$, $j = 0,\ldots,n$) *are orthogonal with respect to* (1.7), *i.e. they are the Hahn orthogonal polynomials. Moreover, in addition to* (4.4) *we have*

$$P_j^{(\alpha,\beta)} = \lambda_{j,n} \sum_{i=0}^{n} H_{j,n}^{(\alpha,\beta)}(i) w_{i,n} N_{i,n}^*, \quad j = 0,\ldots,n, \tag{4.7}$$

where

$$\lambda_{j,n} = \left(\frac{n}{n+j+\alpha+\beta+1}\right)^{(j)} = \frac{\|P_j^{(\alpha,\beta)}\|_{2,w}^2}{\|H_{j,n}^{(\alpha,\beta)}\|_{2,n}^2}. \tag{4.8}$$

Proof. Define on Π_n the following two linear operators:

$$(Tv)(i) = (N_{i,n}^*, v)_w \quad \text{for } i = 0,\ldots,n,$$
$$(T^*v)(i) = (M_{i,n}, v)_w \quad \text{for } i = 0,\ldots,n.$$

It follows by (3.7) and (3.8) that

$$T\Pi_j = \Pi_j \quad \text{and} \quad T^*\Pi_j = \Pi_j \quad \text{for } j = 0,\ldots,n. \tag{4.9}$$

In this notation

$$v = \sum_{i=0}^{n} (Tv)(i) N_{i,n} = \sum_{i=0}^{n} (T^*v)(i) w_{i,n} N_{i,n}^*.$$

Therefore,

$$(u, v)_w = (Tu, T^*v)_{w,n}. \tag{4.10}$$

Thus, if u is orthogonal to Π_{j-1} with respect to $(\cdot,\cdot)_w$, then by (4.9) and (4.10) both Tu and T^*u are orthogonal to Π_{j-1} with respect to $(\cdot,\cdot)_{w,n}$. In particular, since $H_{j,n}^{(\alpha,\beta)} = T(P_j^{(\alpha,\beta)})$, it follows that both $H_{j,n}^{(\alpha,\beta)}$ and $g = T^*(P_j^{(\alpha,\beta)})$ are orthogonal to Π_{j-1} and both are in Π_j. Consequently, there are numbers $\lambda_{j,n}$ such that $g = \lambda_{j,n} H_{j,n}^{(\alpha,\beta)}$. Comparing the coefficients at the highest powers with the help of (3.7) and (3.8) gives (4.8).

PROPOSITION 4.11. *Let w be a Jacobi weight with parameters $\alpha > -1$ and $\beta > -1$ and let*

$$g = \sum_{j=0}^{m} g_j P_j^{(\alpha,\beta)}. \tag{4.12}$$

Then

$$(M_{i,n}, g)_w = \sum_{j=0}^{m} \lambda_{j,n} g_j H_{j,n}^{(\alpha,\beta)}(i) \quad \text{for } i = 0,\ldots,n, \tag{4.13}$$

$$(N_{i,n}^*, g)_w = \sum_{j=0}^{m} g_j H_{j,n}^{(\alpha,\beta)}(i) \quad \text{for } i = 0,\ldots,n,\ n \geq m. \tag{4.14}$$

Proof. Use (4.4) and (4.7).

COROLLARY 4.15. *For $f \in L^1_w(I)$ we have the following formula for the Durrmeyer operators with the Jacobi weight w:*

$$D_n(f) = \sum_{j=0}^{n} \lambda_{j,n}(f, \hat{P}_j^{(\alpha,\beta)})_w \hat{P}_j^{(\alpha,\beta)}.$$

Proof. We apply the operator D_n to both sides of (4.7), and use (1.15) and (4.4) to get $D_n(P_j^{(\alpha,\beta)}) = \lambda_{j,n}(P_j^{(\alpha,\beta)})$.

PROPOSITION 4.16. *Let w be a Jacobi weight. Let $\Phi(t)$, $-\infty < t < \infty$, be a continuous convex function and let $g \in \Pi_m$. Then*

$$\Phi_n(g) \nearrow \Phi_w(g) \swarrow \Phi_n^*(g) \quad \text{as } n \nearrow \infty.$$

Proof. Since by (4.8), for fixed j, $\lambda_{j,n} \to 1$ as $n \to \infty$, the statement follows by Corollary 4.15 and Theorem 2.6.

COROLLARY 4.17. *Let $\alpha > -1$, $\beta > -1$, $1 \le p \le \infty$ and let w be the Jacobi weight on I. Then for $n \ge j$ we have*

$$\lambda_{j,n}\|H_{j,n}^{(\alpha,\beta)}\|_{p,n} \le \lambda_{j,n+1}\|H_{j,n+1}^{(\alpha,\beta)}\|_{p,n+1} \le \|P_j^{(\alpha,\beta)}\|_{p,w} \le \|H_{j,n+1}^{(\alpha,\beta)}\|_{p,n+1} \le \|H_{j,n}^{(\alpha,\beta)}\|_{p,n},$$

or else

$$\sqrt{\lambda_{j,n}}\,\|\hat{H}_{j,n}^{(\alpha,\beta)}\|_{p,n} \le \sqrt{\lambda_{j,n+1}}\,\|\hat{H}_{j,n+1}^{(\alpha,\beta)}\|_{p,n+1} \le \|\hat{P}_j^{(\alpha,\beta)}\|_{p,w} \le \frac{1}{\sqrt{\lambda_{j,n+1}}}\|\hat{H}_{j,n+1}^{(\alpha,\beta)}\|_{p,n+1} \le \frac{1}{\sqrt{\lambda_{j,n}}}\|\hat{H}_{j,n}^{(\alpha,\beta)}\|_{p,n}.$$

To be able to compute the quantities (4.13) and (4.14) which appear in the definitions of $\Phi_n(g)$ and $\Phi_n^*(g)$ one needs some recurrence relations for the Hahn polynomials. The first recurrence relation corresponds to (1.16), and the second one is simply the three term relation for orthogonal polynomials.

