INSTYTUT MATEMATYCZNY POLSKIEJ AKADEMII NAUK

DISSERTATIONES MATHEMATICAE

(ROZPRAWY MATEMATYCZNE)

XCII

TSOY-WO MA

Topological degrees of set-valued compact fields in locally convex spaces

WARSZAWA 1972

PAŃSTWOWE WYDAWNICTWO NAUKOWE

PRINTED IN POLAND

WROCŁAWSKA DRUKARNIA NAUKOWA

CONTENTS

INTRODUCTION

The object of this paper([1]) is to extend the classical Brouwer–Leray–Schauder–Nagumo degree theory from single-valued to set-valued compact fields in separated locally convex spaces. Recently, Granas and Jaworowski [12], [13], [15], [18] obtained some results along this line by applying Vietoris homology. However, the domain of their maps are restricted to the unit balls of Banach spaces. We shall follow their definitions of set-valued compact fields and homotopies. On the other hand, we shall define the topological degrees according to their homotopy invariance. Most of the interesting results in the degree theory for single-valued compact fields will be extended to set-valued compact fields in separated locally convex spaces. Also see [28], [29], [31] for results along the same line.

([1]) This work was supported in part by National Science Foundation Grants GP-8394, GP-20632 and by a University Fellowship. I would like to express my sincere thanks to Professor Ky Fan for his help and guidance during my studies and in the preparation of this dissertation.

CHAPTER I

GENERAL PROPERTIES OF SET-VALUED COMPACT FIELDS

§ 1. Upper semicontinuous maps. Let X, Y be two Hausdorff spaces and $\mathscr{K}Y$ the family of all non-empty compact subsets of Y. A set-valued map F: $X \to \mathscr{K}Y$ is said to be *upper semicontinuous at a given point* $a \in X$ if for each open subset W of Y with $F(a) \subset W$, there exists a neighborhood V of $a \in X$ such that $F(V) \subset W$ where $F(V) = \bigcup\{F(x): x \in V\}$. A set-valued map F: $X \to \mathscr{K}E$ is said to be *upper semicontinuous on* X if F is upper semicontinuous at each point of X. The following facts are well known, e.g. Berge [4], and they will be needed later.

(1.1) Let X, Y be Hausdorff spaces, $\mathscr{K}Y$ the family of all non-empty compact subsets of Y and F: $X \to \mathscr{K}Y$ an upper semicontinuous set-valued map. Then F sends compact subsets of X to compact sets of Y. Also F has a closed graph in $X \times Y$, i.e. the set

$$\{(x, y) \in X \times Y: \ y \in F(x)\}$$

is closed in the product spaces $X \times Y$. If $\{x_\delta : \delta \in D\}$ is a net in X convergent to $x \in X$ and if $\{y_\delta : \delta \in D\}$ is a net in Y convergent to $y \in X$ such that $y_\delta \in F(x_\delta)$ for each δ in the directed set D, then we have $y \in F(x)$.

(1.2) Let X, Y be Hausdorff spaces, $\mathscr{K}Y$ the family of all non-empty compact subsets of Y and F: $X \to \mathscr{K}Y$ a set-valued map. Suppose that A, B are two closed subsets of X with $X = A \cup B$. If $F|_A$, $F|_B$ are upper semicontinuous on A, B respectively, then F is upper semicontinuous on X.

(1.3) Let X, Y, Z be Hausdorff spaces and $\mathscr{K}Y$, $\mathscr{K}Z$ the families of all non-empty compact subsets of Y, Z respectively. Let F: $X \to \mathscr{K}Y$ and G: $Y \to \mathscr{K}Z$ be upper semicontinuous maps on X, Y respectively. The composite set-valued map $G \circ F$: $X \to \mathscr{K}Z$ is defined by

$$G \circ F(x) = G\big(F(x)\big) = \bigcup\{G(y): \ y \in F(x)\} \quad \text{for} \quad x \in X.$$

Then $G \circ F$ is also an upper semicontinuous map on X.

(1.4) Let X be a Hausdorff space, E a saparated locally convex space and $\mathscr{K}E$ the family of all non-empty compact subsets of E. Let F, G: $X \to \mathscr{K}E$ be two upper semicontinuous set-valued maps on X

and $a\colon X \to \boldsymbol{R}$ be a continuous single-valued map on X where $\boldsymbol{R}$ denotes the real field. Then the maps f, g defined by

$$f(x) = F(x) + G(x)$$

and

$$g(x) = a(x)F(x) \quad \text{for} \quad x \in X,$$

are upper semicontinuous on X.

(1.5) Let X, Y be Hausdorff spaces and $\mathscr{K}Y$ the family of all non-empty compact subsets of Y. Then the following statements are equivalent for a set-valued map $F\colon X \to \mathscr{K}Y$.

(a) F is upper semicontinuous on X.

(b) For each open subset W of Y, the set $\{x \in X\colon F(x) \subset W\}$ is open in X.

(c) For each closed subset B of Y, the set $\{x \in X\colon F(x) \cap B \neq \emptyset\}$ is closed in X.

§ 2. Generalization of Dugundji's extension theorem.

(2.1) THEOREM. *Let A be a closed subset of a metrizable space X, E a separated locally convex space, and $\mathscr{K}E$ the family of all non-empty compact convex subsets of E. If $F\colon A \to \mathscr{K}E$ is an upper semicontinuous set-valued map on A, then F has an upper semicontinuous set-valued extension G: $X \to \mathscr{K}E$ such that $G(X)$ is contained in the convex hull of $F(A)$.*

Proof. Let d be a metric compatible with the topology of X. For each $y \in X \setminus A$, let $V(y)$ be a ball with center y and radius $0 < r < \frac{1}{2}d(A, y)$. Let $\mathscr{V} = \{V_j\colon j \in J\}$ be a locally finite open refinement of $\{V(y)\colon y \in X \setminus A\}$ such that $\mathscr{V}$ covers $X \setminus A$. Let $\{a_j\colon j \in J\}$ be a partition of unity on $X \setminus A$ subordinated to $\mathscr{V}$. Let us pick up $x_j \in V_j$ for each $j \in J$. There exists an $a_j \in A$ with $d(a_j, x_j) < 2d(x_j, A)$. Define $G\colon X \to \mathscr{K}E$ as follows:

$$G(x) = \begin{cases} F(x) & \text{for} \quad x \in A, \\ \sum_{j \in J} a_j(x) F(a_j) & \text{for} \quad x \in X \setminus A. \end{cases}$$

Then G is an extension of F and $G(X)$ is contained in the convex hull of $F(X)$. We want to show that G is upper semicontinuous at each point $a \in X$. If $a \in X \setminus A$, since $\mathscr{V}$ is locally finite, there exists an open neighborhood N of $a \in X$ such that $N \subset X \setminus A$ and N meets only a finite number of sets in $\mathscr{V}$. Then by (1.4), $G|_N$ is an upper semicontinuous set-valued map on N and hence G is upper semicontinuous at the point $a \in X \setminus A$. On the other hand, if $a \in A$, let B be an open set in E with $G(a) \subset B$. Since $G(a) = F(a)$ is compact, there exists a convex neighborhood U of $o \in E$ with $G(a) + U \subset B$. By upper semicontinuity of F, there exists an open ball V in X with center $a \in A$ and radius $9\varepsilon > 0$ such that $F(V \cap A) \subset G(a) + U$. It suffices to show that $G(W) \subset G(a) + U$ where W is an open ball in X with center $a \in A$ and radius $\varepsilon > 0$. If $x \in W \cap A$, then $x \in V \cap A$ and hence

$$G(x) = F(x) \subset G(a) + U.$$

If $x \in W \setminus A$, there exists only a finite number of sets in $\mathscr{V}$, say $\{V_{j_1}, V_{j_2}, \ldots, V_{j_m}\}$, containing x. If λ is one of the indices $j_1, j_2, \ldots, j_m$, then $x \in W \cap V_\lambda$. Since $\mathscr{V}$ is a refinement of $\{V(y) \colon y \in X \setminus A\}$, we can write $V_\lambda \subset V(y)$ for some $y \in X \setminus A$. Now

$$\varepsilon > d(a, x) \geqslant d(a, y) - d(y, x) \geqslant d(A, y) - r \geqslant r.$$

Thus

$$d(x, x_\lambda) \leqslant 2r \leqslant 2\varepsilon.$$

Also

$$d(a_\lambda, x_\lambda) < 2d(x_\lambda, A) \leqslant 2d(x_\lambda, a).$$

Hence

$$\begin{aligned} d(a, a_\lambda) &\leqslant d(a, x_\lambda) + d(x_\lambda, a_\lambda) \\ &\leqslant 3d(a, x_\lambda) \\ &\leqslant 3[d(a, x) + d(x, x_\lambda)] \\ &< 3(\varepsilon + 2\varepsilon) \\ &= 9\varepsilon. \end{aligned}$$

Therefore $a_\lambda \in A \cap V$ and $F(a_\lambda) \subset G(a) + U$. Since $G(a) + U$ is convex, we have

$$G(x) = \sum_{j \in J} a_j(x) F(a_j) = \sum_{\nu=1}^{m} a_{j_\nu}(x) F(a_{j_\nu}) \subset G(a) + U.$$

Thus $G(W) \subset G(a) + U \subset B$. Consequently G is upper semicontinuous at $a \in X$. This completes the proof.

We also include the following known facts for reference:

(2.2) Let Q be a non-empty compact subset of a separated locally convex space E. Then for every neighborhood V of $o \in E$, there exists a continuous single-valued map ξ defined on Q into some finite dimensional vector subspace E_1 of E such that for each $x \in Q$, we have $\xi(x) - x \in V$. See for example [25].

(2.3) Let A be an open *convex* neighborhood of the origin of a separated locally convex space E. Let $\bar{A}$ and ∂A denote the closure and boundary of A, respectively. Then

(a) $x \in \bar{A}$ iff for each $0 < \lambda < 1$, we have $\lambda x \in A$;

(b) $x \in \partial A$ iff for each $0 < \lambda < 1 < \mu$, we have $\lambda x \in A$ and $\mu x \notin A$.

As a result, if E_0 is a vector subspace of E, then $\bar{A} \cap E_0$ and $\partial A \cap E_0$ are the closure and boundary of $A \cap E_0$ in E_0 respectively.

§ 3. Set-valued compact fields. Let M be a Hausdorff space, E a separated locally convex space, and $\mathscr{K}E$ the family of all non-empty compact convex subsets of E. An upper semi-continuous set-valued map F: $M \to \mathscr{K}E$ is said to be a *compact map* if $F(M)$ is relatively compact in E; F is said to be *finite dimensional* if $F(M)$ is contained in some finite dimensional vector subspace of E.

(3.1) LEMMA. *Let M be a Hausdorff space, E a separated locally convex space and $\mathscr{K}E$ the family of all non-empty compact convex subsets of E. Let F: $M \to \mathscr{K}E$ be a set-valued compact map. Then for every neighborhood V of $o \in E$, there exists a finite dimensional set-valued compact map G: $M \to \mathscr{K}E$ such that for each $x \in M$, we have $F(x) \subset G(x)+V$ and $G(x) \subset F(x)+V$*

Proof. Without loss of generality, let V be a convex symmetric neighborhood of $o \in E$. The closure Q of $F(M)$ is compact in E. By (2.2), let ξ be a continuous single-valued map defined on Q into some finite dimensional vector subspace E_1 of E such that $\xi(y)-y \in V$ for all $y \in Q$. Now $\xi(Q)$ is compact and hence bounded in the finite dimensional vector subspace E_1. Consequently the closed convex hull $\overline{\text{co}}\,\xi(Q)$ of $\xi(Q)$ is compact in E_1. For each $x \in M$, define $G(x) = \overline{\text{co}}\big(\xi \circ F(x)\big)$. Then $G(M)$ as a subset of $\overline{\text{co}}\,\xi(Q)$ is relatively compact in E. Also $G(x)$ as a closed subset of $\overline{\text{co}}\,\xi(Q)$ is compact convex. To show that G is upper semicontinuous at any $x \in M$, let W be an open subset of E with $G(x) \subset W$. Since $G(x)$ is compact, there exists an open convex neighborhood V_0 of $o \in E$ such that $G(x)+\overline{V_0} \subset W$. Since ξ is continuous, $\xi^{-1}\big(G(x)+V_0\big)$ is an open subset of Q. There exists an open subset B of E such that $\xi^{-1}\big(G(x)+V_0\big) = Q \cap B$. Thus B is an open subset of E with $F(x) \subset B$. By upper semicontinuity of F, there exists a neighborhood U of $x \in M$ such that $F(U) \subset B$. Hence

$$\xi \circ F(U) \subset \xi(B \cap Q) \subset G(x)+\overline{V_0}.$$

Since $G(x)+\overline{V_0}$ is closed convex, we have

$$G(U) \subset G(x)+\overline{V_0} \subset W.$$

Therefore G is upper semicontinuous on M. We conclude that G: $M \to \mathscr{K}E$ is a compact map. Clearly G is finite dimensional. Finally we want to show $G(x) \subset F(x)+V$ for each given $x \in M$. Now for $a \in G(x)$, there exists $a_i \in \xi \circ F(x)$ and $\alpha_i > 0$ with $\sum_{i=1}^{m} \alpha_i = 1$ and $a = \sum_{i=1}^{m} \alpha_i a_i$. Let $b_i \in F(x)$ satisfy $a_i = \xi(b_i)$. Then $\sum_{i=1}^{m} \alpha_i b_i \in F(x)$ and $a - \sum_{i=1}^{m} \alpha_i b_i \in V$ Therefore $G(x) \subset F(x)+V$. Similarly we can show $F(x) \subset G(x)+V$ for all $x \in M$. Our proof is complete.

