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On the Nörlund summability of orthogonal series

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1. Let
$$u_0+u_1+\ldots+u_n+\ldots$$
be a given series with partial sums s_n and let $\{p_n\}$ be a sequence of non-negative real numbers.

The series $\sum u_n$ is said to be (N, p_n)-summable to s, if
$$t_n = \frac{1}{P_n}\sum_{k=0}^{n} p_{n-k}s_k \to s \quad \text{as} \quad n\to\infty,$$
where $P_n = p_0+p_1+\ldots+p_n$, $p_0>0$, $p_n \geqslant 0$. We then write (N, p_n)-$\lim s_n = s$ or (N, p_n)-$\sum u_n = s$. The transforms t_n are called the Nörlund means of the sequence $\{s_n\}$ or of the series $\sum u_n$ (cf. [3], p. 65).

Cesàro's method of summability (C, a), $a>0$, is a special case of the method (N, p_n) if we write
$$p_n = \binom{n+a-1}{a-1} = \frac{\Gamma(n+a)}{\Gamma(n+1)\Gamma(a)}.$$

The sequence $\{p_n\}$ will be said to belong to the class M^a for a certain real $a\geqslant 0$ if

(i) $$0<p_n<p_{n+1} \quad \text{for} \quad n=0,1,2,\ldots, \qquad \text{or} \quad 0<p_{n+1}<p_n \quad \text{for} \quad n=0,1,2,\ldots,$$

(ii) $$p_0+p_1+\ldots+p_n = P_n \nearrow \infty,$$

(iii) $$\lim_{n\to\infty}\frac{np_n}{P_n} = a.$$

Let
$$S_n = \frac{1}{P_n}\sum_{k=0}^{n}\frac{p_k}{k+1}.$$
The sequence $\{p_n\}$ will be said to beolong to the class BVM^a, if $\{p_n\}\in M^a$ and if $\{S_n\}$ is a sequence of bounded variation, i.e.
$$\sum_{n=1}^{\infty}|S_n-S_{n-1}|<\infty.$$

In the case of the methods (C, a), $a > 0$, it is easily seen that $\{p_n\} \in BVM^a$.

It is well known that the method (N, p_n) is regular if and only if

$$\lim_{n\to\infty} \frac{p_n}{P_n} = 0 .$$

Obviously, if $\{p_n\} \in M^a$, the method (N, p_n) is regular.

In this paper we shall deal with (N, p_n)-summability of orthogonal series, restricting ourselves to the case where $\{p_n\} \in M^a$ or $\{p_n\} \in BVM^a$ with $a > \frac{1}{2}$ and in some theorems with $a \geqslant 0$.

2. We now give a number of lemmas concerning the classes M^a and the Nörlund summability of numerical series.

LEMMA 1 (Stolz). *Let $\{A_n\}$ be an arbitrary sequence of real (or complex) numbers and let $\{B_n\}$ be a sequence of positive numbers monotonically increasing to ∞. If*

$$\lim_{n\to\infty} \frac{A_n - A_{n-1}}{B_n - B_{n-1}} = g , \quad \textit{then} \quad \lim_{n\to\infty} \frac{A_n}{B_n} = g ,$$

where $A_{-1} = B_{-1} = 0$.

LEMMA 2 (Pati). *If $\nu \geqslant 0$ and $P_n \nearrow \infty$ as $n \to \infty$, then*

$$\sum_{n=\nu+1}^{\infty} \frac{p_n}{P_n P_{n-1}} = \frac{1}{P_\nu} .$$

This is easily seen, for $p_n = P_n - P_{n-1}$ and $P_n \nearrow \infty$ as $n \to \infty$.

LEMMA 3. *If $\{p_n\} \in M^a$, $a > \frac{1}{2}$, then there is an N such that the sequence $\{P_n^2/n\}$ is increasing for $n \geqslant N$ and tends to ∞. Moreover, $\lim\limits_{n\to\infty} p_n P_n = \infty$.*

Proof. Applying Stolz's lemma we may write for $n \geqslant 2$, $P_1 = 1$,

$$\begin{aligned}
\lim_{n\to\infty} \frac{\log \dfrac{P_n^2}{n}}{\log n} &= \lim_{n\to\infty} \frac{\log \dfrac{P_n^2}{n} - \log \dfrac{P_{n-1}^2}{n-1}}{\log n - \log(n-1)} \\
&= -\lim_{n\to\infty} \frac{\log\left(1 + \dfrac{1}{n-1}\right) + \log\left(1 - \dfrac{p_n}{P_n}\right)^2}{\log\left(1 + \dfrac{1}{n-1}\right)} \\
&= -\lim_{n\to\infty} \frac{\log\left(1 + \dfrac{1}{n-1}\right)^n + \log\left[\left(1 - \dfrac{p_n}{P_n}\right)^{-P_n/p_n}\right]^{-2np_n/P_n}}{\log\left(1 + \dfrac{1}{n-1}\right)^n} \\
&= -(1 - 2a) = 2a - 1 .
\end{aligned}$$

Hence

$$\lim_{n\to\infty} \frac{\log(P_n^2/n)}{\log n^{2a-1}} = 1 \quad \text{for} \quad a > \tfrac{1}{2},$$

i.e.

$$\log \frac{P_n^2}{n} \approx \log n^{2a-1}.$$

Finally,

$$\lim_{n\to\infty} \frac{P_n^2}{n} = \infty \quad \text{for} \quad a > \tfrac{1}{2}.$$

Now we show the sequence $\{P_n^2/n)$ to be increasing beginning with a certain N. We may write

$$\frac{P_n^2}{n} - \frac{P_{n-1}^2}{n-1} = \frac{1}{n(n-1)}[(n-1)P_n^2 - nP_{n-1}^2]$$
$$= \frac{P_n^2}{n(n-1)}\left[\frac{np_n}{P_n^2}(2P_n - p_n) - 1\right] = \frac{P_n^2}{n(n-1)}\left[\frac{2np_n}{P_n} - \frac{np_n^2}{P_n^2} - 1\right].$$

Since the expression in the square brackets tends to $2a-1 > 0$ for $a > \frac{1}{2}$, we have $P_n^2/n > P_{n-1}^2/(n-1)$ for $n > N$.

Moreover, since $p_nP_n = \dfrac{P_n^2}{n}\cdot\dfrac{np_n}{P_n}$, we have $\lim\limits_{n\to\infty} p_nP_n = \infty$ for $a > \frac{1}{2}$.

LEMMA 4. *If* $\{p_n\} \in M^a$, $a > \frac{1}{2}$, *then*

$$\lim_{n\to\infty} \frac{n}{P_n^2} \sum_{k=0}^{n} \frac{P_k^2}{(k+1)^2} = \frac{1}{2a-1}.$$

Proof. Let $\{p_n\} \in M^a$, $a > \frac{1}{2}$. If $0 < p_n \nearrow$, then the sequence $\{P_n/(n+1)\}$ is increasing and the series $\sum\limits_{k=0}^{\infty} P_k^2/(k+1)^2$ is divergent. If $0 < p_n \searrow$, then the sequence $\{P_n/(n+1)\}$ is decreasing and the series $\sum\limits_{k=0}^{\infty} P_k^2/(k+1)^2$ is also divergent, for otherwise, by a well-known theorem, the sequence $\{P_k^2/(k+1)\}$ would tend to zero, in contradiction to Lemma 3. Now, applying Stolz's lemma, we obtain

$$\lim_{n\to\infty} \frac{P_n^2/(n+2)}{\sum\limits_{k=0}^{n-1} P_k^2/(k+1)^2} = \lim_{n\to\infty} \frac{P_n^2/(n+2) - P_{n-1}^2/(n+1)}{P_{n-1}^2/n^2}$$
$$= \lim_{n\to\infty} \frac{n^2}{(n+1)(n+2)}\left[\frac{2(n+1)p_nP_n}{P_{n-1}^2} - \frac{(n+1)p_n^2}{P_{n-1}^2} - 1\right] = 2a-1.$$

Thus,

$$\lim_{n\to\infty}\frac{n+2}{P_n^2}\sum_{k=0}^{n-1}\frac{P_k^2}{(k+1)^2}=\frac{1}{2a-1}.$$

Lemma 4 now follows immediately from the formulae

$$\frac{n}{P_n^2}\sum_{k=0}^{n}\frac{P_k^2}{(k+1)^2}=\frac{n}{n+2}\cdot\frac{n+2}{P_n^2}\sum_{k=0}^{n-1}\frac{P_k^2}{(k+1)^2}+\frac{n}{(n+1)^2}.$$

LEMMA 5. *Let $\{p_n\}\in M^a$, $a>\frac{1}{2}$. If the series $\sum_{k=1}^{\infty}u_k$ is (N,p_n)-summable and if the series $\sum_{k=1}^{\infty}ku_k^2$ is convergent, then the series $\sum_{k=1}^{\infty}u_k$ is convergent.*

Proof. Let $\{p_n\}\in M^a$, $a>\frac{1}{2}$. We write

$$t_n=\frac{1}{P_n}\sum_{k=0}^{n}p_{n-k}s_k=\frac{1}{P_n}\sum_{k=0}^{n}u_k\sum_{\nu=k}^{n}p_{n-\nu}.$$

Hence

$$s_n-t_n=\frac{1}{P_n}\sum_{k=1}^{n}u_k\sum_{\nu=0}^{k-1}p_{n-\nu}.$$

In order to prove the Lemma it is sufficient to prove that the expression on the right side of the last formula tends to zero. Since $\sum_{k=1}^{\infty}ku_k^2<\infty$, for any $\varepsilon>0$ we may find an integer N such that

$$\sum_{k=N+1}^{\infty}ku_k^2<\varepsilon\quad\text{for}\quad n\geqslant N.$$

Moreover, we have $\lim_{n\to\infty}p_n/P_n=0$. Now we may write

$$\frac{1}{P_n}\sum_{k=1}^{n}\sum_{\nu=0}^{k-1}p_{n-\nu}u_k=\frac{1}{P_n}\sum_{k=1}^{N}\sum_{\nu=0}^{k-1}p_{n-\nu}u_k+\frac{1}{P_n}\sum_{k=N+1}^{n}\sum_{\nu=0}^{k-1}p_{n-\nu}u_k=A+B.$$