PROPOSITION 4.18. *The values $H_{j,n}^{(\alpha,\beta)}(0)$ and $H_{j,n}^{(\alpha,\beta)}(1)$ are given by (4.5) for $0 \le j \le n$. For $n > 1$ we have*

$$H_{n,n}^{(\alpha,\beta)}(i) = (-1)^i \frac{\binom{\alpha+n}{i}\binom{\alpha+n}{n-i}}{\binom{n}{i}} \quad \text{for } i = 0, \dots, n. \tag{4.19}$$

Let now $n > j$. Then

$$H_{j,n}^{(\alpha,\beta)}(i) = \frac{i}{n}H_{j,n-1}^{(\alpha,\beta)}(i-1) + \frac{n-i}{n}H_{j,n-1}^{(\alpha,\beta)}(i) \quad \text{for } i = 1, \dots, n-1. \tag{4.20}$$

Proof. It is well known that the Rodrigues formula for $P_n^{(\alpha,\beta)}$ gives

$$P_n^{(\alpha,\beta)} = \sum_{i=0}^{n}(-1)^i \frac{\Gamma(\alpha+n+1)\Gamma(\beta+n+1)}{n!\Gamma(\alpha+i+1)\Gamma(\beta+n-i+1)} N_{i,n},$$

whence by (4.4) formula (4.19) follows. To get (4.20) put $g = P_j^{(\alpha,\beta)}$ in (2.5) and then use (4.4).

The next recurrence relation, the three term relation, gives the possibility to compute the Hahn polynomials by induction on degree.

PROPOSITION 4.21. *For the degrees* 0 *and* 1 *we have*

$$H_{0,n}^{(\alpha,\beta)}(i) = 1 \quad \text{for } i = 0, \ldots, n,$$

$$H_{1,n}^{(\alpha,\beta)}(i) = -\frac{\alpha+\beta+2}{n} i + (\alpha+1) \quad \text{for } i = 0, \ldots, n.$$

For each $j \geq 1$ *we have*

$$H_{j+1,n}^{(\alpha,\beta)}(i) = \frac{i-B}{A} H_{j,n}^{(\alpha,\beta)}(i) - \frac{C}{A} H_{j-1,n}^{(\alpha,\beta)}(i) \quad \text{for } i = 0, \ldots, n,$$

where

$$A = \frac{(j+\alpha+\beta+1)(j-n)(j+1)}{(2j+\alpha+\beta+2)(2j+\alpha+\beta+1)},$$

$$B = \frac{j^2(2n-\alpha+\beta) + j(\alpha+\beta+1)(2n-\alpha+\beta) + n(\alpha(\alpha+\beta+1)+\beta)}{(2j+\alpha+\beta+2)(2j+\alpha+\beta)},$$

$$C = -\frac{(j+\beta)(j+\alpha)(j+n+\alpha+\beta+1)}{(2j+\alpha+\beta+1)(2j+\alpha+\beta)}.$$

5. Comments and bibliographical notes. In bifurcation theory the *degree of a map* plays an important role [8]. For some polynomial maps the degree is equal to the value of an integral of the form

$$\int_{I^d} \max(0, g(\underline{x}))v(\underline{x})\, d\underline{x},$$

where g and v are real polynomials on $\mathbf{R}^d$. The value of the integral is an integer. Thus, numerical approximation of the integral with well controlled errors could help to solve the problem, i.e. to find its exact value. The question in this form, of finding such an algorithm, was explicitly posed to the author by M. Izydorek and S. Rybicki. It stimulated very much the present work. The solution presented here is one-dimensional, but its extension to several variables should cause no essential theoretical difficulties.

Another example of possible application of the algorithms presented here comes from the work [1], where it is shown that the integral

$$\Big(\int_{-1}^{1} |T_n'(x)|^p\,dx\Big)^{1/p}$$

is a norm for a finite-dimensional polynomial operator. Clearly, this norm can be approximated from both sides by the algorithms presented here. As usual, T_n is the Chebyshev polynomial of the first kind.

The dual to the Bernstein basis $(N^*_{i,n})$ was investigated in [3] in the particular case of $w = 1$. For the same special weight w we refer to [6], [5], and in the Jacobi case to [9], [10], for the fundamental properties of the Durrmeyer operators. The relation (4.4) between the Jacobi and Hahn orthogonal polynomials was established in [5] in the case of $w = 1$, and in [4] in the Jacobi case. A part of Theorem 2.6 was established in [2] for the special $\Phi(x) = |x|^p$, $1 \le p \le \infty$, and $w = 1$. Proposition 4.21 is classical (cf. [7]).

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