Let Y be a subset of a separated locally convex space E and $\mathscr{K}E$ the family of all non-empty compact convex subsets of E. Let Z be a compact Hausdorff space. A set-valued map $f: Y\times Z \to \mathcal{K}E$ is called a *set-valued Z-compact field* on $Y\times Z$ if the map $F: Y\times Z \to \mathcal{K}E$ defined by $F(y, z) = y-f(y, z)$ for $(y, z)\in Y\times Z$ is a compact map. The map F is called the *compact map corresponding to the set-valued Z-compact field f*. The set-valued Z-compact fields will be denoted in the sequel by small letters, f, g etc. If some set-valued Z-compact field is denoted by a small letter, then the corresponding compact map will be denoted by the same but capital letter. By (1.4), every set-valued Z-compact field is upper semicontinuous. If the set Y is compact in E, then by (1.1), every upper semicontinuous set-valued map $f: Y\times Z \to \mathcal{K}E$ is a set-valued Z-compact field. We are interested essentially in the case when Z is a singleton and we shall simply call a set-valued compact field. The cases when Z is both the closed unit interval I and the closed unit square $I\times I$, are also used in our proofs. The reason why we introduced the space Z is to avoid repetition of identical arguments. Every set-valued compact field can be considered as a small displacement of the identity map.

(3.2) LEMMA. *Let X be a closed subset of a separated locally convex space E and $\mathcal{K}E$ the family of all non-empty compact convex subsets of E. If Z is a compact Hausdorff space and if $f: X\times Z \to \mathcal{K}E$ is a set-valued Z-compact field, then $f(X\times Z)$ is closed in E.*

Proof. Let $\{u_\delta: \delta\in D\}$ be a net in $f(X\times Z)$ convergent to some $u\in E$. There exists $x_\delta\in X$, $z_\delta\in Z$ such that $u_\delta\in x_\delta-F(x_\delta, z_\delta)$ where F is the compact map corresponding to f. Then $\{x_\delta-u_\delta: \delta\in D\}$ is a net in the relatively compact set $F(X\times Z)$. Replacing by subnets, we may assume that $\{x_\delta-u_\delta: \delta\in D\}$ converges to some $v\in E$ and $\{z_\delta: \delta\in D\}$ converges to some $z\in Z$. Hence $\{x_\delta: \delta\in D\}$ converges to $x = u+v\in X$. By (1.1), we have $x+u\in F(x, z)$, i.e. $u\in f(x, z) \subset f(X\times Z)$. This completes the proof.

§ 4. Reduction to finite dimensional vector spaces. Let $X\subset Y$ be two closed subsets of a separated locally convex space E, $p\in E$ and $\mathcal{K}E$ the family of all non-empty compact convex subsets of E. Let Z be a compact Hausdorff space. A set-valued Z-compact field $f: Y\times Z \to \mathcal{K}E$ is said to be *in $C(X, Y, p)$* if $p\notin f(X\times Z)$. A set-valued map $h: Y\times Z\times I \to \mathcal{K}E$ is called a *set-valued Z-homotopy in $C(X, Y, p)$* if the map H: $Y\times Z\times I \to \mathcal{K}E$ defined by $H(y, z, t) = y-h(y, z, t)$ for $(y, z, t)\in Y\times Z\times I$ is a compact map and $p\notin h(X\times Z\times I)$. The map H is called the *compact map corresponding to the Z-homotopy h*. Again we are interested essentially in the case when Z is a singleton and we shall simply call a set-valued homotopy. Clearly a set-valued Z-homotopy in $C(X, Y, p)$ is the same as a set-valued $Z\times I$-compact field in $C(X, Y, p)$. Two set-valued Z-compact fields f, g in $C(X, Y, p)$ are said to be *Z-homotopic in $C(X, Y, p)$* if there exists a set-valued Z-homotopy in $C(X, Y, p)$ such that $h_0 = f$ and $h_1 = g$ where $h_t(y, z) = h(y, z, t)$ for all $(y, z, t)\in Y\times Z\times I$. We shall denote this fact by $f \simeq g$ in $C(X, Y, p)$. Finally a set-valued Z-compact field g in $C(X, Y, p)$ is said to be *finite dimensional* if the corresponding compact map G is finite dimensional. A set-valued Z-homotopy h in $C(X, Y, p)$ is said to be *finite dimensional* if the corresponding compact map H is finite dimensional.

(4.1) THEOREM. *Let $X \subset Y$ be two closed subsets of a separated locally convex space E, $p \in E$ and $\mathcal{K}E$ the family of all non-empty compact convex subsets of E. Let Z be a compact Hausdorff space and $f\colon Y \times Z \to \mathcal{K}E$ a set-valued Z-compact field in $C(X, Y, p)$. Then for every given neighborhood V of $o \in E$, there exists a finite dimensional Z-compact field g such that*

$$f \simeq g \quad in \quad C(X, Y, p)$$

and

$$g(y, z) \subset f(y, z) + V,$$

$$f(y, z) \subset g(y, z) + V \quad for \quad (y, z) \in Y \times Z.$$

Proof. By (3.2), we may assume that V is convex symmetric with $(p+V) \cap f(X \times Z) = \emptyset$. Let F be the compact map corresponding to f. By (3.1), there exist a finite dimensional vector subspace E_1 of E and a compact map $G\colon Y \times Z \to \mathcal{K}E$ such that $G(Y \times Z) \subset E_1$, $G(y, z) \subset F(y, z) + V$ and $F(y, z) \subset G(y, z) + V$ for all $(y, z) \in Y \times Z$. Let $g(y, z) = y - G(y, z)$ for $(y, z) \in Y \times Z$. Clearly $g(y, z) \subset f(y, z) + V$ and $f(y, z) \subset g(y, z) + V$ for $(y, z) \in Y \times Z$. Let $h(y, z, t) = (1-t)f(y, z) + tg(y, z)$ for $(y, z, t) \in Y \times Z \times I$. Then $h_0 = f$, $h_1 = g$. Furthermore for every $(x, z, t) \in X \times Z \times I$, we have

$$h(x, z, t) \subset (1-t)f(x, z) + t[f(x, z) + V] \subset f(x, z) + V.$$

Since $p \notin f(y, z) + V$, we have $p \notin h(X \times Y \times I)$. Hence f, g are homotopic under the set-valued Z-homotopy h in $C(X, Y, p)$. This completes the proof.

(4.2) THEOREM. *Let $X \subset Y$ be two closed subsets of a separated locally convex space E, $p \in E$ and $\mathcal{K}E$ the family of all non-empty compact convex subsets of E. Let f_0, f_1 be two finite dimensional set-valued compact fields in $C(X, Y, p)$. If f_0, f_1 are homotopic in $C(X, Y, p)$, then f_0, f_1 are homotopic under a finite dimensional set-valued homotopy in $C(X, Y, p)$.*

Proof. Let g be a homotopy for $f_0 \simeq f_1$ in $C(X, Y, p)$. By (3.2), let V be a convex symmetric neighborhood of $o \in E$ such that $(p+V) \cap \cap g(X \times I) = \emptyset$. We can consider g as a set-valued I-compact field in $C(X, Y, p)$. By (4.1), let h be a finite dimensional set-valued I-compact field in $C(X, Y, P)$ such that $h(y, t) \subset g(y, t) + V$ for $(y, t) \in Y \times I$. Then for $(x, t, \lambda) \in X \times I \times I$, we have

(a) $(1-\lambda)h(x, t) + \lambda g(x, t) \subset (1-\lambda)[g(x, t) + V] + \lambda g(x, t) \subset g(x, t) + V.$

Let F_0, F_1, H be compact maps corresponding to f_0, f_1, h respectively. Define

$$H^*(y,t) = \begin{cases} (1-3t)F_0(y)+3tH(y,0) & \text{for} \quad (y,t)\in Y\times[0,\frac{1}{3}], \\ H(y,3t-1) & \text{for} \quad (y,t)\in Y\times[\frac{1}{3},\frac{2}{3}], \\ (3-3t)H(y,1)+(3t-2)F_1(y) & \text{for} \quad (y,t)\in Y\times[\frac{2}{3},1] \end{cases}$$

and

$$h^*(y,t) = y-H^*(y,t) \quad \text{for} \quad (y,t)\in Y\times I.$$

By (1.2), H^* is a compact map. The condition (a) shows $p\notin h^*(X\times I)$. Clearly $h_0^* = f_0$ and $h_1^* = f_1$. Since F_0, F_1, H are all finite dimensional, H^* is also finite dimensional. This completes the proof.

§ 5. Reduction to single-valued compact fields. In this section, we shall restrict ourselves to finite dimensional vector spaces. The following lemma plays the crucial role for reducing set-valued to single-valued compact fields. By a polyhedron, we mean the underlying space of a finite simplicial complex. Again Z will be either a singleton, or the closed unit interval I, or the square $I\times I$.

(5.1) LEMMA. *Let D be a closed subset of a polyhedron K in a finite dimensional vector space E, $p\in E$ and $\mathscr{K}E$ the family of all non-empty compact convex subsets of E. Let Z be a polyhedron and $f\colon K\times Z\to\mathscr{K}E$ a set-valued Z-compact field in $C(D,K,p)$. Then there exists a single-valued Z-compact field g in $C(D,K,p)$ satisfying the following conditions:*

(a) $p\notin(1-t)f(x,z)+tg(x,z)$ *for* $(x,z,t)\in D\times Z\times I$.