If $0<p_n\nearrow$, then applying Schwartz's inequality we obtain for $n>N$

$$A^2\leqslant\left(\frac{1}{P_n}\sum_{k=1}^{N}|u_k|\sum_{\nu=0}^{k-1}p_{n-\nu}\right)^2\leqslant\frac{p_n^2}{P_n^2}\sum_{k=1}^{N}ku_k^2\sum_{k=1}^{N}k=O(\varepsilon^2),$$

whence

(2.1) $$A=o(1)\quad\text{as}\quad n\to\infty.$$

Since

$$B^2 \leqslant \frac{p_n^2}{P_n^2} \sum_{k=N+1}^{n} k \sum_{k=N+1}^{n} k u_k^2 < \frac{n^2 p_n^2}{P_n^2} \cdot \varepsilon = O(\varepsilon),$$

we have

(2.2) $$B = o(1) \quad \text{as} \quad n \to \infty .$$

Now let $0 < p_n \searrow$. Then for $n > 2N$

$$A^2 \leqslant \left(\frac{1}{P_n} \sum_{k=1}^{N} k p_{n-k} |u_k|\right)^2$$

$$\leqslant \frac{1}{P_n^2} \sum_{k=1}^{N} k p_{n-k}^2 \sum_{k=1}^{N} k u_k^2 < \frac{N^2 p_{n-N}^2}{P_{n-N}^2} \sum_{k=1}^{N} k u_k^2 = O(\varepsilon^2),$$

whence

(2.3) $$A = o(1) \quad \text{as} \quad n \to \infty .$$

By Lemma 4 we obtain

$$\frac{n}{P_n^2} \sum_{k=0}^{n} p_{n-k}^2 = \frac{n}{P_n^2} \sum_{k=0}^{n} p_k^2 = \frac{n}{P_n^2} \sum_{k=0}^{n} \frac{(k+1)^2 p_k^2}{P_k^2} \cdot \frac{P_k^2}{(k+1)^2}$$

$$= O(1) \frac{n}{P_n^2} \sum_{k=0}^{n} \frac{P_k^2}{(k+1)^2} = O(1) \cdot O(1) = O(1).$$

Hence

$$B^2 \leqslant \frac{1}{P_n^2} \sum_{k=N+1}^{n} k p_{n-k}^2 \sum_{k=N+1}^{n} k u_k^2 = O(1) \cdot O(\varepsilon) = O(\varepsilon),$$

and consequently,

(2.4) $$B_n = o(1) \quad \text{as} \quad n \to \infty .$$

It follows from (2.1), (2.2), (2.3) and (2.4) that

$$A + B = o(1) \quad \text{as} \quad n \to \infty ,$$

if $\{p_n\} \in M^a$, $a > \frac{1}{2}$, and we conclude the lemma.

Remark. Lemma 5 belongs to the Tauberian theorems; it is an analogon in the case of Nörlund's methods of a well-known theorem of Fejér on Abel's method (cf. [1], p. 71). However, (N, p_n)-$\sum u_n = s$ does not imply in general A-$\sum u_n = s$, for the series $\sum u_n x^n$ is not necessarily convergent for $0 < x < 1$ at all (cf. [3], p. 66).

LEMMA 6. *If an increasing sequence of indices $\{n_k\}$ is lacunary, then the sequence $\{P_{n_k}\}$ where $P_n = p_1 + \ldots + p_n$, $\{p_n\} \in M^a$, $a > 0$, is also lacunary.*

Proof. Assuming

$$\frac{n_{k+1}}{n_k} \geqslant q > 1 \quad \text{for} \quad k = 0, 1, 2, \ldots$$

we shall prove the existence of a constant $q_1 > 1$ such that

$$\frac{P_{n_{k+1}}}{P_{n_k}} \geqslant q_1 > 1 \quad \text{for} \quad k = 0, 1, 2, \ldots$$

Since $\{p_n\} \in M^a$, $a > 0$, for any $0 < \varepsilon < a$ there is an integer k_0 such that

$$\frac{n_k p_{n_k}}{P_{n_k}} > a - \varepsilon \quad \text{for} \quad k \geqslant k_0 .$$

If $0 < p_n \nearrow$, then we have

$$\frac{P_{n_{k+1}}}{P_{n_k}} > 1 + \frac{p_{n_k}(n_{k+1} - n_k)}{P_{n_k}} = 1 + \frac{n_k p_{n_k}}{P_{n_k}} \left(\frac{n_{k+1}}{n_k} - 1 \right) > 1 + (a - \varepsilon)(q - 1)$$

for $k \geqslant k_0$. If $0 \leqslant k < k_0$, then

$$\frac{P_{n_{k+1}}}{P_{n_k}} \geqslant 1 + \frac{p_{n_0}}{P_{n_{k_0}}} .$$

Hence we may take $q_1 = \min[1 + (a - \varepsilon)(q - 1), 1 + p_{n_0}/P_{n_{k_0}}]$.

Now, if $0 < p_n \searrow$, then we have

$$\begin{aligned} \frac{P_{n_{k+1}}}{P_{n_k}} &= 1 + \frac{p_{n_k+1} + \ldots + p_{n_{k+1}}}{P_{n_k}} \\ &> 1 + \frac{p_{n_{k+1}}}{P_{n_k}} (n_{k+1} - n_k) > 1 + \frac{n_{k+1} p_{n_{k+1}}}{P_{n_{k+1}}} \left(1 - \frac{n_k}{n_{k+1}} \right) \\ &> 1 + (a - \varepsilon)(1 - 1/q) \end{aligned}$$

for $k \geqslant k_0$. If $0 \leqslant k < k_0$, then

$$\frac{P_{n_{k+1}}}{P_{n_k}} > 1 + \frac{p_{n_{k_0}+1} + \ldots + p_{n_{k_0+1}}}{P_{n_{k_0+1}}} > 1 + (a - \varepsilon)(1 - 1/q) .$$

Hence we may take $q_1 = 1 + (a - \varepsilon)(1 - 1/q) > 1$. Then, in both cases there exists such a number q_1 that

$$\frac{P_{n_{k+1}}}{P_{n_k}} > q_1 > 1 \quad \text{for} \quad k = 0, 1, 2, \ldots$$

3. Let $ON\{\varphi_n(x)\}$ be an orthonormal system of real functions defined in the interval $\varphi_n \in L^2(0, 1)$. We shall denote by

$$\sum_{n=0}^{\infty} c_n \varphi_n(x) \tag{3.1}$$

orthogonal series corresponding to the system $ON\{\varphi_n(x)\}$ with real coefficients c_n satisfying the condition $\{c_n\} \in l^2$, i.e.

$$\sum_{n=0}^{\infty} c_n^2 < \infty . \tag{3.2}$$

In order to formulate further lemmas we introduce the following notation:

$$\begin{aligned} s_k(x) &= c_0\varphi_0(x) + c_1\varphi_1(x) + \dots + c_k\varphi_k(x) , \\ s_k^*(x) &= c_0\lambda_0\varphi_0(x) + c_1\lambda_1\varphi_1(x) + \dots + c_k\lambda_k\varphi_k(x) , \\ t_n(x) &= \frac{1}{P_n}\sum_{k=0}^{n} p_{n-k}s_k(x) , \\ t_n^*(x) &= \frac{1}{P_n}\sum_{k=0}^{n} p_{n-k}s_k^*(x) , \end{aligned} \tag{3.3}$$

where $\{\lambda_n\}$ is a sequence (dependent on $\{c_n\}$), increasing to ∞ and satisfying the condition

$$\sum_{n=0}^{\infty} c_n^2\lambda_n^2 < \infty . \tag{3.4}$$

As is well known, such a sequence $\{\lambda_n\}$ exists.

LEMMA 7. *If* $0 < w(n) \nearrow \infty$, $\{p_n\} \in M^a$, $a \geqslant 0$, *and if*

$$t_n^*(x) \doteq O[w(n)] \quad \textit{as} \quad n \to \infty \text{ (}^1\text{)} ,$$

then

$$t_n(x) \doteq o[w(n)] \quad \textit{as} \quad n \to \infty .$$

Proof. Applying Abel's transformation twice and omitting the argument x for the sake of brevity, we obtain

$$\begin{aligned} t_n &= \frac{1}{P_n}\sum_{k=0}^{n} p_{n-k}\sum_{\nu=0}^{k} c_\nu\varphi_\nu\frac{\lambda_\nu}{\lambda_\nu} \\ &= \frac{1}{P_n}\sum_{k=0}^{n} p_{n-k}\frac{s_k^*}{\lambda_k} + \frac{1}{P_n}\sum_{k=0}^{n} p_{n-k}\sum_{\nu=0}^{k-1} s_\nu^*\Delta\frac{1}{\lambda_\nu} \\ &= \frac{1}{\lambda_n P_n}\sum_{k=0}^{n} p_{n-k}\sum_{\nu=0}^{k} c_\nu\lambda_\nu\varphi_\nu + \end{aligned}$$

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[1] If the sequence $\{f_n(x)/g_n(x)\}$ tends to zero, resp. is bounded in the interval $\langle 0, 1\rangle$ a.e. (almost everywhere), we shall write $f_n(x) \doteq o[g_n(x)]$, resp. $f_n(x) \doteq O[g_n(x)]$.