(b) *The set $G(K\times Z)$ is contained in the convex hull of $F(K\times Z)$ where F, G are the compact maps corresponding to f, g respectively.*

Proof. Let $\mathscr{A}, \mathscr{B}$ be any triangulations of K and Z respectively. For each vertices a of $\mathscr{A}$, b of $\mathscr{B}$, let $u(a,b)$ be a point of $F(a,b)$. Since K, Z are the underlying spaces of the finite simplicial complexes $\mathscr{A}, \mathscr{B}$ respectively, any point $y\in K$ and $z\in Z$ can be written in the forms:

$$y = \sum_i \alpha_i a_i, \quad \alpha_i>0, \quad \sum_i \alpha_i = 1,$$

$$z = \sum_j \beta_j b_j, \quad \beta_j>0, \quad \sum_j \beta_j = 1,$$

where $\{a_0, a_1, \ldots\}$ and $\{b_0, b_1, \ldots\}$ are vertices of the carrier simplixes of y, z in $\mathscr{A}, \mathscr{B}$ respectively. Define

(c) $$G(y,z) = \sum_i \sum_j \alpha_i\beta_j u(a_i, b_j),$$

(d) $$g(y,z) = y-G(y,z) \quad \text{for} \quad (y,z)\in K\times Z.$$

Then g is a continuous single-valued map and $G(K\times Z)$ is contained in the convex hull of $F(K\times Z)$. We claim that for sufficiently fine triangulations $\mathscr{A}$, $\mathscr{B}$, the condition (a) also holds and consequently g is a single-valued compact field in $C(D, K, p)$. Suppose that this is false. For each integer $m>0$, there exist triangulations $\mathscr{A}^m$, $\mathscr{B}^m$ of K and Z respectively such that

$$\text{(e)} \qquad \operatorname{mesh}\mathscr{A}^m \leqslant \frac{1}{m}, \quad \operatorname{mesh}\mathscr{B}^m \leqslant \frac{1}{m}.$$

There exists $u^m(a, b)\in F(a, b)$ for vertices a, b of $\mathscr{A}^m, \mathscr{B}^m$ respectively. There exist $x^m\in D$ and $z^m\in K$ which can be written in the forms:

$$\text{(f)} \qquad \begin{aligned} x^m &= \sum_{i=0}^{r_m} \alpha_i^m a_i^m \in D, \quad \alpha_i^m > 0, \quad \sum_{i=0}^{r_m} \alpha_i^m = 1, \\ z^m &= \sum_{j=0}^{s_m} \beta_j^m b_j^m \in Z, \quad \beta_j^m > 0, \quad \sum_{j=0}^{s_m} \beta_j^m = 1, \end{aligned}$$

where $\{a_0^m, \ldots, a_{r_m}^m\}$ and $\{b_0^m, \ldots, b_{s_m}^m\}$ are vertices of the carrier simplexes of x^m, z^m in $\mathscr{A}^m$, $\mathscr{B}^m$ respectively. Assuming that (a) does not hold, there exists $t^m\in I$ satisfying

$$\text{(g)} \qquad p\in(1-t^m)f(x^m, z^m)+t^m g(x^m, z^m).$$

Let r, s be the dimensions of the polyhedra K, Z respectively. For $r_m < i \leqslant r$, $s_m < j \leqslant s$, let

$$\alpha_i^m = \beta_j^m = 0, \quad a_i^m = a_0^m, \quad b_j^m = b_0^m.$$

Then (f) can be written as

$$\text{(h)} \qquad \begin{aligned} x^m &= \sum_{i=0}^{r} \alpha_i^m a_i^m \in D, \quad \alpha_i^m \geqslant 0, \quad \sum_{i=0}^{r} \alpha_i^m = 1, \\ z^m &= \sum_{j=0}^{s} \beta_j^m b_j^m \in Z, \quad \beta_j^m \geqslant 0, \quad \sum_{j=0}^{s} \beta_j^m = 1. \end{aligned}$$

By (g), (c), (d), there exists $w^m\in F(x^m, z^m)$ satisfying

$$\text{(i)} \qquad p = x^m-(1-t^m)w^m-t^m\sum_{i=0}^{r}\sum_{j=0}^{s}\alpha_i^m\beta_j^m u^m(a_i^m, b_j^m).$$

Since K, Z, $F(K\times Z)$, I are compact, replacing by subsequences, we may assume

$$\begin{aligned} &\lim_m x^m = x, && \lim_m t^m = t, && \lim_m w^m = w, \\ &\lim_m \alpha_i^m = \alpha_i, && \lim_m \beta_j^m = \beta_j, && \lim_m z^m = z, \\ &\lim_m a_i^m = a_i, && \lim_m b_j^m = b_j, && \lim_m u^m(a_i^m, b_j^m) = u_{ij}. \end{aligned}$$

By (e) and (h), we have

$$x = a_0 = a_1 = \ldots = a_r \in D \quad \text{and} \quad z = b_0 = b_1 = \ldots = b_s \in Z.$$

By (i), we have

$$p = x - (1-t)w - t\sum_{i=0}^{r}\sum_{j=0}^{s} a_i\beta_j u_{ij}.$$

By (1.1), we have $w, u_{ij} \in F(x, y)$. Since $x-p$ is a convex combination of points in the convex set $F(x, z)$, we have

$$p \in x - F(x, z) \subset f(D\times Z).$$

This contradiction completes the proof.

(5.2) THEOREM. *Let $X \subset Y$ be two closed subsets of a finite dimensional vector space E, $p \in E$ and $\mathscr{K}E$ the family of all non-empty compact convex subsets of E. Let Z be a polyhedron and $f\colon Y\times Z \to \mathscr{K}E$ a set-valued Z-compact field in $C(X, Y, p)$. Then there exists a bounded open neighborhood A of $o \in E$ and a single-valued Z-compact field g in $C(X, Y, p)$ such that*

(a) $$p \notin (1-t)f(x, z) + tg(x, z) \quad \textit{for} \quad (x, z, t) \in X\times Z\times I,$$

(b) $$g(y, z) = y \quad \textit{for} \quad y \in Y\setminus A,$$

(c) *If E_1 is a vector subspace containing $F(Y\times Z)$, then $G(Y\times Z) \subset E_1$ where F, G are compact maps corresponding to f, g respectively.*

Proof. Let K be a bounded closed convex symmetric neighborhood of $o \in E$ such that

(d) $$p \in \tfrac{1}{4}K \quad \text{and} \quad F(Y\times Z) \subset \tfrac{1}{4}K.$$

By (2.1), there exists a compact map $F^*\colon K\times Z \to \mathscr{K}E$ such that

(e) $$F^*(K\times Z) \subset \tfrac{1}{4}K\cap E_1, \quad F^*|_{(Y\cap K)\times Z} = F|_{(Y\cap K)\times Z}.$$

Define $f^*(a, z) = a - F^*(a, z)$ for $(a, z) \in K\times Z$. Let $\|\cdot\|$ be the gauge of K. Then $\|\cdot\|$ is a norm on E with the closed unit ball K. Applying (5.1) (with $D = X\cap K$), let g^* be a single-valued Z-compact field in $C(X\cap K, K, p)$ such that

(f) $$p \notin (1-\lambda)f^*(x, z) + \lambda g^*(x, z) \quad \text{for} \quad (x, z) \in (X\cap K)\times Z\times I.$$

(g) The set $G^*(K\times Z)$ is contained in the convex hull of $F^*(K\times Z)$ where G^* is the compact map corresponding to g^*.

Define for $(y, z) \in Y\times Z$,

$$G(y, z) = \begin{cases} G^*(y, z) & \text{if} \quad \|y\| \leqslant 1, \\ (2-\|y\|)G^*\left(\dfrac{y}{\|y\|}, z\right) & \text{if} \quad 1 \leqslant \|y\| \leqslant 2, \\ 0 & \text{if} \quad 2 \leqslant \|y\|; \end{cases}$$

and

$$g(y, z) = y - G(y, z).$$

By (1.2), (1.3), G is a compact map. By (f), the condition (a) holds for $\|x\| \leqslant 1$. For $(x, z, t) \in X \times Z \times I$ and $1 \leqslant \|x\| \leqslant 2$, let $u \in F(x, z)$ and $v \in G(x, z)$. Then by (d), (e) and (g), we have

$$\|p\| \leqslant \tfrac{1}{4} < \tfrac{1}{2} \leqslant \|x\| - (1-t)\|u\| - t\|v\| \leqslant \|x-(1-t)u - tv\|.$$

Thus

$$p \notin x - (1-t)F(x, z) - tG(x, z) = (1-t)f(x, z) + tg(x, z).$$

Finally for $(x, z, t) \in X \times Z \times I$ and $2 \leqslant \|x\|$, let $u \in F(x, z)$. Then

$$\|p\| \leqslant \tfrac{1}{4} < \tfrac{1}{2} \leqslant \|x\| - (1-t)\|u\| \leqslant \|x - (1-t)u\|.$$

Thus

$$p \notin (1-t)f(x, z) + tg(x, z).$$

Therefore, the condition (a) holds in all cases. In particular, g is a Z-compact field in $C(X, Y, p)$. Clearly g is single-valued since G^* is. Also the condition (b) holds if A is a bounded open set containing $2K$. By (e), (g), we have $G(Y) \subset E_1$. This completes the proof.

(5.3) THEOREM. *Let $X \subset Y$ be two closed subsets of a finite dimensional vector space E, $p \in E$ and $\mathscr{K}E$ the family of all non-empty compact convex subsets of E. If f_0, f_1 are two single-valued compact fields homotopic in $C(X, Y, p)$, then f_0, f_1 are homotopic under some single-valued homotopy h in $C(X, Y, p)$. Furthermore if there exists a bounded neighborhood A of $o \in E$ such that $f_0(y) = f_1(y) = y$ for $y \in Y \setminus A$, then there exists some bounded neighborhood B of $o \in E$ such that $h(y, t) = y$ for $(y, t) \in (Y \setminus B) \times I$.*

Proof. Let f be a homotopy for $f_0 \simeq f_1$ in $C(X, Y, p)$. Then f can be considered as a set-valued I-compact field in $C(X, Y, p)$. By (5.2), there exists a bounded neighborhood B of $o \in E$ and a single-valued I-compact field g in $C(X, Y, p)$ such that

$$p \notin (1-t)f(x, \lambda) + tg(x, \lambda) \quad \text{for} \quad (x, \lambda, t) \in X \times I \times I,$$

and

$$g(y, \lambda) = y \quad \text{for} \quad y \in Y \setminus B,\ \lambda \in I.$$

Define for $(y, t) \in Y \times I$,

$$h(y, t) = \begin{cases} (1-3t)f_0(y) + 3tg(y, 0) & \text{for} \quad 0 \leqslant t \leqslant \frac{1}{3}, \\ g(y, 3t-1) & \text{for} \quad \frac{1}{3} \leqslant t \leqslant \frac{2}{3}, \\ (3-3t)g(y, 1) + (3t-2)f_1(y) & \text{for} \quad \frac{2}{3} \leqslant t \leqslant 1. \end{cases}$$

Then h is a single-valued homotopy for $f_0 \simeq f_1$ in $C(X, Y, p)$. Furthermore if A is given as in our theorem, we may choose B containing A and the last condition can be easily proved.

CHAPTER II

TOPOLOGICAL DEGREES OF SET-VALUED COMPACT FIELDS IN LOCALLY CONVEX SPACES

§ 6. Basic known facts about Brouwer's degrees. Let A be a *bounded* open subset of a *finite dimensional* vector space E. For a continuous *single-valued* map $f\colon \bar{A} \to E$ and a point $p \in E \setminus f(\partial A)$, let $d(A, p, f)$ denote the classical Brouwer's degree. The following facts are listed here for the convenience of our reference:

(6.1) If $f\colon \bar{A} \to \bar{A}$ is the identity map and if $p \in A$, then $d(A, p, f) = 1$.

(6.2) Let $f\colon \bar{A} \to E$ be a continuous single-valued map. If $p \in E \setminus f(A)$ or if A is empty, then $d(A, p, f) = 0$.

(6.3) If $h\colon \bar{A} \times I \to E$ is a continuous single-valued map such that $p \in E \setminus h(\partial A \times I)$, then $d(A, p, h_0) = d(A, p, h_1)$.

(6.4) Let $\{A_j\colon j \in J\}$ be a family of disjoint open subsets of a bounded open set A, $f\colon \bar{A} \to E$ a continuous single-valued map and $p \in E \setminus f(\bar{A} \setminus \bigcup_j A_j)$. Then $d(A_j, p; f|_{\bar{A}_j}) = 0$ for all except only a finite number of indices $j \in J$ and $d(A, p, f) = \sum_{j \in J} d(A_j, p, f|_{\bar{A}_j})$.

(6.5) Let A be a bounded open convex symmetric neighborhood of $o \in E$ and $f\colon \bar{A} \to E$ an antipodal continuous single-valued map, i.e. $f(-x) = -f(x)$ for all $x \in \partial A$. If $o \in E \setminus f(\partial A)$, then $d(A, o, f)$ is an odd integer.

(6.6) Let A be a bounded open convex symmetric neighborhood of $o \in E$ and $f, g\colon \bar{A} \to E$ be continuous single-valued maps with $o \in E \setminus [f(\partial A) \cup g(\partial A)]$. If $d(A, o, f) = d(A, o, g)$, then there exists a homotopy $h\colon \bar{A} \times I \to E$ such that $o \notin h(\partial A \times I)$ and $h_0 = f$, $h_1 = g$.