$$+\frac{1}{P_n}\sum_{k=0}^{n-1}\Big(\sum_{\nu=0}^{k}p_{n-\nu}s_\nu^*\Big)\Delta\frac{1}{\lambda_k}+\frac{1}{P_n}\sum_{k=0}^{n}p_{n-k}\sum_{\nu=0}^{k-1}s_\nu^*\Delta\frac{1}{\lambda_\nu}$$

$$=\frac{1}{\lambda_n P_n}\sum_{k=0}^{n}p_{n-k}s_k^*+\frac{1}{P_n}\sum_{k=0}^{n-1}\Big(\sum_{\nu=0}^{k}p_{n-\nu}s_\nu^*\Big)\Delta\frac{1}{\lambda_k}+\frac{1}{P_n}\sum_{k=0}^{n}p_{n-k}\sum_{\nu=0}^{k-1}s_\nu^*\Delta\frac{1}{\lambda_\nu}.$$

Hence

$$t_n=\frac{t_n^*}{\lambda_n}+\frac{1}{P_n}\sum_{k=0}^{n-1}\Big(\sum_{\nu=0}^{k}p_{n-\nu}s_\nu^*\Big)\Delta\frac{1}{\lambda_k}+\frac{1}{P_n}\sum_{k=0}^{n}p_{n-k}\sum_{\nu=0}^{k-1}s_\nu^*\Delta\frac{1}{\lambda_\nu}. \tag{3.5}$$

If $0<p_n\nearrow$, then $p_{n-\nu}\leqslant p_n$ for $0\leqslant\nu\leqslant n$. However, it is easily seen that the series

$$\sum_{k=0}^{\infty}\frac{\Delta\frac{1}{\lambda_k}}{k+1}\sum_{\nu=0}^{k}|s_\nu^*|$$

is convergent a.e. Hence

$$\Big|\frac{1}{P_n}\sum_{k=0}^{n-1}\Big(\sum_{\nu=0}^{k}p_{n-\nu}s_\nu^*\Big)\Delta\frac{1}{\lambda_k}\Big|\leqslant\frac{1}{P_n}\sum_{k=0}^{n}\Big(\sum_{\nu=0}^{k}p_{n-\nu}|s_\nu^*|\Big)\Delta\frac{1}{\lambda_k}$$

$$\leqslant\frac{p_n}{P_n}\sum_{k=0}^{n}\Delta\frac{1}{\lambda_k}\sum_{\nu=0}^{k}|s_\nu^*|=\frac{(n+1)p_n}{P_n}\cdot\frac{1}{(n+1)}\sum_{k=0}^{n}\Delta\frac{1}{\lambda_k}\sum_{\nu=0}^{k}|s_\nu^*|\doteq o(1)\,.$$

Now, if $0<p_n\nearrow$, then the second term on the right side of (3.5) tends to zero a.e. We now show the same to be true if $0<p_n\searrow$. Namely, we then have $0<p_{n-\nu}\leqslant p_{k-\nu}$ for $0\leqslant\nu\leqslant k\leqslant n$. Hence

$$\Big|\frac{1}{P_n}\sum_{k=0}^{n-1}\Big(\sum_{\nu=0}^{k}p_{n-\nu}s_\nu^*\Big)\Delta\frac{1}{\lambda_k}\Big|\leqslant\frac{1}{P_n}\sum_{k=0}^{n}\Big(\sum_{\nu=0}^{k}p_{n-\nu}|s_\nu^*|\Big)\Delta\frac{1}{\lambda_k}$$

$$\leqslant\frac{1}{P_n}\sum_{k=0}^{n}P_k\Big(\frac{1}{P_k}\sum_{\nu=0}^{k}p_{k-\nu}|s_\nu^*|\Big)\Delta\frac{1}{\lambda_k}.$$

But the last expression tends to zero a.e., for the series

$$\sum_{k=0}^{\infty}\frac{1}{P_k}\Big(\sum_{\nu=0}^{k}p_{k-\nu}|s_\nu^*|\Big)\Delta\frac{1}{\lambda_k}$$

is convergent a.e.

Finally, the third term on the right side of (3.5) converges a.e., for the series

$$\sum_{\nu=0}^{\infty}|s_\nu^*|\Delta\frac{1}{\lambda_\nu}$$

is convergent a.e.

Now, dividing both sides of (3.5) by $w(n)$ and taking into account the assumptions of the Lemma and the properties of the sequence $\{\lambda_n\}$, we conclude the proof of Lemma 7.

Applying the results of T. Pati (cf. [6], p. 155), we now prove

LEMMA 8. (²) *Let $t_n(x)$ denote the n-th (N, p_n)-means of the orthogonal series* (3.1) *with coefficients c_n satisfying condition* (3.2) *and let $\{p_n\} \in BVM^a$, $a > \frac{1}{2}$. Then the series*

$$\sum_{n=1}^{\infty} n[t_n(x)-t_{n-1}(x)]^2$$

is convergent a.e.

Proof. Writing $p_{-1} = P_{-1} = 0$, we have

$$\begin{aligned} t_n - t_{n-1} &= \frac{1}{P_n}\sum_{k=0}^{n} p_{n-k}s_k - \frac{1}{P_{n-1}}\cdot\sum_{k=0}^{n-1} p_{n-1-k}s_k \\ &= \frac{p_0 s_n}{P_n} + \frac{1}{P_nP_{n-1}}\sum_{k=0}^{n-1} s_k(p_{n-k}P_{n-1} - p_{n-k-1}P_n) \\ &= \frac{1}{P_nP_{n-1}}\sum_{k=0}^{n} c_k\varphi_k \sum_{\nu=k}^{n}(p_{n-\nu}P_{n-1} - p_{n-\nu-1}P_n) \\ &= \frac{1}{P_nP_{n-1}}\sum_{k=0}^{n} c_k\varphi_k[(P_n-p_n)P_{n-k} - P_{n-k-1}P_n] \\ &= \frac{1}{P_nP_{n-1}}\sum_{k=0}^{n} c_k\varphi_k(P_np_{n-k} - p_nP_{n-k}) \\ &= \frac{1}{P_nP_{n-1}}\sum_{k=0}^{n} c_{n-k}\varphi_{n-k}(p_kP_n - p_nP_k)\,. \end{aligned}$$

Thus, writing $c_{n,k} = c_{n-k}\varphi_{n-k}$, we get

$$t_n - t_{n-1} = \frac{1}{P_nP_{n-1}}\sum_{k=0}^{n} c_{n,k}(p_kP_n - p_nP_k)\,.$$

The last expression may be written in the form

$$t_n - t_{n-1} = \frac{n+1}{P_nP_{n-1}}\sum_{k=0}^{n} c_{n,k}\left(\frac{p_kP_n}{n+1} - \frac{p_nP_k}{k+1} + \frac{p_nP_k}{k+1} - \frac{p_nP_k}{n+1}\right)$$

$$= \frac{n+1}{P_nP_{n-1}}\sum_{k=0}^{n}\left(\frac{p_kP_n}{n+1} - \frac{p_nP_k}{k+1}\right)c_{n,k} + \frac{p_n}{P_nP_{n-1}}\sum_{k=0}^{n}\frac{P_k}{k+1}(n-k)c_{n,k} = \Sigma_1 + \Sigma_2\,.$$

(²) Lemma 8 is a generalization of the theorem known as the Kaczmarz-Zygmund lemma (see e.g. [5], p. 184-185).

Since

$$\frac{P_n}{n+1} = P_n S_n - P_{n-1} S_{n-1} = -P_{n-1}\Delta S_{n-1} + p_n S_n ,$$

where

$$S_n = \frac{1}{P_n}\sum_{k=0}^{n}\frac{P_k}{k+1},$$

we have

$$\Sigma_1 = \frac{n+1}{P_n P_{n-1}}\Big[-P_{n-1}\Delta S_{n-1}\sum_{k=0}^{n} p_k c_{n,k} + p_n S_n \sum_{k=0}^{n} p_k c_{n,k} +$$

$$+ p_n \sum_{k=0}^{n} P_{k-1}\Delta S_{k-1} c_{n,k} - p_n \sum_{k=0}^{n} p_k S_k c_{n,k}\Big]$$

$$= \frac{n+1}{P_n P_{n-1}}\Big[p_n \sum_{k=0}^{n} P_{k-1}\Delta S_{k-1} c_{n,k} - P_{n-1}\Delta S_{n-1}\sum_{k=0}^{n} p_k c_{n,k} -$$

$$- p_n \sum_{\nu=0}^{n-1}\Big(\sum_{k=0}^{\nu} p_k c_{n,k}\Big)\Delta S_\nu\Big].$$

Similarly,

$$\Sigma_2 = \frac{p_n}{P_n P_{n-1}}\sum_{k=0}^{n} p_k S_k (n-k) c_{n,k} - \frac{p_n}{P_n P_{n-1}}\sum_{k=0}^{n} P_{k-1}\Delta S_{k-1}(n-k) c_{n,k} .$$

Since

$$(n+1)\sum_{k=0}^{\nu} p_k c_{n,k} = \sum_{k=0}^{\nu} p_k (n-k) c_{n,k} + \sum_{k=0}^{\nu} p_k (k+1) c_{n,k}$$

for $0 \leqslant \nu \leqslant n$, we obtain after reduction

$$\Sigma_1 + \Sigma_2 = \frac{1}{P_n P_{n-1}}\Big[-p_n \sum_{\nu=0}^{n-1}\Delta S_\nu \sum_{k=0}^{\nu} p_k (n-k) c_{n,k} -$$

$$- p_n \sum_{\nu=0}^{n-1}\Delta S_\nu \sum_{k=0}^{\nu}(k+1) p_k c_{n,k} - P_{n-1}\Delta S_{n-1}\sum_{k=0}^{n} p_k (n-k) c_{n,k} -$$

$$- P_{n-1}\Delta S_{n-1}\sum_{k=0}^{n} p_k (k+1) c_{n,k} + p_n \sum_{k=0}^{n} P_{k-1} S_{k-1}(k+1) c_{n,k} +$$

$$+ p_n \sum_{k=0}^{n} p_k S_k (n-k) c_{n,k}\Big].$$

Replacing the first term in the square brackets by

$$-p_n \sum_{k=0}^{n-1} p_k(n-k)c_{n,k} \sum_{\nu=k}^{n-1} \Delta S_\nu$$
$$= -p_n \sum_{k=0}^{n-1} p_k S_k(n-k)c_{n,k} + p_n S_n \sum_{k=0}^{n-1} p_k(n-k)c_{n,k},$$

and noticing that the last but one term in the last formula may be reduced with the last term in the square brackets and then changing the order of summation, we may finally write

$$t_n - t_{n-1} = \frac{1}{P_n P_{n-1}} \Bigg[p_n S_n \sum_{k=0}^{n-1} p_k(n-k)c_{n,k} -$$
$$-p_n \sum_{k=0}^{n-1} (k+1) p_k c_{n,k} \Big(\sum_{\nu=k}^{n-1} \Delta S_\nu \Big) - P_{n-1} \Delta S_{n-1} \sum_{k=0}^{n} p_k(n-k)c_{n,k} -$$
$$- P_{n-1} \Delta S_{n-1} \sum_{k=0}^{n} p_k(k+1)c_{n,k} + p_n \sum_{k=0}^{n} P_{k-1} \Delta S_{k-1}(k+1)c_{n,k} \Bigg].$$

Hence we find

$$\sum_{n=1}^{\infty} n \int_0^1 (t_n - t_{n-1})^2 dx = O(1) \Bigg[\sum_{n=1}^{\infty} \frac{n p_n^2 S_n^2}{P_n^2 P_{n-1}^2} \sum_{k=1}^{n-1} k^2 c_k^2 p_{n-k}^2 +$$
$$+ \sum_{n=1}^{\infty} \frac{n p_n^2}{P_n^2 P_{n-1}^2} \sum_{k=0}^{n-1} (n-k-1)^2 p_{n-k}^2 \Big(\sum_{\nu=n-k}^{n-1} |\Delta S_\nu| \Big)^2 c_k^2 +$$
$$+ \sum_{n=1}^{\infty} \frac{n (\Delta S_{n-1})^2}{P_n^2} \sum_{k=1}^{n} k^2 p_{n-k}^2 c_k^2 +$$
$$+ \sum_{n=1}^{\infty} \frac{n (\Delta S_{n-1})^2}{P_n^2} \sum_{k=1}^{n} p_{n-k}^2 (n-k+1)^2 c_k^2 +$$
$$+ \sum_{n=1}^{\infty} \frac{n p_n^2}{P_n^2 P_{n-1}^2} \sum_{k=1}^{n} P_{n-k-1}^2 (\Delta S_{n-k-1})^2 (n-k+1)^2 c_k^2 \Bigg]$$
$$= O(1)[\mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV} + \mathrm{V}].$$

Since

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{P_n/(n+1)}{p_n} = \frac{1}{\alpha},$$

we have by lemma 4

$$\mathrm{I} = O(1)\Big[\sum_{k=1}^{\infty} k^2c_k^2 \sum_{n=k+1}^{2k} \frac{p_nS_n^2}{P_nP_{n-1}^2}p_{n-k}^2 + \sum_{k=1}^{\infty} k^2c_k^2 \sum_{n=2k+1}^{\infty} \frac{p_nS_n^2}{P_nP_{n-1}^2}p_{n-k}^2\Big]$$

$$= O(1)\Big[\sum_{k=1}^{\infty} \frac{kc_k^2}{P_k^2} \sum_{n=1}^{k} \frac{P_n^2}{n^2} + \sum_{k=1}^{\infty} k^2c_k^2 \sum_{n=k}^{\infty} \frac{1}{n^3}\Big] = O(1)\,.$$

Similarly, we estimate the next two expressions, II and III, taking into account the relation $\Delta S_n = O(1/n)$ and Lemma 3:

$$\mathrm{II} = O(1)\Big[\sum_{k=1}^{\infty} k^2c_k^2 \sum_{n=k+1}^{2k} \frac{p_np_{n-k}^2}{P_nP_{n-1}^2} + \sum_{k=1}^{\infty} k^2c_k^2 \sum_{n=k}^{\infty} \frac{1}{n^3}\Big] = O(1)\,,$$

$$\mathrm{III} = O(1)\Big[\sum_{k=1}^{\infty} k^2c_k^2 \sum_{n=k}^{2k} \frac{p_{n-k}^2}{nP_n^2} + \sum_{k=1}^{\infty} k^2c_k^2 \sum_{n=2k+1}^{\infty} \frac{p_{n-k}^2}{nP_n^2}\Big] = O(1)\,.$$

We estimate the last two expressions, IV and V, applying the additional assupmtion $\{p_n\} \epsilon\, BVM^a$, $a > \frac{1}{2}$:

$$\mathrm{IV} = O(1)\Big[\sum_{k=1}^{\infty} c_k^2 \sum_{n=k+1}^{\infty} (n-k-1)^2p_{n-k}^2 \frac{n(\Delta S_{n-1})^2}{P_n^2}\Big]$$

$$= O(1)\sum_{k=0}^{\infty} c_k^2 \sum_{n=k+1}^{\infty} \frac{P_{n-k}^2}{P_n^2}|\Delta S_{n-1}| = O(1)\,,$$

$$\mathrm{V} = O(1)\Big[\sum_{k=0}^{\infty} P_k^2c_k^2 \sum_{n=k+1}^{2k} \frac{1}{nP_{n-1}^2} + \sum_{k=0}^{\infty} c_k^2 \sum_{n=2k+1}^{\infty} |\Delta S_{n-k-1}|\Big] = O(1)\,.$$

Thus we have proved the series given in Lemma 8 to be convergent a.e.

LEMMA 9. (³) *Let $t_n(x)$ be the n-th (N, p_n)-means of the orthogonal series* (3.1) *with coefficients c_n satisfying condition* (3.2) *and let $\{p_n\} \epsilon\, M^a$, $a > 0$. Finally, let $\{n_k\}$ be an arbitrary increasing sequence of indices satisfying the condition*

$$1 < q \leqslant \frac{n_{k+1}}{n_k} \quad \textit{for} \quad k = 1, 2, \ldots, \quad q = a \textit{ constant}\,,$$

of lacunarity. Then the series

$$\sum_{k=0}^{\infty} [s_{n_k}(x) - t_{n_k}(x)]^2$$

is convergent a.e.

(³) Lemma 9 is a generalization of a theorem of Kolmogoroff (see e.g. [1], p. 111-113, 2.7.1).

Proof. We have

$$s_n - t_n = \sum_{k=0}^{n} c_k \varphi_k - \frac{1}{P_n} \sum_{k=0}^{n} p_{n-k} s_k$$
$$= \sum_{k=0}^{n} c_k \varphi_k - \frac{1}{P_n} \sum_{k=0}^{n} c_k \varphi_k \sum_{\nu=k}^{n} p_{n-\nu},$$

i.e.

$$s_n - t_n = \frac{1}{P_n} \sum_{k=0}^{n} c_k \varphi_k \Big(\sum_{i=n-k+1}^{n} p_i \Big).$$

Hence

$$\int_0^1 (s_n - t_n)^2 dx = \frac{1}{P_n^2} \sum_{k=0}^{n} c_k^2 \Big(\sum_{i=n-k+1}^{n} p_i \Big)^2. \tag{3.6}$$

If $0 < p_n \nearrow$, then $p_i \leqslant p_n$ for $n-k+1 \leqslant i \leqslant n$, whence

$$\int_0^1 (s_n - t_n)^2 dx < \frac{p_n^2}{P_n^2} \sum_{i=1}^{n} i^2 c_i^2. \tag{3.7}$$

Replacing n by n_k in the last formula, we obtain after summation from $k = 1$ to ∞

$$\sum_{k=1}^{\infty} \int_0^1 (s_{n_k} - t_{n_k})^2 dx = \sum_{k=1}^{\infty} \frac{n_k^2 p_{n_k}^2}{P_{n_k}^2} \frac{1}{n_k^2} \sum_{i=1}^{n_k} i^2 c_i^2$$
$$= O(1) \sum_{k=1}^{\infty} \frac{1}{n_k^2} \sum_{i=1}^{n_k} i^2 c_i^2 = O(1) \sum_{i=1}^{\infty} i^2 c_i^2 \sum_{n_k \geqslant i} \frac{1}{n_k^2}$$
$$= O(1) \frac{q^2}{q^2-1} \sum_{i=1}^{\infty} c_i^2 = O(1).$$

If $0 < p_n \searrow$, we decompose the sum in formula (3.6) into two sums, writing

$$\int_0^1 (s_n - t_n)^2 dx < \frac{1}{P_n^2} \sum_{\nu=0}^{[n/2]} c_\nu^2 \Big(\sum_{i=n-\nu+1}^{n} p_i \Big)^2 + \sum_{\nu=[n/2]+1}^{n} c_\nu^2 \; (4).$$

Since $n-\nu+1 \geqslant \nu+1$ for $\nu \leqslant [n/2]$ and $p_i \leqslant p_{n-\nu+1} \leqslant p_{\nu+1} < p_\nu$ for $n-\nu+$ $+1 \leqslant i \leqslant n$, we have

$$\int_0^1 (s_n - t_n)^2 dx < \frac{1}{P_n^2} \sum_{i=1}^{[n/2]} i^2 p_i^2 c_i^2 + \sum_{i=[n/2]+1}^{n} c_i^2. \tag{3.8}$$

(4) The symbol $[x]$ denotes the greatest integer not greater than x.