(6.7) Let A be a bounded open subset of E, $f\colon \bar{A} \to E$ a continuous single-valued map and $p \in E \setminus f(\partial A)$. If $g\colon \bar{A} \to E$ is defined by $g(x) = f(x) - p$ for all $x \in \bar{A}$, then $o \in E \setminus g(\partial A)$ and $d(A, p, f) = d(A, o, g)$.

(6.8) For $j = 1, 2$, let A_j be a bounded open subset of a finite dimensional vector space E_j, $f_j\colon \bar{A}_j \to E_j$ a continuous single-valued map and $p_j \in E_j \setminus f_j(\partial A_j)$. Let $E = E_1 \times E_2$, $A = A_1 \times A_2$, $p = (p_1, p_2)$ and

$f(x) = (f_1(x_1), f_2(x_2))$ for $x = (x_1, x_2) \in \bar{A}$. Then
$$d(A, p, f) = d_1(A_1, p_1, f_1)\, d_2(A_2, p_2, f_2).$$

(6.9) Let A be a bounded open subset of E, $f\colon \bar{A} \to E$ a continuous single-valued map and $p \in E \setminus f(\partial A)$. Let $F\colon \bar{A} \to E$ be defined by $F(x) = x - f(x)$ for $x \in \bar{A}$. If E_1 is a vector subspace containing p and $F(\bar{A})$, then $d(A, p, f) = d_1(A \cap E_1, p, f|_{\overline{A \cap E_1}})$.

(6.10) Let A, B be bounded open subsets in a finite dimensional vector space E, $g\colon \bar{B} \to E$ and $f\colon \bar{A} \to E$ be continuous single-valued maps such that $g(\bar{B}) \subset A$. If $\{A_j\colon j \in J\}$ is the family of all components of $A \setminus g(\partial B)$, $a_j \in A_j$ for $j \in J$ and $p \in E \setminus [f \circ g(\partial B) \cup f(\partial A)]$, then
$$d(B, p, f \circ g) = \sum_{j \in J} d(A_j, p, f|_{\bar{A}_j})\, d(B, a_j, g),$$
and the right-hand side is finite sum independent of the choice of $a_j \in A_j$.

For the case of single-valued compact fields in locally convex spaces, see [5], [6], [9], [21], [22], [23], [24], [25], [26], [27].

§ 7. Definition of topological degree and its homotopy invariance. In this section, we shall define the topological degree for set-valued compact fields in separated locally convex spaces and prove a theorem on homotopy invariance. It will be divided into two parts. The first part is for finite dimensional vector spaces and the second for general case.

(7.1) Let A be an open subset of a *finite dimensional* vector space E, $p \in E$ and $\mathscr{K}E$ the family of all non-empty compact convex subsets of E. Let $f\colon \bar{A} \to \mathscr{K}E$ be a set-valued compact field in $C(\partial A, A, p)$. By (5.2), there exist a single-valued compact field g in $C(\partial A, \bar{A}, p)$ and a bounded open set B in E such that

(a) $$f \simeq g \text{ in } C(\partial A, \bar{A}, p),$$

(b) $$g(y) = y \text{ for } y \in \bar{A} \setminus B,$$

(c) $$p \in B.$$

Then $A \cap B$ is a bounded open subset of E and $g|_{\overline{A \cap B}}$ a continuous single-valued map with $p \in E \setminus g(\partial(A \cap B))$. Define the *degree* $d(A, p, f)$ to be the Brouwer's degree $d(A \cap B, p, g|_{\overline{A \cap B}})$.

We want to show that our definition is independent of the choice of g and B. Let g_1 be any single-valued compact field in $C(\partial A, \bar{A}, p)$ and B_1 any bounded open set in E such that (a), (b), (c) hold for g_1 and B_1. By (a), $g \simeq f \simeq g_1$ in $C(\partial A, \bar{A}, p)$ and by (b), $g(y) = g_1(y) = y$ for $y \in \bar{A} \setminus (B \cup B_1)$. By (5.3), there exist a single-valued homotopy h for $g \simeq g_1$ in $C(\partial A, \bar{A}, p)$ and a bouded open set B^* such that

(d) $$h(y, t) = y \text{ for } (y, t) \in (\bar{A} \setminus B^*) \times I,$$

(e) $$B \cup B_1 \subset B^*.$$

By (b), (c),

$$p \notin g(\overline{A \cap B^* \setminus B}).$$

By (6.4),

$$d(A \cap B^*, p, g|_{\overline{A \cap B^*}}) = d(A \cap B, p, g|_{\overline{A \cap B}}).$$

Similarly,

$$d(A \cap B^*, p, g_1|_{\overline{A \cap B^*}}) = d(A \cap B_1, p, g_1|_{\overline{A \cap B_1}}).$$

Now $h|_{\overline{A \cap B^*} \times I}$ is a single-valued homotopy for the continuous single-valued maps $g|_{\overline{A \cap B^*}}$ and $g_1|_{\overline{A \cap B^*}}$. By (c), (d), (e),

$$p \notin h(\partial(A \cap B^*) \times I).$$

By (6.3),

$$d(A \cap B^*, p, g|_{\overline{A \cap B^*}}) = d(A \cap B^*, p, g_1|_{\overline{A \cap B^*}}).$$

Combining all equalities in this paragraph, we have

$$d(A \cap B, p, g|_{\overline{A \cap B}}) = d(A \cap B_1, p, g_1|_{\overline{A \cap B_1}}).$$

This shows that $d(A, p, f)$ is independent of the choice of g and B.

(7.2) Suppose that A is a *bounded* open subset of a finite dimensional vector space E and $f: \bar{A} \to E$ is a continuous single-valued map with $p \in E \setminus f(\partial A)$. If we choose $B \supset \bar{A}$ and $g = f$ in (7.1), it follows that the degree defined in (7.1) is equal to the Brouwer's degree in § 6.

(7.3) Lemma. *Let A be an open subset of a finite dimensional vector space E, $p \in E$ and $\mathcal{K}E$ the family of all non-empty compact convex subsets of E. If $f_0, f_1: \bar{A} \to \mathcal{K}E$ are set-valued compact fields homotopic in $C(\partial A, \bar{A}, p)$ then $d(A, p, f_0) = d(A, p, f_1)$.*

Proof. Let B be a bounded open set in E and g a single-valued compact field in $C(\partial A, \bar{A}, p)$ such that $f_0 \simeq g$ in $C(\partial A, \bar{A}, p)$, $g(y) = y$ for $y \in \bar{A} \setminus B$ and $p \in B$. Then $f_1 \simeq g$ in $C(\partial A, \bar{A}, p)$. Consequently,

$$d(A, p, f_0) = d(A \cap B, p, g|_{\overline{A \cap B}}) = d(A, p, f_1).$$

(7.4) Lemma. *Let A be an open subset of a finite dimensional vector space E, $p \in E$ and $\mathcal{K}E$ the family of all non-empty compact convex subsets of E. Let $f: \bar{A} \to \mathcal{K}E$ be a compact field in $C(\partial A, \bar{A}, p)$ with the corresponding compact map F. If E_1 is a vector subspace containing both $F(\bar{A})$ and the point p, then*

$$d(A, p, f) = d_1(A \cap E_1, p, f|_{\overline{A \cap E_1}}).$$

Proof. By (5.2), there exist a single-valued compact field g in $C(\partial A, \bar{A}, p)$ and a bounded open set B such that

$$p \notin (1-t)f(x) + tg(x) \quad \text{for} \quad (x, t) \in \partial A \times I,$$

$$g(y) = y \quad \text{for} \quad y \in \bar{A} \setminus B,$$

$$G(\bar{A}) \subset E_1 \quad \text{and} \quad p \in B.$$

Then f, g are homotopic in $C(\partial A, \bar{A}, p)$. By (7.1),

$$d(A, p, f) = d(A \cap B, p, g|_{\overline{A \cap B}}).$$

Since $F(\bar{A}) \cup G(\bar{A}) \subset E_1$, when we restrict ourselves to E_1, we have

$$f|_{\overline{A \cap E_1}} \simeq g|_{\overline{A \cap E_1}} \quad \text{in} \quad C\big(\partial(A \cap E_1), \overline{A \cap E_1}, p\big),$$
$$g(y) = y \quad \text{for} \quad y \in \overline{A \cap E_1} \setminus (B \cap E_1),$$

and p is a point of the bounded open subset $B \cap E_1$ of E_1. Hence

$$d_1(A \cap E_1, p, f|_{\overline{A \cap E_1}}) = d_1(A \cap B \cap E_1, p, g|_{\overline{A \cap B \cap E_1}}).$$

By (6.9),

$$d(A \cap B, p, g|_{\overline{A \cap B}}) = d_1(A \cap B \cap E_1, p, g|_{\overline{A \cap B \cap E_1}}).$$

Consequently $d(A, p, f) = d_1(A \cap E_1, p, f|_{\overline{A \cap E_1}})$.

(7.5) Now we are ready to define the notion of topological degree for the general case. Let A be an open subset of a separated locally convex space E, $p \in E$, $\mathscr{K}E$ the family of all non-empty compact convex subsets of E and $f\colon \bar{A} \to \mathscr{K}E$ a set-valued compact field in $C(\partial A, \bar{A}, p)$. By (4.1), there exists a finite dimensional set-valued compact field g homotopic to f in $C(\partial A, \bar{A}, p)$. Let G be the compact map corresponding to g, and E_1 any finite dimensional vector subspace containing $G(\bar{A})$ and p. Then the degree $d(A, p, f)$ is defined to be $d_1(A \cap E_1, p, g|_{\overline{A \cap E_1}})$ in the finite dimensional vector space E_1.

We want to show that our definition is independent of the choice of g and E_1. Let g_2 be any finite dimensional set-valued compact field homotopic to f in $C(\partial A, \bar{A}, p)$ and E_2 any vector subspace containing $G_2(\bar{A})$ and p where G_2 is the compact map corresponding to g_2. Then g, g_2 are finite dimensional set-valued compact fields homotopic in $C(\partial A, \bar{A}, p)$. By (4.2), there exists a finite dimensional homotopy h for $g \simeq g_2$ in $C(\partial A, \bar{A}, p)$. Let E_3 be a finite dimensional vector subspace containing $E_1 \cup E_2$ and $H(\bar{A} \times I)$ where H is the compact map corresponding to h. By (7.4),

$$d_1(A \cap E_1, p, g|_{\overline{A \cap E_1}}) = d_3(A \cap E_3, p, g|_{\overline{A \cap E_3}}),$$

and

$$d_2(A \cap E_2, p, g_2|_{\overline{A \cap E_2}}) = d_3(A \cap E_3, p, g_2|_{\overline{A \cap E_3}}).$$

Since $H(\bar{A} \times I) \subset E_3$ and $p \in E_3$, we have

$$g|_{\overline{A \cap E_3}} \simeq g_2|_{\overline{A \cap E_3}} \quad \text{in} \quad C\big(\partial(A \cap E_3), \overline{A \cap E_3}, p\big).$$

By (7.3),

$$d_3(A \cap E_3, p, g|_{\overline{A \cap E_3}}) = d_3(A \cap E_3, p, g_2|_{\overline{A \cap E_3}}).$$

Combining all equalities in this paragraph, we have

$$d_1(A \cap E_1, p, g|_{\overline{A \cap E_1}}) = d_2(A \cap E_2, p, g_2|_{\overline{A \cap E_2}}).$$

This shows that our definition is well-defined.

(7.6) Suppose that A is an open subset of a *finite dimensional* vector space E, $p \in E$, $\mathscr{K}E$ the family of all non-empty compact convex subsets of E and $f\colon \bar{A} \to \mathscr{K}E$ a set-valued compact field in $C(\partial A, \bar{A}, p)$. If we choose $E_1 = E$ and $g = f$ in (7.5), it follows that the degrees defined in (7.1) and (7.5) are the same.