Now let s be the least positive integer satisfying the inequality $q^s \geqslant 2$. Then we may assume that the inequality $n_k/n_{k-s} \geqslant 2$ holds for every positive integer $k \geqslant s$. Replacing n by n_k in inequality (3.8), we obtain by Lemma 6, after the summation of both sides from $k = s$ to ∞,

$$\sum_{k=s}^{\infty} \int_0^1 (s_{n_k} - t_{n_k})^2 dx < \sum_{k=s}^{\infty} \frac{1}{P_{n_k}^2} \sum_{i=1}^{n_k} i^2 p_i^2 c_i^2 + s \sum_{k=s}^{\infty} \sum_{i=n_{k-1}+1}^{n_k} c_i^2$$

$$< \sum_{i=1}^{\infty} i^2 p_i^2 c_i^2 \sum_{n_k \geqslant i} \frac{1}{P_{n_k}^2} + s \sum_{i=1}^{\infty} c_i^2 \ (^5)$$

$$< \frac{q_1^2}{q_1^2 - 1} \sum_{i=1}^{\infty} \frac{i^2 p_i^2 c_i^2}{P_i^2} + s \sum_{i=1}^{\infty} c_i^2 = O(1) \sum_{i=1}^{\infty} c_i^2 = O(1).$$

Hence, the series

$$\sum_{k=s}^{\infty} [s_{n_k}(x) - t_{n_k}(x)]^2$$

is convergent a.e., which completes the proof of Lemma 9.

4. THEOREM 1. [6] *Let $s_n(x)$ be the n-th partial sum of the orthogonal series* (3.1) *with coefficients c_n satisfying condition* (3.2), *and let $\{n_k\}$ be an arbitrary increasing sequence of indices satisfying the condition*

$$(4.1) \qquad 1 < q \leqslant \frac{n_{k+1}}{n_k} \leqslant r \quad \text{for} \quad k = 0, 1, 2, \dots,$$

where r and q are constants. Finally, let $\{p_n\} \in BVM^a$, $a > \frac{1}{2}$. Then the orthogonal series (3.1) *is (N, p_n)-summable a.e. if and only if the sequence $\{s_{n_k}(x)\}$ is convergent a.e.*

Proof. Necessity. Suppose that the orthogonal series (3.1) satisfying condition (3.2) is (N, p_n)-summable a.e., where $\{p_n\} \in BVM^a$, $a > \frac{1}{2}$. Hence the sequence $\{t_{n_k}(x)\}$ converges a.e. for an arbitrary increasing sequence of indices satisfying condition (4.1), and by Lemma 9 so does the sequence $\{s_{n_k}(x)\}$.

Sufficiency. Assuming the sequence $\{s_{n_k}(x)\}$ to be convergent a.e. for an increasing sequence of indices satisfying condition (4.1), the sequence $\{t_{n_k}(x)\}$ is also convergent a.e., by Lemma 9.

Now let p be an arbitrary positive integer satisfying the condition

$$n_k < p < n_{k+1} \quad \text{for} \quad k = 0, 1, 2, \dots$$

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(5) We have $1 + [n_k/2] > n_{k-s}$, for otherwise we should have $n_{k/2} < [n_k/2] + 1 \leqslant n_{k-s}$, i.e. $n_k/n_{k-s} < 2$, a contradiction.

(6) Theorem 1 is a generalization of the Kaczmarz-Zygmund theorem concerning the method (C, a), $a > 0$, in the case $a > \frac{1}{2}$.

Then

$$(t_p - t_{n_k})^2 = \Big[\sum_{n=n_k+1}^{p} (t_n - t_{n-1})\Big]^2 \leqslant \sum_{n=n_k+1}^{n_{k+1}} n(t_n - t_{n-1})^2 \sum_{n=n_k+1}^{n_{k+1}} \frac{1}{n}.$$

The expression on the right side of this inequality tends to zero a.e., as follows from Lemma 8 and from the estimation

$$\sum_{n=n_k+1}^{n_{k+1}} \frac{1}{n} < \frac{1}{n_k}(n_{k+1} - n_k) = \frac{n_{k+1}}{n_k} - 1 \leqslant r - 1 .$$

The sequence $\{t_{n_k}\}$ being convergent a.e., it follows that $\{t_n\}$ is also convergent a.e.

In order to formulate further theorems, we extend the sequence $\{n_k\}$ satisfying the condition of lacunarity (4.1) to a continuous and strictly increasing function $n(x)$, assuming the value $n(k) = n_k$ at $x = k$ for $k = 0, 1, 2, \ldots$, by means of linear interpolation. We denote the inverse of the function $n(x)$ by $l(x)$. Evidently, the function $l(x)$ is continuous and strictly increasing.

THEOREM 2. (7) *If*

$$\sum_{n=1}^{\infty} c_n^2 \log^2[l(n)] < \infty , \tag{4.2}$$

then the orthogonal series (3.1) *is* (N, p_n)*-summable a.e. for every* $\{p_n\} \in BVM^a$, $a > \frac{1}{2}$.

Proof. Let us write

$$A_k = \sqrt{\sum_{\nu=n_k+1}^{n_{k+1}} c_\nu^2}, \quad \Phi_k(x) = \frac{1}{A_k} \sum_{\nu=n_k+1}^{n_{k+1}} c_\nu \varphi_\nu(x) .$$

Obviously, the system $\{\Phi_k(x)\}$ is orthonormal and

$$s_{n_k}(x) = s_{n_0}(x) + \sum_{\nu=n_0+1}^{k-1} A_\nu \Phi_\nu(x) .$$

Condition (4.1) and the definition of the function $l(x)$ imply

$$\sum_{k=2}^{\infty} A_k^2 \log^2 k = \sum_{k=2}^{\infty} \log^2 k \sum_{\nu=n_k+1}^{n_{k+1}} c_\nu^2$$
$$= \sum_{k=2}^{\infty} \log^2[ln(k)] \sum_{\nu=n_k+1}^{n_{k+1}} c_\nu^2 < \sum_{\nu=n_0+1}^{\infty} c_\nu^2 \log^2[l(\nu)] < \infty .$$

(7) Theorems 2 and 3 constitute a generalization of the Kaczmarz-Menchoff theorem concerning the method (C, a), $a > 0$, in the case $a > \frac{1}{2}$ (see e.g. [1], p. 114-116, 2.8.1-2.8.2).

By the theorem of Rademacher and Menchoff, the series $\sum_{k=1}^{\infty} A_k \Phi_k(x)$ is convergent a.e. Hence the sequence $\{s_{n_k}(x)\}$ is convergent a.e. Moreover, we obviously have $\{c_n\} \in l^2$. Thus by Theorem 1 the series (3.1) is (N, p_n)-summable a.e. if $\{p_n\} \in BVM^a$, $a > \frac{1}{2}$.

THEOREM 3. *Let $\{v(n)\}$ be an arbitrary sequence of numbers satisfying the conditions $v(n) = o\{\log^2[l(n)]\}$, $0 \leqslant v(n) \leqslant v(n+1) \to \infty$. Moreover, let $\{p_n\} \in BVM^a$, $a > \frac{1}{2}$. Then there exists a system $ON\{\psi_n(x)\}$ and a sequence of numbers $\{b_n\}$ such that*

1° $\sum_{n=0}^{\infty} b_n^2 v(n) < \infty$,

2° *the series $\sum_{n=0}^{\infty} b_n \psi_n(x)$ is not (N, p_n)-summable at any point of the interval $\langle 0, 1\rangle$.*

Proof. We shall base ourselves here on the following theorem of D. Menchoff.

If $w(n) = o(\log^2 n)$, $0 \leqslant w(n) \leqslant w(n+1) \to \infty$, then there exist a system $ON\{\varphi_n(x)\}$ and a sequence of numbers $\{a_n\}$ such that

(a) $\sum_{n=0}^{\infty} a_n^2 w(n) < \infty$,

(b) the series $\sum_{n=0}^{\infty} a_n \varphi_n(x)$ is divergent at every point of the interval $\langle 0, 1\rangle$.

Proceeding to the proof of Theorem 3, we write $w(k) = v(n_k)$, where $\{n_k\}$ is an arbitrary sequence of indices satisfying the condition of lacunarity (4.1). Hence

$$0 \leqslant w(k) = v(n_k) = o\{\log^2[l(n_k)]\} = o(\log^2 k),$$

and by Menchoff's theorem there exist an orthonormal system $\{\varphi_n(x)\}$ and a sequence of numbers $\{a_n\}$ such that conditions (a) and (b) are satisfied.

Let $\{N_k\}$ be the increasing sequence of indices defined in Menchoff's proof (cf. [4], p. 195). We define the sequence $\{\psi_n(x)\}$ as follows:

$$\psi_{n_m}(x) = \varphi_n(x), \quad b_{n_m} = a_m \quad \text{for} \quad N_{k-1} \leqslant m < N_k,$$
$$\psi_n(x) = \varphi_{N_k}(x), \quad b_n = 0 \quad \text{for} \quad n \neq m.$$

Evidently, the sequence $\{\psi_n(x)\}$ is orthonormal in $\langle 0, 1\rangle$. Obviously, we have

$$\sum_{n=0}^{\infty} b_n^2 v(n) = \sum_{m=0}^{\infty} b_{n_m}^2 v(n_m) = \sum_{m=0}^{\infty} a_m^2 w(m) < \infty.$$

Thus condition 1° is satisfied. We now prove that the series $\sum_{n=0}^{\infty} b_n \psi_n(x)$ ist not (N, p_n)-summable at any point of the interval $\langle 0, 1\rangle$. Denoting by $s_n(x)$, resp. $\bar{s}_n(x)$, the n-th partial sums of the series (b), resp. the series 2°, we may write $\bar{s}_{n_m}(x) = s_m(x)$. Since the sequence $\{s_m(x)\}$ is divergent at every point of the interval $\langle 0, 1\rangle$, the sequence $\{\bar{s}_{n_m}(x)\}$ is also divergent at every point of this interval.

If $\{\bar{t}_n(x)\}$ is the sequence of (N, p_n)-means of the series 2°, then by Lemma 9 the sequence $\{\bar{t}_{n_m}(x)\}$ is divergent a.e. in the interval $\langle 0, 1\rangle$, i.e. in a set $\langle 0, 1\rangle - E$, where E is a set of measure zero. Now, defining $\psi_{n_m}(x) = 1/b_{n_m}$ for $x \in E$, we conclude that the sequence $\{\bar{t}_{n_m}(x)\}$ is divergent at every point of the interval $\langle 0, 1\rangle$. Hence the sequence $\{\bar{t}_n(x)\}$ is divergent in the whole interval $\langle 0, 1\rangle$.

Thus we have proved that condition (4.2) cannot be improved in the sense of Menchoff.

THEOREM 4. ([8]) *If* $\{p_n\} \in BVM^a$, $a > \frac{1}{2}$, *and if* $t_n(x)$ *denote* (N, p_n)-*means of the orthogonal series* (3.1) *with coefficients* c_n *satisfying condition* (3.2), *then*

$$t_n(x) \doteq o\{\log[l(n)]\} \quad \textit{as} \quad n \to \infty .$$

Proof. Let $\{\mu_n\}$ be an arbitrary sequence of positive numbers increasing to ∞. Let us write

$$\bar{s}_n^*(x) = \sum_{\nu=0}^{n} \frac{c_\nu \lambda_\nu \varphi_\nu(x)}{\mu_\nu}, \qquad \bar{t}_n^*(x) = \frac{1}{P_n} \sum_{k=0}^{n} p_{n-k} \bar{s}_k^*(x) , \tag{4.3}$$

where $\{\lambda_n\}$ is the sequence defined by condition (3.4). By formula (3.5) we may write

$$\frac{t_n^*}{\mu_n} = \bar{t}_n^* - \frac{1}{P_n} \sum_{k=1}^{n-1} p_{n-k} \sum_{\nu=0}^{k-1} s_\nu^* \Delta \frac{1}{\mu_\nu} - \frac{1}{P_n} \sum_{k=0}^{n} \Big(\sum_{\nu=0}^{k} p_{n-\nu} s_\nu^* \Big) \Delta \frac{1}{\mu_k} . \tag{4.4}$$

Now, if we take $\mu_n = \log[l(n)]$, Theorem 2 shows that the first term on the right side of (4.4) is bounded a.e. But the second and the third term are also bounded a.e., for the series

$$\sum_{\nu=0}^{\infty} |s_\nu^*| \Delta \frac{1}{\mu_\nu} \quad \text{and} \quad \sum_{k=0}^{\infty} \frac{1}{P_k} \Big(\sum_{\nu=0}^{k} p_{k-\nu} |s_\nu^*| \Big) \Delta \frac{1}{\mu_k}$$

are obviously convergent a.e.

Thus, we have

$$t_n^* \doteq O(\mu_n) \quad \text{as} \quad n \to \infty .$$

Hence it follows by Lemma 7 that

$$t_n \doteq o\{\log[l(n)]\} \quad \text{as} \quad n \to \infty .$$

([8]) Theorems 4 and 5 generalize theorems VII and VIII of K. Tandori concerning the method (C, a), $a > 0$, in the case $a > \frac{1}{2}$ (see [7], p. 101-111, or [1], p. 116).

THEOREM 5. *Let $\{v(n)\}$ be an arbitrary sequence of positive numbers increasing monotonically to ∞ and satisfying the condition*

$$v(n) = o\{\log[l(n)]\} \quad as \quad n \to \infty ,$$

and let $\{p_n\} \in BVM^a$, $a > \frac{1}{2}$. Then there exist a system $ON\{\Phi_n(x)\}$ and a sequence of numbers $\{a_n\} \in l^2$ satisfying the condition

$$\overline{\lim_{n\to\infty}} \frac{1}{v(n)} \left| \frac{1}{P_n} \sum_{k=0}^{n} p_{n-k} \sum_{\nu=0}^{k} a_\nu \Phi_\nu(x) \right| = \infty$$

at every point of the interval $\langle 0, 1\rangle$.

Proof. Without loss of generality we may assume that $v(n) \geqslant 1$, $\lambda_n \geqslant 1$, where $\{\lambda_n\}$ is the sequence defined by (3.4). Since $v(n) = o\{\log[l(n)]\}$, $v(n) \nearrow \infty$, and $\{p_n\} \in BVM^a$, $a > \frac{1}{2}$, from Theorem 4 follows the existence of a system $ON\{\Phi_n(x)\}$ and of a sequence of numbers $\{b_n\}$ such that $\sum_{n=0}^{\infty} b_n v^2(n) < \infty$ and that theseries $\sum_{n=0}^{\infty} b_n \Phi_n(x)$ is not (N, p_n)-summable at any point of the interval $\langle 0, 1\rangle$.

Let us choose $c_n = b_n v(n)$ and $\mu_n = \lambda_n v(n)$. By (3.3), (4.3) and (4.4) we have

$$\frac{t_n^*}{v(n)} = \lambda_n \left[\tilde{t}_n^* - \frac{1}{P_n} \sum_{k=0}^{n-1} p_{n-k} \sum_{\nu=0}^{k-1} s_\nu^* \Delta \frac{1}{\mu_\nu} - \frac{1}{P_n} \sum_{k=0}^{n} \left(\sum_{\nu=0}^{k} p_{n-\nu} s_\nu^* \right) \Delta \frac{1}{\mu_k} \right].$$

Arguments analogous to those used in the proof of Theorem 4 lead to the conclusion that the second and the third term in the square brackets tend to zero a.e. By Theorem 3, the first term is divergent at every point of the interval $\langle 0, 1\rangle$. Since $\lambda_n \nearrow \infty$, we have

$$\overline{\lim_{n\to\infty}} \frac{|t_n^*(x)|}{v(n)} = \infty \tag{4.5}$$

a.e. in $\langle 0, 1\rangle$.

We now show that (4.5) holds for every $x \in \langle 0, 1\rangle$ if we change in a suitable way the definition of $\{\Phi_n(x)\}$ in a set of measure zero. Let E be the set of all $x \in \langle 0, 1\rangle$ for which (4.5) does not hold and let $\{n_m\}$ be a lacunary sequence of indices satisfying the condition

$$\frac{n_{m+1}}{n_m} \geqslant a+1 > 1 \quad \text{for} \quad m = 0, 1, 2, \ldots$$

Further, let $\{\varphi_n(x)\}$, resp. $\{N_k\}$, be the orthonormal system, resp. the sequence of numbers defined in the proof of Menchoff's theorem (cf. e.g. [4], p. 195). Now, for $x \in E$, we put

$$\Phi_{n_m}(x) = 1/b_{n_m} \quad \text{for} \quad N_{k-1} < m < N_k, \quad k = 1, 2, \ldots,$$
$$\Phi_n(x) = \varphi_{N_k}(x), \quad b_n = 0 \quad \text{for} \quad n \neq n_m .$$

Evidently, the system $\{\Phi_n(x)\}$ is orthonormal in $\langle 0, 1\rangle$. Hence we obtain the inequality

$$(4.6)\qquad \frac{t_n^*(x)}{v(x)} > \frac{1}{v(n)}\,\frac{1}{P_n}\sum_{k=10}^{n} p_{n-k}\sum_{\mu=4}^{s}\sum_{N_{\mu-1}<m<N_\mu}\lambda_{n_m}v(n_m)\,;$$

where m_0 is the least positive integer satisfying the inequalities

$$N_{s-1} < k \leqslant n_{m_0} < N_s\,.$$

It follows from the definition of the sequence $\{N_k\}$ that $N_{k-1} > 2k$ for $k \geqslant 4$. Hence for $s \geqslant 4$,

$$m_0+1 > l(k) > N_{s-1} > 2s\,.$$

Denoting by p the number of terms in the inner sum on the right side of the coefficient p_{n-k} in the formula (4.6), we have

$$p = m_0-(s-4) > m_0+1-s > \tfrac{1}{2}(m_0+1)\,.$$

The lacunarity of the sequence $\{n_m\}$ implies

$$n_{m_0+1} \geqslant (a+1)n_{m_0} \geqslant (a+1)k \quad \text{for} \quad k \geqslant 10\,,$$

i.e.

$$n_{m_0+1} \geqslant (a+1)k\,.$$

Hence

$$m_0+1 \geqslant l[(a+1)k]\,.$$

Thus we have

$$p > \tfrac{1}{2}l[(a+1)k]\,.$$

Now let $\{p_n\} \in M^a$, $a > \frac{1}{2}$. If $0 < p_n\searrow$, then we have for sufficiently large n

$$\frac{t_n^*(x)}{v(n)} > \frac{1}{v(n)}\cdot\frac{1}{P_n}\sum_{k=[n/(a+1)]+1}^{n} p_{n-k}\frac{l[(a+1)k]}{2}$$

$$> \frac{1}{2}\cdot\frac{l(n)}{v(n)}\cdot\frac{np_n}{P_n}\cdot\frac{a}{a+1} > \frac{1}{4}\cdot\frac{a}{a+1}\cdot\frac{\log l(n)}{v(n)} \to \infty\,.$$

If $0 < p_n\nearrow$, then for sufficiently large n we have

$$\frac{t_n^*(x)}{v(n)} \geqslant \frac{1}{2v(n)}\cdot\frac{1}{P_n}\sum_{k=[n/(a+1)]}^{n} l[(a+1)k]\,p_{n-k}$$

$$> \frac{l(n)}{2v(n)}\cdot\frac{1}{P_n}\sum_{k=[n/(a+1)]}^{n} p_{n-k} = \frac{l(n)}{2v(n)}\cdot\frac{1}{P_n}\sum_{k=0}^{n-[n/(a+1)]} p_k$$

$$= \frac{l(n)}{2v(n)}\Bigl(1-\frac{1}{P_n}\sum_{k=n-[n/(a+1)]+1}^{n} p_k\Bigr)$$

$$> \frac{l(n)}{2v(n)}\Bigl(1-\frac{np_n}{P_n}\cdot\frac{1}{a+1}\Bigr) > \frac{a}{4(a+1)}\cdot\frac{\log l(n)}{v(n)} \to \infty\,.$$

Hence it follows that, assuming $\{p_n\} \in M^a$, $a > \frac{1}{2}$, we have

$$\overline{\lim_{n\to\infty}} \frac{|t_n^*(x)|}{v(n)} = \infty \tag{4.7}$$

for $x \in E$, and this together with the first part of the proof implies that condition (4.7) holds for every $x \in \langle 0, 1 \rangle$. Now our theorem is obtained by choosing $a_n = b_n \lambda_n v(n)$.