(7.7) Theorem. *Let A be an open subset of a separated locally convex space E, $p \in E$ and $\mathscr{K}E$ the family of all non-empty compact convex subsets of E. If f_0, f_1 are set-valued compact fields homotopic in $C(\partial A, \bar{A}, p)$, then $d(A, p, f_0) = d(A, p, f_1)$.*

Proof. Let g be a finite dimensional set-valued compact field homotopic to f_0 in $C(\partial A, \bar{A}, p)$. Let E_1 be a finite dimensional vector subspace containing both $G(\bar{A})$ and p where G is the compact map corresponding to g. Then $g \simeq f_1$ in $C(\partial A, \bar{A}, p)$. Consequently,

$$d(A, p, f_0) = d(A \cap E_1, p, g|_{\overline{A \cap E_1}}) = d(A, p, f_1).$$

(7.8) Corollary. *Let A be an open subset of a separated locally convex space E, $p \in E$ and $\mathscr{K}E$ the family of all non-empty compact convex subsets of E.*

(a) *For each given set-valued compact field f in $C(\partial A, \bar{A}, p)$, there exists a neighborhood V of $o \in E$ such that the following conditions holds:*

For every set-valued compact field g in $C(\partial A, \bar{A}, p)$ with $g(x) \subset f(x) + V$ for all $x \in \partial A$, we have

$$d(A, p, f) = d(A, p, g).$$

(b) *If f, g are set-valued compact fields in $C(\partial A, \bar{A}, p)$ and if p does not belong to the convex hull of $f(x) \cup g(x)$ for all $x \in \partial A$, then*

$$d(A, p, f) = d(A, p, g).$$

Proof. If we choose V to be a convex neighborhood of $o \in E$ with $(p + V) \cap f(\partial A) = \emptyset$, then (a) is reduced to (b). Define $h(x, t) = (1 - t) \times \times f(x) + tg(x)$ for $(x, t) \in \bar{A} \times I$. Then h is a homotopy for $f \simeq g$ in $C(\partial A, \bar{A}, p)$. The result follows from (7.7).

§ 8. Sum theorem.

(8.1) Theorem. *Let A be an open subset of a separated locally convex space E, $p \in E$ and $\mathscr{K}E$ the family of all non-empty compact convex subsets of E. Let $\{A_j\colon j \in J\}$ be a family of disjoint open subsets of A and $f\colon \bar{A} \to \mathscr{K}E$ a set-valued compact field with $p \in E \setminus f(\bar{A} \setminus \bigcup_j A_j)$. Then $d(A_j, p, f_{\bar{A}_j}) = 0$ for all except only a finite number of indices $j \in J$. Furthermore,*

$$d(A, p, f) = \sum_{j \in J} d(A_j, p, f|_{\bar{A}_j}).$$

Proof. Consider first the case when E is finite dimensional. Let $X = \bar{A} \setminus \bigcup_{j \in J} A_j$ and $Y = \bar{A}$. By (5.2), there exists a single-valued compact field g in $C(X, Y, p)$ and a bounded open set B such that

(a) $$f \simeq g \text{ in } C(X, Y, p),$$

(b) $$g(y) = y \text{ for } y \in Y \setminus B,$$

(c) $$p \in B.$$

By (7.1),

$$d(A, p, f) = f(A \cap B, p, g|_{\overline{A \cap B}}),$$

$$d(A_j, p, f|_{\bar{A}_j}) = d(A_j \cap B, p, g|_{\overline{A_j \cap B}}).$$

Now $A \cap B$ is bounded open subset of E and $g|_{\overline{A \cap B}}: \overline{A \cap B} \to E$ is a continuous single-valued map. Also $\{A_j \cap B: j \in J\}$ is a family of disjoint open subsets of $A \cap B$. Suppose that $x \in \overline{A \cap B} \setminus \bigcup_j (A_j \cap B)$ and $p = g(x)$. By (a), $x \notin X$, i.e., $x \in \bigcup_j A_j$. Thus $x \notin B$ and by (b), (c), $x = g(x) = p \in B$. This contradiction shows that $p \notin g(\overline{A \cap B} \setminus \bigcup_j (A_j \cap B))$. By (6.4), we have $d(A_j \cap B, p, g|_{\overline{A_j \cap B}}) = 0$ for all except only a finite number of indices $j \in J$ and

$$d(A \cap B, p, g|_{\overline{A \cap B}}) = \sum_{j \in J} d(A_j \cap B, p, g|_{\overline{A_j \cap B}}).$$

Combining all equalities, the proof of the finite dimensional case is complete.

Now, let E be a separated locally convex space. Let $X = \bar{A} \setminus \bigcup_j A_j$ and $Y = \bar{A}$. By (4.1), there exists a finite dimensional set-valued compact field g homotopic to f in $C(X, Y, p)$. Let G be the compact map corresponding to g and let E_1 be a finite dimensional vector subspace containing $G(\bar{A})$ and p. Then by (7.5), we have

$$d(A, p, f) = d_1(A \cap E_1, p, g|_{\overline{A \cap E_1}}),$$

$$d(A_j, p, f|_{\bar{A}_j}) = d_1(A_j \cap E_1, p, g|_{\overline{A_j \cap E_1}}).$$

By the result of the finite dimensional case,

$$d_1(A_j \cap E_1, p, g|_{\overline{A_j \cap E_1}}) = 0$$

for all except only a finite number of indices $j \in J$ and

$$d_1(A \cap E_1, p, g|_{\overline{A \cap E_1}}) = \sum_{j \in J} d_1(A_j \cap E_1, p, g|_{\overline{A_j \cap E_1}}).$$

The result follows if we combine all equalities of this paragraph.

(8.2) COROLLARY. *Let A be an open subset of a separated locally convex space E, $p \in E$, $\mathcal{K}E$ the family of all non-empty compact subsets of E and f a set-valued compact field in $C(\partial A, \bar{A}, p)$. If B is an open subset of A*

such that $p \in E \setminus f(\bar{A} \setminus B)$, *then*

$$d(A, p, f) = d(B, p, f|_{\bar{B}}).$$

Proof. This is a special case of (8.1) when J consists of only one element.

§ 9. The case of odd degrees.

(9.1) Theorem. *Let A be an open subset of a separated locally convex space E, $f: \bar{A} \to \bar{A}$ the identity map and $p \in A$. Then $d(A, p, f) = 1$.*

Proof. Let E_1 be a finite dimensional vector space containing p. Let B be a bounded open subset in E_1 with $p \in B$. Then by (6.1), we have

$$d(A, p, f) = d_1(A \cap E_1, p, f|_{\overline{A \cap E_1}}) = d_1(A \cap B, p, f|_{\overline{A \cap B}}) = 1.$$

(9.2) Theorem. *Let A be an open neighborhood of the origin in a separated locally convex space E and $\mathscr{K}E$ the familly of all non-empty compact convex subsets of E. If f is a set-valued compact field in $C(\partial A, \bar{A}, o)$ such that for every $x \in \partial A$ and $\lambda > 0$, we have $-\lambda x \notin f(x)$, then $d(A, o, f) = 1$.*

Proof. Define $h(y, t) = y - tF(y)$ for $(y, t) \in \bar{A} \times I$. Then h is a homotopy in $C(\partial A, \bar{A}, o)$ such that $h_1 = f$ and h_0 is the identity map. The result follows from (7.7) and (9.1).

(9.3) Theorem. *Let A be an open convex symmetric neighborhood of the origin in a separated locally convex space E, $\mathscr{K}E$ the family of all non-empty compact convex subsets of E and f a set-valued compact field in $C(\partial A, \bar{A}, o)$. If for each $x \in \partial A$ and $\lambda > 0$, we have $f(x) \cap \lambda f(-x) = \emptyset$, then $d(A, o, f)$ is an odd integer.*

When A is the unit ball in a Banach space, see Granas [13]. Before we prove this theorem, we shall need the following lemmas. The first lemma reduces the theorem from locally convex space to finite dimensional vector space; the second reduces the domain of the set-valued compact field from unbounded to bounded set; and the last lemma reduces the set-valued map to single-valued map.

Let B be a symmetric neighborhood of a separated locally convex space E, and $\mathscr{K}E$ the family of all non-empty compact convex subsets of E. A set-valued map $g: \bar{B} \to \mathscr{K}E$ is said to be *antipodal* if for each $x \in \partial A$, we have $f(-x) = -f(x)$.

(9.4) Lemma. *Let A be an open convex symmetric neighborhood of the origin in a separated locally convex space E and $\mathscr{K}E$ the family of all non-empty compact convex subsets of E. If f is a set-valued compact field in $C(\partial A, \bar{A}, o)$ with $f(x) \cap \lambda f(-x) = \emptyset$ for all $x \in \partial A$ and $\lambda > 0$, then f is homotopic to some finite dimensional antipodal set-valued compact field in $C(\partial A, \bar{A}, o)$.*

Proof. Let $h(y,t) = \frac{1}{1+t}[f(y)-tf(-y)]$ for $(y,t)\in \bar{A}\times I$. Then h is a homotopy in $C(\partial A, \bar{A}, o)$ such that $h_0 = f$ and h_1 is antipodal. By (3.2), let V be a convex symmetric neighborhood of $o\in E$ such that $V\cap h_1(\partial A) = \emptyset$. Let Q^* be the closure of $H_1(\bar{A})$ where H_1 is the compact map corresponding to h_1. The set $Q = Q^*\cup(-Q^*)$ is compact. By (2.2), let ξ^* be a continuous single-valued map defined on Q into some finite dimensional vector subspace E_1 of E such that $\xi^*(z)-z\in V$ for all $z\in Q$. Define $\xi(z) = \frac{1}{2}[\xi^*(z)-\xi^*(-z)]$ for $z\in Q$. Then $\xi(z)-z\in V$ and $\xi(-z) = -\xi(z)$ for all $z\in Q$. Define $G(y) = \overline{\mathrm{co}}(\xi\circ H_1(y))$ and $g(y) = y-G(y)$ for $y\in\bar{A}$. By (4.1) and (3.1), h_1 and g are homotopic in $C(\partial A, \bar{A}, o)$. Hence f, g are homotopic in $C(\partial A, \bar{A}, o)$. Clearly g is antipodal according to our construction. This completes the proof.

(9.5) Lemma. *Let A be an open convex symmetric neighborhood of the origin in a finite dimensional vector space E and $\mathscr{K}E$ the family of all non-empty compact convex subsets of E. If f is a set-valued compact field in $C(\partial A, \bar{A}, o)$ with $f(x)\cap\lambda f(-x) = \emptyset$ for all $x\in\partial A$ and $\lambda > 0$, then there exists a bounded open convex symmetric neighorhood B of $o\in E$ and a set-valued compact field g in $C(\partial B, \bar{B}, o)$ such that the following conditions hold:*

(a) $$g(y) = f(y) \quad \text{for all} \quad y\in\bar{A}\cap\bar{B};$$

(b) $$0\notin f(\bar{A}\setminus B)\cup g(\bar{B}\setminus A);$$

(c) $$g(x)\cap\lambda g(-x) = \emptyset \quad \text{for all} \quad x\in\partial B \quad \text{and} \quad \lambda > 0.$$

Proof. Let B be a bounded open convex symmetric neighborhood of $o\in E$ such that $F(\bar{A})\subset\frac{1}{2}B$ where F is the compact map corresponding to f. Let α, β bo the gauges of A, B respectively. Define

$$G(y) = \begin{cases} F(y) & \text{for} \quad y\in\bar{A}\cap\bar{B}, \\ \alpha(y)F\left(\dfrac{y}{\alpha(y)}\right) & \text{for} \quad y\in\bar{B}\setminus A, \end{cases}$$

and

$$g(y) = y-G(y) \quad \text{for} \quad y\in\bar{B}.$$

By (1.2) and (1.3), G is a compact map. Clearly the condition (a) holds. If $x\in\bar{A}\setminus B$ and $o\in f(x)$, then

$$1\leqslant\beta(x)\leqslant\max\{\beta(y): y\in f(\bar{A})\}\leqslant\tfrac{1}{2}.$$

If $x\in\bar{B}\setminus A$ and $o\in g(x)$, then $x/\alpha(x)\in\partial A$ and $o\in f(x/\alpha(x))$ which is again impossible. This proves (b) and in particular g is a set-valued compact field in $C(\partial B, \bar{B}, o)$. Finally suppose that $g(x)\cap\lambda g(-x)\neq\emptyset$ for some

$x \in \partial B$ and $\lambda > 0$. If $x \in A \cap \partial B$, there exist z, $z' \in F(A)$ such that $x - z = \lambda(-x - z')$, i.e.,

$$1 = \beta(x) = \frac{1}{1+\lambda}\beta(z - \lambda z') \leqslant \frac{1}{1+\lambda}\big(B(z) + \lambda B(z')\big) \leqslant \tfrac{1}{2}.$$

If $x \in \partial B \setminus A$, we have $x/\alpha(x) \in \partial A$ and

$$f\left(\frac{x}{\alpha(x)}\right) \cap \lambda f\left(\frac{-x}{\alpha(x)}\right) = \frac{1}{\alpha(x)}[g(x) \cap \lambda g(-x)] \neq \emptyset.$$

In both cases, we have a contradiction. This proves the condition (c).