5. THEOREM 6. *Let $\{p_n\} \in BVM^a$, $a > \frac{1}{2}$. If $\{T_n(x)\}$ is the sequence of (N, p_n)-means of the orthogonal series* (3.1) *with coefficients c_n satisfying condition* (3.2), *and if the limit*

$$\lim_{n\to\infty} \frac{1}{Q_n} \sum_{k=0}^{n} q_{n-k} T_k(x)$$

exists a.e. for a sequence $\{q_n\} \in M^\beta$, $\beta > \frac{1}{2}$, then the series (3.1) *is (N, p_n)-summable a.e.*

Proof. Without loss of generality we may assume $T_0(x) \equiv 0$. By the assumption, the series

$$\sum_{n=1}^{\infty} [T_n(x) - T_{n-1}(x)] \tag{5.1}$$

is then (N, q_n)-summable a.e., $\{q_n\} \in M^\beta$, $\beta > \frac{1}{2}$. By Lemma 8, the series

$$\sum_{n=1}^{\infty} n[T_n(x) - T_{n-1}(x)]^2$$

is convergent a.e. Hence, by Lemma 5, series (5.1) is convergent a.e., i.e. series (3.1) is (N, p_n)-summable a.e.

THEOREM 7. *If $\{p_n\} \in M^a$, $a > \frac{1}{2}$, and if*

$$(N, p_n)\text{-}\lim \sigma_n^\beta(x) \doteq s(x),$$

where $\sigma_n^\beta(x)$ are the n-th Cesàro means of order $\beta > 0$ of the orthogonal series (3.1) with coefficients c_n satisfying condition (3.2), then series (3.1) is (C, β)-summable a.e. for every $\beta > 0$.

Proof. Without loss of generality we may assume $\sigma_0^\beta(x) \equiv 0$. Then the assumption of Theorem 7 implies the series

$$\sum_{n=1}^{\infty} [\sigma_n^\beta(x) - \sigma_{n-1}^\beta(x)]$$

to be (N, p_n)-summable a.e., where $\{p_n\} \in M^a$, $a > \frac{1}{2}$. By a lemma of Kaczmarz-Zygmund (see e.g. [5], p. 184) the series

$$\sum_{n=1}^{\infty} n[\sigma_n^{\beta}(x) - \sigma_{n-1}^{\beta}(x)]^2$$

is convergent a.e. for $\beta > \frac{1}{2}$. Hence and from Lemma 5 it follows that the orthogonal series (3.1) is (C, β)-summable a.e. for every $\beta > \frac{1}{2}$. Thus, by the theorem of Kaczmarz-Zygmund (see e.g. [4], p. 219, 5.8.3) the orthogonal series (3.1) is (C, β)-summable a.e. for every $\beta > 0$.

In order to formulate the next theorem we extend the definition of the Nörlund transforms (N, q_n) to the case where $\sum q_n$ is a series with partial sums $Q_n = q_0 + q_1 + \ldots + q_n \neq 0$. (Obviously, it may happen that certain of the values q_n are negative.)

THEOREM 8. (9) *Let $T_n(x)$ and $t_n(x)$ denote the (N, p_n)-means, resp. (N, q_n)-means of the orthogonal series* (3.1) *with coefficients c_n satisfying condition* (3.2), *where $p_0 > 0$, $p_n \geqslant 0$, $np_n/P_n = O(1)$ and $Q_n = q_0 + q_1 + \ldots + q_n \neq 0$, $\lim_{n\to\infty} nq_n/Q_n = a-1$, where $a > \frac{1}{2}$. If by these assumptions the orthogonal series* (3.1) *is (N, p_n)-summable to a function $s(x)$ a.e., then*

$$\sum_{k=0}^{n} [t_k(x) - s(x)]^2 \doteq o(n+1) \qquad as \qquad n \to \infty .$$

Proof. Proceeding as in the proofs of Lemmas 3 and 4, we state first that

(a) the sequence $\{nQ_n^2\}$ increases to infinity beginning with a certain N,

(b) $\displaystyle\lim_{n\to\infty} \frac{1}{nQ_n^2} \sum_{v=0}^{n} Q_v^2 = \frac{1}{2a-1} \quad (a > \tfrac{1}{2})$.

Evidently we have

$$nQ_n^2 - (n-1)Q_{n-1}^2 = Q_n^2 \left[\frac{2nq_n}{Q_n} + \left(1 - \frac{q_n}{Q_n}\right)^2 - \frac{nq_n^2}{Q_n^2}\right].$$

According to our assumptions, the expression in the square brackets tends to $2a-1$, whence it follows that the sequence $\{nQ_n^2\}$ is increasing for sufficiently large n. In order to show that $nQ_n^2 \to \infty$ we observe, in virtue of Stolz's lemma, that

$$\lim_{n\to\infty} \frac{\log nQ_n^2}{\log n} = \lim_{n\to\infty} \frac{\log(n-1)Q_{n-1}^2 - \log nQ_n^2}{\log(n-1) - \log n}$$
$$= 1 + \lim_{n\to\infty} \frac{\log[(1 - q_n/Q_n)^{Q_n/q_n}]^{2nq_n/Q_n}}{\log(1-1/n)^n} = 2a-1 .$$

(9) Theorem 8 generalizes a theorem of Zygmund-Borgen (see e.g. [1], p. 102-103, 2.6.2) concerning the strong (C, a)-summability (for $a > \frac{1}{2}$) of orthogonal series.

Hence

$$\log nQ_n^2 \approx \log n^{2a-1}$$

and therefore

$$nQ_n^2 \to \infty \quad \text{as} \quad n \to \infty \quad (a > \tfrac{1}{2}).$$

Since $nQ_n^2 \nearrow +\infty$, applying Stolz's lemma again, we get

$$\lim_{n\to\infty} \frac{\sum_{v=0}^{n} Q_v^2}{nQ_n^2} = \lim_{n\to\infty} \frac{1}{2nq_n/Q_n + (1-q_n/Q_n)^2 - nq_n^2/Q_n^2} = \frac{1}{2a-1}$$

which shows property (b) of the sequence $\{Q_n\}$.

Passing to the proof of our Theorem, we observe that the second term on the right side of the inequality

$$\frac{1}{n+1}\sum_{k=0}^{n}[t_k(x)-s(x)]^2$$
$$\leqslant \frac{2}{n+1}\sum_{k=0}^{n}[t_k(x)-T_k(x)]^2 + \frac{2}{n+1}\sum_{k=0}^{n}[T_k(x)-s(x)]^2$$

tends to zero a.e. according to our assumptions. Hence it will be enough to show that the first term of the last inequality also tends to zero a.e. Changing the order of summation. we may write

$$T_n - t_n = \sum_{k=0}^{n} s_k\left(\frac{p_{n-k}}{P_n} - \frac{q_{n-k}}{Q_n}\right)$$
$$= \sum_{v=0}^{n} c_v\varphi_v \sum_{n=v}^{n}\left(\frac{p_{n-k}}{P_n} - \frac{q_{n-k}}{Q_n}\right)$$

and therefore

$$T_n - t_n = \sum_{v=0}^{n} c_v\varphi_v\left(\frac{P_{n-v}}{P_n} - \frac{Q_{n-v}}{Q_n}\right).$$

Hence

$$\sum_{n=0}^{\infty}\int_0^1 \frac{(T_n-t_n)^2}{n+1}dx = \sum_{n=0}^{\infty}\frac{1}{n+1}\sum_{v=0}^{n} c_v^2\left(\frac{P_{n-v}}{P_n} - \frac{Q_{n-v}}{Q_n}\right)^2.$$

Changing the order of summation in the last expression and decomposing the inner sum into two sums from $n = v$ to $n = 2v$ and from $n = 2v+1$ to $n = \infty$, we can write

$$\sum_{n=N}^{\infty}\int_0^1 \frac{(T_n-t_n)^2}{n+1}dx \leqslant 2\sum_{v=N}^{\infty} c_v^2 \sum_{n=v}^{2v}\frac{1}{n+1}\left(\frac{P_{n-v}^2}{P_n^2} + \frac{Q_{n-v}^2}{Q_n^2}\right) +$$
$$+ \sum_{v=N}^{\infty} c_v^2 \sum_{n=2v+1}^{\infty}\frac{1}{n+1}\left[\frac{P_{n-v}-P_n}{P_n} - \frac{Q_{n-v}-Q_n}{Q_n}\right]^2 = A + B.$$

From the properties of the sequences $\{P_n\}$ and $\{Q_n\}$ it follows that

$$A \leqslant 2 \sum_{v=N}^{\infty} c_v^2 + 2 \sum_{v=N}^{\infty} \frac{c_v^2}{(v+1)Q_v^2} \sum_{k=0}^{v} Q_k^2 = O(1) \sum_{v=N}^{\infty} c_v^2 = O(1)\,.$$

To estimate the expressions B we notice that

$$|q_{n-r}| = \max_{n-v+1 \leqslant k \leqslant n} |q_k|\,, \quad p_{n-s} = \max_{n-v+1 \leqslant k \leqslant n} p_k\,,$$

where $0 \leqslant r \leqslant v-1$, $0 \leqslant s \leqslant v-1$. (Of course, it may happen that one or both of the values q_{n-r}, p_{n-s} vanish.) Now we can write

$$B \leqslant 2 \sum_{v=0}^{\infty} c_v^2 \sum_{n=2v+1}^{\infty} \frac{1}{n+1} \left[\frac{(p_{n-v+1} + \ldots + p_n)^2}{P_n^2} + \frac{(|q_{n-v+1}| + \ldots + |q_n|)^2}{Q_n^2} \right]$$

$$\leqslant 2 \sum_{v=0}^{\infty} (v+1)^2 c_v^2 \sum_{n=2v+1}^{\infty} \left[\frac{q_{n-r}^2}{(n-r)Q_{n-r}^2} + \frac{p_{n-s}^2}{(n-s)P_{n-s}^2} \right]$$

$$= O(1) \sum_{v=0}^{\infty} (v+1)^2 c_v^2 \sum_{n=2v+1}^{\infty} \frac{1}{(n-v+1)^3} = O(1) \sum_{v=0}^{\infty} c_v^2 = O(1)\,.$$

Thus we have

$$\sum_{k=0}^{n} [T_k(x) - t_k(x)]^2 \doteq o(n+1) \qquad \text{as} \qquad n \to \infty\,,$$

which completes the proof of our Theorem.