(9.6) Lemma. *Let E_n be an n-dimensional vector space and $\mathscr{K}E_n$ the family of all non-empty compact convex subsets of E_n. Let Y be a bounded closed convex symmetric neighborhood of the origin in E_n and X a closed subset of Y with $\partial Y \subset X$. If f is a set-valued compact field in $C(X, Y, o)$ and if f is antipodal, then there exists a single-valued antipodal compact field g in $C(X, Y, o)$ such that*

$$o \notin (1-t)f(x) + tg(x) \quad \textit{for} \quad (x, t) \in X \times I.$$

Proof. Without loss of generality, we may assume $E_n = \boldsymbol{R}^n$ where $\boldsymbol{R}$ denotes the real field. Let $\{e_1, e_2, \ldots, e_n\}$ be the standard base of E_n. Let $\|x\| = \sum_{i=1}^{n} |x_i|$ for each $x = \sum_{i=1}^{n} x_i e_i$ and $x_i \in \boldsymbol{R}$. Without loss of generality we may assume that Y is the unit ball with the above norm. Let M be the family of all faces of the simplexes of the form $\operatorname{co}\{0, \varepsilon_1 e_1, \varepsilon_2 e_2, \ldots, \varepsilon_n e_n\}$ where $\varepsilon_i = \pm 1$. Then M is a finite simplicial complex triangulating Y. Let $\delta > 0$ and let $\mathscr{A}$ be a successive barycentric subdivision of M with mesh $\mathscr{A} \leqslant \delta$. Since f is antipodal and $\mathscr{A}$ is centrally symmetric, there exists a point $G(a) \in F(a)$ for each vertex a of $\mathscr{A}$ such that $G(-x) = -G(x)$ for all vertex $x \in \mathscr{A}$ on the boundary of Y. Let G be linearly extended over every simplex of M and define $g(y) = y - G(y)$ for $y \in Y$. Then g is a single-valued antipodal compact field on Y. By (5.1), when we choose sufficiently small $\delta > 0$, we have

$$0 \notin (1-t)f(x) + tg(x) \quad \text{for} \quad (x, t) \in X \times I.$$

In particular, when $t = 1$, g is a single-valued antipodal compact field in $C(X, Y, o)$.

Proof of (9.3). By (9.4), let g be a finite dimensional set-valued antipodal compact field homotopic to f in $C(\partial A, \overline{A}, o)$. Let E_1 be a finite dimensional vector subspace containing $G(\overline{A})$ where G is the compact map corresponding to g. By (7.5),

$$d(A, o, f) = d_1(A \cap E_1, o, g|_{\overline{A \cap E_1}}).$$

By (9.5), there exist a bounded open convex symmetric neighborhood B of $o \in E_1$ and a set-valued compact field g_0 in $C(\partial B, \bar{B}, o)$ such that the following conditions hold:

(a) $$g_0(\bar{B}) \subset E_1,$$

(b) $$g_0(y) = g(y) \quad \text{for all} \quad y \in \overline{A \cap E_1} \cap \bar{B},$$

(c) $$0 \notin g(\overline{A \cap E_1} \setminus B) \cup g_0(\bar{B} \setminus A),$$

(d) $$g_0(x) \cap \lambda g_0(-x) = \emptyset \quad \text{for all} \quad x \in \partial B \quad \text{and} \quad \lambda > 0.$$

By (8.2) and (c), we have

$$d_1(A \cap E_1, o, g|_{\overline{A \cap E_1}}) = d_1(A \cap B, o, g|_{\overline{A \cap B}}),$$

and

$$d_1(B, o, g_0) = d_1(A \cap B, o, g_0|_{\overline{A \cap B}}).$$

By (b),

$$d_1(A \cap B, o, g|_{\overline{A \cap B}}) = d_1(A \cap B, o, g_0|_{\overline{A \cap B}}).$$

Define $h(x, t) = [g_0(x) - t g_0(-x)]/(1+t)$ for $(x, t) \in \bar{B} \times I$. Then (d) implies that h is a homotopy in $C(\partial B, \bar{B}, o)$ such that $h_0 = g_0$ and h_1 is antipodal. By (7.7),

$$d_1(B, o, g_0) = d_1(B, o, h_0) = d_1(B, o, h_1).$$

By (9.6), there exists a single-valued antipodal compact field g^* homotopic to h_1 in $C(\partial B, \bar{B}, o)$ and $g^*(\bar{B}) \subset E_1$. By (7.7), we have

$$d_1(B, o, h_1) = d_1(B, o, g^*).$$

By (6.5), $d_1(B, o, g^*)$ is an odd integer. If we trace back all the equalities, $d(A, o, f)$ is an odd integer.

§ 10. The case of non-vanishing degrees.

(10.1) THEOREM. *Let A be an open subset of a separated locally convex space E, $p \in E$, $\mathscr{K}E$ the family of all non-empty compact convex subsets of E, and f a set-valued compact field in $C(\partial A, \bar{A}, p)$. If $d(A, p, f) \neq 0$, then $p \in f(A)$.*

Proof. Suppose that $p \notin f(A)$. Then f is a set-valued compact field in $C(\bar{A}, \bar{A}, p)$. By (4.1), there exists a finite dimensional set-valued compact field g homotopic to f in $C(\bar{A}, \bar{A}, p)$. Let E_1 be a finite dimensional vector subspace containing $G(\bar{A})$ and p where G is the compact map corresponding to g. By (7.5), we have

$$d(A, p, f) = d_1(A \cap E_1, p, g|_{\overline{A \cap E_1}}).$$

Since $g|_{\overline{A \cap E_1}}$ is in $C(\overline{A \cap E_1}, \overline{A \cap E_1}, p)$, by (5.2), there exist a single-valued compact field g^* in $C(\overline{A \cap E_1}, \overline{A \cap E_1}, p)$ and a bounded open set B in E_1 such that

$$g|_{\overline{A\cap E_1}} \simeq g^* \quad \text{in} \quad C(\overline{A\cap E_1}, \overline{A\cap E_1}, p),$$

$$g^*(x) = y \quad \text{for} \quad y \in \overline{A\cap E_1} \setminus B,$$

and

$$p \in B.$$

By (7.1),

$$d_1(A\cap E_1, p, g|_{\overline{A\cap E_1}}) = d_1(A\cap B, p, g^*|_{\overline{A\cap B}}).$$

Now $p \notin g^*(\overline{A\cap E_1})$ and hence by (6.2), we have

$$d_1(A\cap B, p, g^*|_{\overline{A\cap B}}) = 0.$$

Therefore $d(A, p, f) = 0$ and the proof is complete.

The following theorem is a generalization of the classical result: If $f\colon S_n \to S_n$ is a continuous single-valued map on the n-dimensional unit sphere with non-vanishing degree, then f must be surjective.

(10.2) THEOREM. *Let A be an open convex symmetric neighborhood of the origin in a separated locally convex space E and $\mathscr{K}E$ the family of all non-empty compact convex subsets of E. If f is a set-valued compact field in $C(\partial A, \bar{A}, o)$ with $d(A, o, f) \neq 0$, then for every $b \in \partial A$, there exists $\lambda > 0$ such that $\lambda b \in f(\partial A)$.*

Before we prove this theorem, the following lemmas will be needed.

(10.3) LEMMA. *Let A be an open convex symmetric neighborhood of the origin in a separated locally convex space F, $b \in \partial A$, $L = \{\lambda b\colon \lambda > 0\}$ and $\mathscr{K}E$ the family of all non-empty compact convex subsets of E. If f is a set-valued compact field in $C(\partial A, \bar{A}, o)$ with $f(\partial A) \cap L = \emptyset$, then f is homotopic to some finite dimensional set-valued compact field g in $C(\partial A, \bar{A}, o)$ with $g(\partial A) \cap L = \emptyset$.*

Proof. First of all, we claim that there exists a neighborhood V of $o \in E$ such that $f(\partial A) \cap (L + V) = \emptyset$. Suppose that it is false. Let N be the family of all convex symmetric neighborhoods W of $o \in E$ with $W \subset A$. Let N be indiced by $\{W_j\colon j \in J\}$. For each $j \in J$, there exist

$$\text{(a)} \qquad x_j \in \partial A, \quad t_j > 0, \quad z_j \in f(x_j) \cap (t_j b + W_j).$$

Define $j_1 \leqslant j_2$ in J if $W_{j_2} \subset W_{j_1}$. Then J becomes a directed set. Let $y_j = x_j - z_j$ for $j \in J$. Since $\{y_j\colon j \in J\}$ is a net in the relatively compact set $F(\bar{A})$, it has a subnet $\{y_{j(\delta)}\colon \delta \in D\}$ convergent to some $y \in E$. Let μ be the gauge of A. Since μ is continuous at y, there exists $\delta_1 \in D$ such that

$$\mu(y_{j(\delta)}) \leqslant \mu(y) + 1 \quad \text{for all} \quad \delta \geqslant \delta_1.$$

By (a),

$$\begin{aligned} t_{j(\delta)} &= \mu(t_{j(\delta)}b) \\ &\leqslant \mu(t_{j(\delta)}b - z_{j(\delta)}) + \mu(x_{j(\delta)}) + \mu(y_{j(\delta)}) \\ &\leqslant 1+1+\mu(y)+1 \quad \text{for all} \quad \delta \geqslant \delta_1. \end{aligned}$$

Thus, $\{t_{j(\delta)}\colon \delta\in D\}$ is eventually bouded. Without loss of generality, let $\{t_{j(\delta)}\colon \delta\in D\}$ converge to some t. By (a), $\{z_{j(\delta)}\colon \delta\in D\}$ converges to $tb\in E$. As a result $\{x_{j(\delta)}\colon \delta\in D\}$ converges to $y+tb$. Since ∂A is closed, we have $y+tb\in\partial A$. By (1.1), we have $tb\in f(y+tb)\subset f(\partial A)$ contrary to $f(\partial A)\cap L = \emptyset$. Hence there exists a convex symmetric neighborhood V of $o\in E$ such that $f(\partial A)\cap(L+V) = \emptyset$. By (4.1), f is homotopic to some finite dimensional set-valued compact field g in $C(\partial A, \bar{A}, o)$ with $g(y)\subset f(y)+V$ for all $y\in\bar{A}$. Clearly $g(\partial A)\cap L = \emptyset$ and g is the required set-valued compact field.

(10.4) Lemma. *Let A be an open convex symmetric neighborhood of the origin in a finite dimensional vector space E, $b\in\partial A$, $L = \{\lambda b\colon \lambda > 0\}$ and $\mathscr{K}E$ the family of all non-empty compact convex subsets of E. If f is a set-valued compact field in $C(\partial A, \bar{A}, o)$ such that $f(\partial A)\cap L = \emptyset$, then there exists a bounded open convex symmetric neighborhood B of $o\in E$ such that $b\in\partial B$, $B\subset A$, $o\notin f(\bar{A}\setminus B)$ and $f|_{\bar{B}}$ is homotopic to some non-zero constant map in $C(\partial B, \bar{B}, o)$.*

Proof. Since $F(\bar{A})$ is bounded, there exists $\lambda > 0$ such that $F(\bar{A})\subset\lambda A$. Let D be a bounded open convex symmetric neighborhood of $o\in E$ such that $F(\bar{A})\subset\frac{1}{4}D$ and $4(1+\lambda)b\in D$. Let α, δ be the gauges of A and D respectively and let $B = A\cap D$. Then B is a bounded open convex symmetric neighborhood of $o\in E$ such that $b\in\partial B$, $B\subset A$ and $o\notin f(\bar{A}\setminus B)$. We claim that $f(\partial B)\cap L = \emptyset$. Since $\partial B\subset\partial A\cup\partial D$, it suffices to show that for $x\in\partial B\cap\partial D$, $u\in F(x)$, $t > 0$, the relation $x-u = tb$ will give a contradiction. In fact,

$$t = \alpha(tb)\leqslant\alpha(x)+\alpha(u)\leqslant 1+\lambda,$$

and

$$1 = \delta(x)\leqslant\delta(u)+\delta(tb)\leqslant\tfrac{1}{4}+\frac{t}{4(1+\lambda)}\leqslant\tfrac{1}{2}.$$

Hence we have $f(\partial B)\cap L = \emptyset$. Define

$$h(x, t) = (1-t)f(x)-tb,$$

and $H(x, t) = x-h(x, t)$ for $(x, t)\in\bar{B}\times I$. Since H is upper semicontinuous on the compact set $\bar{B}\times I$, H is a compact map. Since $f(\partial B)\cap L = \emptyset$, h is a homotopy in $C(\partial B, \bar{B}, o)$. Now clearly h_1 is a non-zero constant map. This completes the proof.