Choosing in Theorem 8 $p_n = \binom{n+a-1}{n}$, $q_n = \binom{n+a-2}{n}$, $a > \frac{1}{2}$, we get the theorem of Zygmund-Borgen (see e.g. [1], p. 102, 2.6.2) concerning the strong (C, a)-summability, for $a > \frac{1}{2}$, of orthogonal series.

Taking $q_0 > 0$, $q_n = 0$ for $n = 1, 2, \ldots$, in Theorem 8, we obtain as a corollary the following theorem:

THEOOEM 9. *If orthogonal series* (3.1) *with coefficients* c_n *satisfying condition* (3.2) *and with partial sums* $s_n(x)$ *is* (N, p_n)*-summable to a function* $s(x)$ *a.e., where* $np_n/P_n = O(1)$, *then*

$$\sum_{k=0}^{n} [s_k(x) - s(x)]^2 \doteq o(n+1) \qquad \textit{as} \qquad n \to \infty\,.$$

This theorem is a generalization of a special case of the theorem of Zygmund-Borgen.

THEOREM 10. (10) *If the orthogonal series* (3.1) *with coefficients* c_n *satisfying condition* (3.2) *is* (N, p_n)*-summable to* $s(x)$ *a.e., where* $\{p_n\} \in M^a$, $a > 0$, *then*

$$\sum_{k=0}^{n} [s_{n_k}(x) - s(x)]^2 \doteq o(n) \quad \text{as} \quad n \to \infty ,$$

for an arbitrary convex sequence of indices $\{n_k\}$.

Proof. If $0 < p_n \nearrow$, then we may apply inequality (3.7). Replacing in this inequality n by n_k and dividing both sides by $k+1$, we obtain after summation from $k = 0$ to ∞ $(n_{-1} = 0)$

$$\sum_{k=0}^{\infty} \frac{1}{k+1} \int_0^1 (s_{n_k} - t_{n_k})^2 dx < \sum_{k=0}^{\infty} \frac{1}{k+1} \frac{p_{n_k}^2}{P_{n_k}^2} \sum_{i=0}^{k} n_i^2 \sum_{\nu=n_{i-1}+1}^{n_i} c_\nu^2 .$$

Since the sequence $\{n_k^2\}$ is also convex, we have

$$n_0^2 + n_{2k}^2 \geqslant 2n_k^2 \quad \text{for} \quad k = 1, 2, \ldots$$

Hence there exists a constant $q > 1$ such that

$$n_{2k}^2 \geqslant q n_k^2 \quad \text{for} \quad k = 1, 2, \ldots$$

Thus we have

$$n_i^2 \sum_{k=i}^{\infty} \frac{1}{(k+1) n_k^2} < \frac{1}{i+1} + 1 + \frac{1}{q^2} + \frac{1}{q^4} + \ldots = \frac{1}{i+1} + \frac{q^2}{q^2 - 1} ,$$

whence

$$\sum_{k=0}^{\infty} \frac{1}{k+1} \int_0^1 (s_{n_k} - t_{n_k})^2 dx = O(1) \sum_{i=0}^{\infty} \left(\frac{1}{i+1} + \frac{q^2}{q^2 - 1} \right) \sum_{\nu=n_{i-1}+1}^{n_i} c_\nu^2$$

$$= O(1) \sum_{i=0}^{\infty} \sum_{\nu=n_{i-1}+1}^{n_i} c_\nu^2 = O(1) .$$

Hence it follows that the series

$$\sum_{k=1}^{\infty} \frac{1}{k} (s_{n_k} - t_{n_k})^2 \tag{5.2}$$

is convergent a.e. when $0 < p_n \nearrow$.

Now assume $0 < p_n \searrow$ and $n_{2k}/n_k \geqslant q > 1$ for $k = 1, 2, \ldots$ Proceeding as in the proof of Lemma 6 we may easily show that there is a constant $q_1 > 1$ such that $P_{n_{2k}}/P_{n_k} \geqslant q_1 > 1$ for $k = 1, 2, \ldots$

(10) Theorem 10 is an analogon of a similar theorem on Abel's method (cf. [1], p. 105-106, 2.6.4).

Applying inequality (3.8), we may write

$$\sum_{k=1}^{\infty}\frac{1}{k}\int_0^1 (s_{n_k}-t_{n_k})^2dx < \sum_{k=1}^{\infty}\frac{1}{kP_{n_k}^2}\sum_{\nu=0}^{n_k}\nu^2p_\nu^2c_\nu^2+\sum_{k=1}^{\infty}\frac{1}{k}\sum_{\nu=[n_k/2]+1}^{n_k}c_\nu^2 .$$

Let s be the least positive integer satisfying the inequality $q^s \geqslant 2$. Then

$$n_k \geqslant qn_{[k/2]} \geqslant q^2n_{[k/2^2]} \geqslant \ldots \geqslant q^sn_{[k/2^s]} \geqslant 2n_{[k/2^s]} ,$$

where k is an arbitrary positive integer $\geqslant 2^s$. Writing $p = [k/2^s]$ for $k \geqslant 2^s$ we have

$$\sum_{k=1}^{\infty}\frac{1}{k}\int_0^1 (s_{n_k}-t_{n_k})^2dx < \sum_{\nu=1}^{\infty}\nu^2p_\nu^2c_\nu^2\sum_{n_k\geqslant\nu}\frac{1}{kP_{n_k}^2}+\sum_{k=1}^{2^s-1}\frac{1}{k}\sum_{\nu=[n_k/2]+1}^{n_k}c_\nu^2+$$

$$+\sum_{k=2^s}^{\infty}\frac{1}{k}\Big[\sum_{\nu=n_p+1}^{n_{p+1}}c_\nu^2+\sum_{\nu=n_{p+1}+1}^{n_{p+2}}c_\nu^2+\ldots+\sum_{\nu=n_{k-1}+1}^{n_k}c_\nu^2\Big].$$

Obviously, the second and the third term on the right side of this inequality are bounded. We show the first term to be also finite. This is obtained by applying the following estimation:

$$\sum_{n_k\geqslant\nu}\frac{1}{kP_{n_k}^2}=\frac{1}{kP_{n_k}^2}+\frac{1}{(k+1)P_{n_{k+1}}^2}+\ldots+\frac{1}{2kP_{n_{2k}}^2}+\ldots+\frac{1}{4kP_{4k}^2}+\ldots$$

$$<\frac{1}{P_{n_k}^2}\left(1+\frac{1}{q_1^2}+\frac{1}{q_1^4}+\ldots\right)=\frac{q_1^2}{q_1^2-1}\cdot\frac{1}{P_\nu^2}.$$

Hence

$$\sum_{\nu=1}^{\infty}\nu^2p_\nu^2c_\nu^2\sum_{n_k\geqslant\nu}\frac{1}{kP_{n_k}^2}\leqslant\frac{q_1^2}{q_1^2-1}\sum_{\nu=1}^{\infty}\frac{\nu^2p_\nu^2}{P_\nu^2}c_\nu^2=O(1)\sum_{\nu=1}^{\infty}c_\nu^2=O(1).$$

Thus we have proved the series (5.2) to be convergent a.e. in the case $0<p_n\nearrow$. This, together with the first part of the proof, shows for every $\{p_n\}\in M^a$, $a>0$, the convergence of series (5.2) a.e. Now, by Kronecker's theorem,

$$\frac{1}{n}\sum_{k=1}^{n}(s_{n_k}-t_{n_k})^2 = o(1) \quad \text{as} \quad n\to\infty .$$

Theorem 9 now follows from the inequalities

$$\frac{1}{n}\sum_{k=1}^{n}(s_{n_k}-s)^2\leqslant\frac{2}{n}\sum_{k=1}^{n}(t_{n_k}-s_{n_k})^2+\frac{2}{n}\sum_{k=1}^{n}(t_{n_k}-s)^2$$

and from the assumptions.

References

[1] G. Alexits, *Konvergenzprobleme der Orthogonalreihen*, Budapest 1960.

[2] G. M. Fichtengolc, *A cours of differential and integral calculus* (in Russian), Moscow 1951.

[3] G. H. Hardy, *Divergent series*, Oxford 1959.

[4] S. Kaczmarz and H. Steinhaus, *Theory of orthogonal series* (in Russian), Moscow 1958.

[5] J. Meder, *On the summability almost everywhere of orthonormal series by the method of Euler-Knopp*, Annales Pol. Math. 5 (1958), p. 135-148.

[6] T. Pati, *On the absolute Nörlund summability of a Fourier series*, Journal London Math. Soc. 34 (1959), p. 153-160.

[7] K. Tandori, *Über die orthogonalen Funktionen I*, Acta Sci. Math. 18 (1957), p. 57-130.

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