Proof of (10.2). By (7.7) and (10.3), we may assume that f is finite dimensional. Let E_1 be a finite dimensional vector subspace containing $F(\bar{A})$ and b where F is the compact map corresponding to f. Let $A_1 = A \cap E_1$ and $\partial_1 A_1$ the boundary of A_1 in the vector subspace E_1. By (2.3), we have $\partial_1 A_1 = \partial A \cap E_1$ and $\bar{A}_1 = \bar{A} \cap E_1$. By (10.4), there exists a bounded open convex symmetric neighborhood B of $o \in E_1$ such that $B \subset A_1$, $0 \notin f(\bar{A}_1 \setminus B)$ and $f|_{\bar{B}}$ is homotopic to some non-zero constant map g in $C(\partial B, \bar{B}, o)$. By (7.5), we have

$$d(A, o, f) = d_1(A_1, o, f|_{\bar{A}_1}).$$

By (8.2),

$$d_1(A_1, o, f|_{\bar{A}_1}) = d_1(B, o, f|_{\bar{B}}).$$

By (7.7),

$$d_1(B, o, f|_{\bar{B}}) = d_1(B, o, g).$$

By (10.1),

$$d(B, o, g) = 0.$$

This completes the proof.

§ 11. Reduction formula.

(11.1) THEOREM. *Let A be an open subset of a separated locally convex space E, $p \in E$ and $\mathscr{K}E$ the family of all non-empty compact convex subsets of E. Let f be a set-valued compact field in $C(\partial A, \bar{A}, p)$ and F its corresponding compact map. If E_0 is a vector subspace containing both the closure of $F(\bar{A})$ and p, then $d(A, p, f) = d_0(A \cap E_0, p, f|_{\overline{A \cap E_0}})$ where $\overline{A \cap E_0}$ is the closure of $A \cap E_0$ in E_0 and the vector subspace E_0 need not be finite dimensional.*

Proof. By (3.2), let V be a convex symmetric neighborhood of $o \in E$ such that $(p+V) \cap f(\partial A) = \emptyset$. By (3.1), let E_1 be a finite dimensional vector subspace and $G: \bar{A} \to \mathscr{K}E$ a compact map such that the following conditions hold:

(a) $$p \in E_1 \subset E_0,$$

(b) $$G(\bar{A}) \subset E_1,$$

(c) $$G(y) \subset F(y) + V \quad \text{for all} \quad y \in \bar{A}.$$

Define

$$g(y) = y - G(y) \quad \text{for} \quad y \in \bar{A},$$

and

$$h(y, t) = (1-t) f(y) + t g(y) \quad \text{for} \quad (y, t) \in \bar{A} \times I.$$

Then h is a homotopy for $f \simeq g$ in $C(\partial A, \bar{A}, p)$ and $h|_{\overline{A \cap E_0} \times I}$ is a homotopy for $f|_{\overline{A \cap E_0}} \simeq g|_{\overline{A \cap E_0}}$ in $C(\partial_0(A \cap E_0), \overline{A \cap E_0}, p)$ with respect to the vector subspace E_0. Therefore, by (7.5), we have

$$d(A, p, f) = d_1(A \cap E_1, p, g|_{\overline{A \cap E_1}}) = d_0(A \cap E_0, p, f|_{\overline{A \cap E_0}}).$$

§ 12. Translation invariance and component dependence.

(12.1) THEOREM. *Let A be an open subset of E, $p \in E$, $\mathscr{K}E$ the family of all non-empty compact convex subsets of E, and f a set-valued compact field in $C(\partial A, \bar{A}, p)$. If $g(x) = f(x) - p$ for all $x \in \bar{A}$, then g is a set-valued compact field in $C(\partial A, \bar{A}, o)$ and we have*

$$d(A, p, f) = d(A, o, g).$$

Proof. Consider first the case when E is finite dimensional. Let F, G be the compact maps corresponding to f, g respectively. Let D be a bounded open set containing $p + F(\bar{A})$ and let $B = A \cap D$. Then we have $p \notin f(\bar{A} \setminus B)$ and $o \notin g(\bar{A} \setminus B)$. By (8.2), we have

$$d(A, p, f) = d(B, p, f|_{\bar{B}}),$$

and

$$d(A, o, g) = d(B, o, g|_{\bar{B}}).$$

By (5.2), let h be a homotopy in $C(\partial B, \bar{B}, p)$ such that $h_0 = f|_{\bar{B}}$ and h_1 is a single-valued compact field in $C(\partial B, \bar{B}, p)$. Define $h^*(x, t) = h(x, t) - p$ for $(x, t) \in I \times I$. Then h^* is a homotopy in $C(\partial B, \bar{B}, o)$ such that $h_0^* = g|_{\bar{B}}$ and h_1^* is a single-valued compact field in $C(\partial B, \bar{B}, o)$. By (7.7), we have

$$d(B, p, f|_{\bar{B}}) = d(B, p, h_0) = d(B, p, h_1),$$

and

$$d(B, o, g|_{\bar{B}}) = d(B, o, h_0^*) = d(B, o, h_1^*).$$

By (6.7),

$$d(B, p, h_1) = d(B, o, h_1^*).$$

Combining all equalities, the proof of the finite dimensional case is complete.

Now let E be a separated locally convex space. Let $h: \bar{A} \times I \to \mathscr{K}E$ be a homotopy in $C(\partial A, \bar{A}, p)$ such that $h_0 = f$ and h_1 is finite dimensional set-valued compact field in $C(\partial A, \bar{A}, p)$. Let E_1 be a finite dimensional vector subspace containing $H_1(\bar{A})$ and p where H_1 is the compact map corresponding to h_1. Define $h^*(x, t) = h(x, t) - p$ for $(x, t) \in \bar{A} \times I$. Then h^* is a homotopy in $C(\partial A, \bar{A}, o)$ such that $h_0^* = g$ and h_1^* is a finite dimensional set-valued compact field in $C(\partial A, \bar{A}, o)$. Hence we have

$$d(A, p, f) = d_1(A \cap E_1, p, h_1|_{\overline{A \cap E_1}}),$$

and

$$d(A, o, g) = d_1(A \cap E_1, o, h_1^*|_{\overline{A \cap E_1}}).$$

Since $h_1^*(x) = h_1(x) - p$ for $x \in \bar{A}$, by the first case, we have

$$d_1(A \cap E_1, p, h_1|_{\overline{A \cap E_1}}) = d_1(A \cap E_1, o, h_1^*|_{\overline{A \cap E_1}}).$$

Combining all equalities, the proof is complete.

(12.2) THEOREM. *Let A be an open subset of a separated locally convex space E, $\mathscr{K}E$ the family of all non-empty compact convex subsets of E and f: $\bar{A} \to \mathscr{K}E$ a set-valued compact field. If a, b belong to the same component of $E \setminus f(\partial A)$, then*

$$d(A, a, f) = d(A, b, f).$$

Proof. Since E is locally connected, there exists a continuous single-valued map s: $I \to E \setminus f(\partial A)$ such that $s(0) = a$, $s(1) = b$. Define $h(x, t) = f(x) - s(t)$ for $(x, t) \in \bar{A} \times I$. Then h is a homotopy in $C(\partial A, \bar{A}, o)$. By (12.1) and (7.7), we have

$$d(A, a, f) = d(A, o, h_0) = d(A, o, h_1) = d(A, b, f).$$

§ 13. Product of domains.

(13.1) THEOREM. *For $j = 1, 2$, let A_j be an open subset of a separated locally convex space E_j, $p_j \in E_j$, $\mathscr{K}E_j$ the family of all non-empty compact convex subsets of E_j and f_j a set-valued compact field in $C(\partial A_j, \bar{A}_j, p_j)$. If $E = E_1 \times E_2$, $A = A_1 \times A_2$, $p = (p_1, p_2)$ and $f(x) = f_1(x_1) \times f_2(x_2)$ for $x = (x_1, x_2) \in \bar{A}$, then f is a set-valued compact field in $C(\partial A, \bar{A}, p)$ and*

$$\text{(a)} \qquad d(A, p, f) = d_1(A_1, p_1, f_1)\, d_2(A_2, p_2, f_2).$$

Proof. Since $\partial A \subset [(\partial A_1) \times \bar{A}_2] \cup [\bar{A}_1 \times \partial A_2]$, we have $p \in E \setminus f(\partial A)$. Let F_j be the compact map corresponding to f_j. Define $F(x) = F_1(x_1) \times F_2(x_2)$ for $x = (x_1, x_2) \in \bar{A}$. Then F: $\bar{A} \to \mathscr{K}E$ is the compact map corresponding to f where $\mathscr{K}E$ is the family of all non-empty compact convex subsets of E. Hence f is a set-valued compact field in $C(\partial A, \bar{A}, p)$.

In order to prove (a), suppose that both E_1 and E_2 are finite dimensional. Let B_j be a bounded open subset of A_j such that $p_j \notin f_j(\bar{A}_j \setminus B_j)$. Then by (8.2), we have

$$d_j(A_j, p_j, f_j) = d_j(B_j, p_j, f_j|_{\bar{B}_j}) \quad \text{for} \quad j = 1, 2.$$

By (5.2), let h^j be a homotopy in $C(\partial B_j, \bar{B}_j, p_j)$ such that $h_0^j = f_j|_{\bar{B}_j}$ and h_1^j is single-valued compact field in $C(\partial B_j, \bar{B}_j, p_j)$. By (7.7), we have

$$d_j(B_j, p_j, f_j|_{\bar{B}_j}) = d_j(B_j, p_j, h_0^j) = d_j(B_j, p_j, h_1^j) \quad \text{for} \quad j = 1, 2.$$

Define $B = B_1 \times B_2$ and $h(x, t) = h^1(x_1, t) \times h^2(x_2, t)$ for $x = (x_1, x_2) \in B$, $t \in I$. Then $p \in E \setminus f(\bar{A} \setminus B)$ and by (8.2),

$$d(A, p, f) = d(B, p, f|_{\bar{B}}).$$

Also h is a homotopy in $C(\partial B, \bar{B}, p)$ and $h_0 = f|_{\bar{B}}$. By (7.7), we have

$$d(B, p, f|_{\bar{B}}) = d(B, p, h_0) = d(B, p, h_1).$$

Now h_1 is single-valued, B is bounded in E and $h_1(x) = (h_1^1(x_1), h_1^2(x_2))$ for $x = (x_1, x_2) \in \bar{B}$. By (6.8)

$$d(B, p, h_1) = d_1(B_1, p_1, h_1^1) d_2(B_2, p_2, h_1^2).$$

Combining all equalities, the proof of the finite dimensional case is complete.

Now let both E_1 and E_2 be separated locally convex spaces. Let h^j be a homotopy in $C(\partial A_j, \bar{A}_j, p_j)$ such that $h_0^j = f_j$ and h_1^j is a finite dimensional set-valued compact field in $C(\partial A_j, \bar{A}_j, p_j)$. Let E_j^* be a finite dimensional vector subspace of E_j with $H_1^j(\bar{A}_j) \subset E_j^*$ and $p_j \in E_j^*$ where H_1^j is the compact map corresponding to h_1^j. Define $h(x, t) = h^1(x_1, t) \times \times h^2(x_2, t)$ for $x = (x_1, x_2) \in \bar{A}$ and $t \in I$. Then h is a homotopy in $C(\partial A, \bar{A}, p)$ such that $h_0 = f$ and h_1 is a finite dimensional set-valued compact field in $C(\partial A, \bar{A}, p)$. In fact, $E^* = E_1^* \times E_2^*$ is a finite dimensional vector subspace of E with $p \in E^*$ and $H_1(\bar{A}) \subset E^*$ where H_1 is the compact map corresponding to h_1. By (7.5), we have

$$d(A, p, f) = d^*(A \cap E^*, p, h_1|_{\overline{A \cap E^*}}),$$

and

$$d_j(A_j, p, f_j) = d_j^*(A_j \cap E_j^*, p_j, h_1^j|_{\overline{A_j \cap E_j^*}}).$$

Applying the result of the first case, we have

$$d^*(A \cap E^*, p, h_1|_{\overline{A \cap E^*}}) = d_1^*(A_1 \cap E_1^*, p_1, h_1^1|_{\overline{A_1 \cap E_1^*}}) d_2^*(A_2 \cap E_2^*, p_2, h_1^2|_{\overline{A_2 \cap E_2^*}}).$$

Combining all the equalities, the proof is complete.

§ 14. Generalized Hopf theorem for metrizable locally convex spaces.

(14.1) THEOREM. *Let A be an open convex symmetric neighborhood of a metrizable locally convex space E such that for every finite dimensional vector subspace E_1, the set $A \cap E_1$ is bounded in E_1. If f, g are two set-valued compact fields in $C(\partial A, \bar{A}, o)$ with $d(A, o, f) = d(A, o, g)$, then f, g are homotopic in $C(\partial A, \bar{A}, o)$.*

When A is the unit ball in a Banach space and f, g are single-valued, see Krasnoselskii [21]. Before we prove (14.1), the following lemmas will be needed.

(14.2) LEMMA. *Let $X \subset Y$ be two closed subsets of a metrizable locally convex space E, $p \in E$ and $\mathscr{K}E$ the family of all non-empty compact convex subsets of E. Let f, g be two set-valued compact fields in $C(X, Y, p)$ and let E_0 be a complete vector subspace containing $F(Y) \cup G(Y)$ and p, where F, G are compact maps corresponding to f, g respectively. If $f|_{Y \cap E_0}$ and $g|_{Y \cap E_0}$ are homotopic in $C(X \cap E_0, Y \cap E_0, p)$, then f, g are homotopic in $C(X, Y, p)$.*

Proof. Let h^* be a homotopy for $f|_{Y \cap E_0} \simeq g|_{Y \cap E_0}$ in $C(X \cap E_0, Y \cap E_0, p)$

and let $H^*(y,t) = y - h^*(y,t)$ for $(y,t) \in (Y \cap E_0) \times I$. Then $H^*[(Y \cap E_0) \times I]$ is a relatively compact subset of E_0. Let

$$T = Y \times \{0\} \cup Y \times \{1\} \cup (Y \cap E_0) \times I.$$

Define $H_0^*: T \to \mathscr{K}E$ by

$$H_0^*(y,t) = \begin{cases} F(y) & \text{for} \quad y \in Y,\ t = 0, \\ H^*(y,t) & \text{for} \quad y \in Y \cap E_0,\ t \in I, \\ G(y) & \text{for} \quad y \in Y,\ t = 1. \end{cases}$$

By (1.2), H_0^* is a compact map and $H_0^*(T) \subset E_0$. By (2.1), there exists an upper semicontinuous set-valued map H: $Y \times I \to \mathscr{K}E$ such that $H|_T = H_0^*$ and $H(Y \times I) \subset \operatorname{co} H_0^*(T)$. Now the convex hull of the precompact set $H_0^*(T)$ is precompact in the complete space E_0. Hence $\operatorname{co} H_0^*(T)$ is relatively compact in E_0 and H is a compact map. Define $h(y,t) = y - H(y,t)$ for $(y,t) \in Y \times I$. Then for $(x,t) \in (X \cap E_0) \times I$, we have

$$p \notin x - H^*(x,t) = x - H(x,t) = h(x,t).$$

Also for $(x,t) \in (X \setminus E_0) \times I$, we have $p - H(x,t) \subset E_0$, $x \notin E_0$ and hence $x \notin h(x,t)$. Clearly we have $h_0 = f$ and $h_1 = g$. This proves that h is a homotopy for $f \simeq g$ in $C(X, Y, p)$.

(14.3) Lemma. *Let $X \subset Y$ be two closed subsets of a metrizable locally convex space E, $p \in E$ and $\mathscr{K}E$ the family of all non-empty compact convex subsets of E. Then every set-valued compact field f in $C(X, Y, p)$ is homotopic to a finite dimensional single-valued compact field in $C(X, Y, p)$.*

Proof. By (4.1), we may assume that f is finite dimensional. Let E_1 be a finite dimensional vector subspace containing $F(Y)$ and p where F is the compact map corresponding to f. By (5.2), let h^* be a homotopy in $C(X \cap E_1, Y \cap E_1, p)$ such that $h_0^* = f|_{Y \cap E_1}$, h_1^* is a single-valued compact field in $C(X \cap E_1, Y \cap E_1, p)$ and $H^*\big((Y \cap E_1) \times I\big) \subset E_1$. By (2.1), there exists a continuous single-valued map G: $Y \to E$ such that $G|_{Y \cap E_1} = H_1^*$ and $G(Y) \subset \operatorname{co} H_1^*(Y \cap E_1)$. Since E_1 is finite dimensional, G is a compact map. Define $g(y) = y - G(y)$ for $y \in Y$. Clearly g is a single-valued compact field in $C(X, Y, p)$. By (14.2), we have $f \simeq g$ in $C(X, Y, p)$.

Proof of (14.1). By (14.3), let f^*, g^* be finite dimensional single-valued compact fields homotopic to f, g respectively in $C(\partial A, \bar{A}, o)$. Let E_1 be a finite dimensional vector subspace containing $F^*(\bar{A}) \cup G^*(\bar{A})$ where F^*, G^* are compact maps corresponding to f^*, g^* respectively. Then we have

$$\begin{aligned} d(A \cap E_1, o, f^*|_{\overline{A \cap E_1}}) &= d(A, o, f) = d(A, o, g) \\ &= d(A \cap E_1, o, g^*|_{\overline{A \cap E_1}}). \end{aligned}$$

Since $A\cap E_1$ is a bounded open convex symmetric neighborhood of $o\in E_1$, by (6.6), we have

$$f^*|_{\overline{A\cap E_1}} \simeq g^*|_{\overline{A\cap E_1}}$$

in $C(\partial A\cap E_1, \bar{A}\cap E_1, o)$. By (14.2), we have $f^*\simeq g^*$ in $C(\partial A, \bar{A}, o)$. Consequently $f\simeq g$ in $C(\partial A, \bar{A}, o)$.

§ 15. Product theorem for composite maps. The object of this section is to prove (15.1). The composite of two set-valued compact fields may not take non-empty compact convex sets as values. That is why we shall assume that one of the compact fields in (15.1) is single-valued. The homotopy invariance theorem allows us to change the image of a given set-valued compact field but the product theorem for composite maps allows us to change the domain of a given set-valued compact field.

(15.1) THEOREM. *Let A, B be open subsets in a separated locally convex space E and $\mathscr{K}E$ the family of all non-empty compact convex subsets of E. Let $g\colon \bar{B}\to\bar{A}$ be a single-valued compact field and $f\colon \bar{A}\to\mathscr{K}E$ a set-valued compact field. Clearly the composite $f\circ g\colon \bar{B}\to\mathscr{K}E$ is a set-valued compact field on $\bar{B}$. If $\{A_j\colon j\in J\}$ is the family of all components of $A\setminus g(\partial B)$, $a_j\in A_j$ for each $j\in J$ and $p\in E\setminus[f\circ g(\partial B)\cup f(\partial A)]$, then we have*

$$\text{(a)}\qquad d(B, p, f\circ g) = \sum_{j\in J} d(A_j, p, f|_{\bar{A}_j})\,d(B, a_j, g).$$

Note that the right-hand side of this equality is a finite sum independent of the choice of $\{a_j\in A_j\colon j\in J\}$.

Before we prove (15.1), we shall need the following lemmas. In (15.3), we prove a particular case of (15.1), in which we have a finite dimensional vector space E and a stronger condition $g(\bar{B})\subset A$.

(15.2) LEMMA. *Let $A = \bigcup_{j\in J} A_j$ be a union of disjoint open sets in a separated locally convex space E, $p\in E$, $\mathscr{K}E$ the family of all non-empty compact convex subsets of E and f a set-valued compact field in $C(\partial A, \bar{A}, p)$. Then $p\in E\setminus f(A_j)$ for all except only a finite number of indices $j\in J$.*

Proof. Suppose that it is not the case. There exists a sequence of *distinct* indices $\{j_n\in J\colon n\geqslant 1\}$ such that $p\in f(A_{j_n})$. Let $x_n\in A_{j_n}$ such that $p\in f(x_n)$. Then $\{x_n - p\colon n\geqslant 1\}$ is a sequence in the relatively compact set $F(\bar{A})$ where F is the compact map corresponding to f. Let $\{y_\delta\colon \delta\in D\}$ be a subnet of $\{x_n - p\colon n\geqslant 1\}$ and $\{y_\delta\colon \delta\in D\}$ converges to some point $y\in E$. Then the corresponding subnet $\{x_\delta\colon \delta\in D\}$ converges to the point $x = y+p\in\bar{A}$. By (1.1), we have $p\in f(x)$. Since $p\notin f(\partial A)$, there exists $j_0\in J$ such that $x\in A_{j_0}$. There exist distinct indices $j_m, j_n\in J$ such that $x_m, x_n\in A_{j_0}$. This contradicts to the fact that $\{A_j\colon j\in J\}$ is a family of disjoint sets. The proof is complete.

(15.3) LEMMA. *Let A, B be open subsets in a finite dimensional vector space E and $\mathscr{K}E$ the family of all non-empty compact convex subsets of E. Let $g\colon \bar{B} \to A$ be a single-valued compact field and $f\colon \bar{A} \to \mathscr{K}E$ a set-valued compact field. If $\{A_j : j \in J\}$ is the family of all components of $A \setminus g(\partial B)$, $a_j \in A_j$ for each $j \in J$ and $p \in E \setminus [f \circ g(\partial B) \cup f(\partial A)]$, then we have*

$$\text{(a)} \qquad d(B, p, f\circ g) = \sum_{j\in J} d(A_j, p, f|_{\bar{A}_j}) d(B, a_j, g).$$

Note that the right-hand side of this equality is a finite sum independent of the choice of $\{a_j \in A_j : j \in J\}$.

Proof. By (15.2), there exists a finite subset J^* of J such that $p \notin f(A_j)$ for $j \in J \setminus J^*$. By (10.1), $d(A_j, p, f|_{\bar{A}_j}) = 0$ for all $j \in J \setminus J^*$ Hence the right hand side of (15.3) (a) is only a finite sum. By (12.2), it is independent of the choice of $\{a_j \in A_j : j \in J\}$.

Consider first the case when A, B are *bounded* open subsets of E. The set $X = \bar{A} \setminus \bigcup_{j\in J^*} A_j$ is closed and $p \notin f(X)$. By (5.2), there exists an upper semicontinuous set-valued map $h\colon \bar{A}\times I \to \mathscr{K}E$ such that $h_0 = f$, h_1 is single-valued and $p \notin h(X \times I)$. Then $h|_{\bar{A}_j\times I}$ is a homotopy for $f|_{\bar{A}_j}$ to $h_1|_{\bar{A}_j}$ such that $p \notin h(\partial A_j \times I)$. By (7.7), we have

$$d(A_j, p, f|_{\bar{A}_j}) = d(A_j, p, h_1|_{\bar{A}_j}).$$

Define $k(b, t) = h(g(b), t)$ for $(b, t) \in \bar{B}\times I$. Since $g(\partial B) \subset X$, by (1.3), $k\colon \bar{B}\times I \to \mathscr{K}E$ is a homotopy from $f\circ g$ to $h_1 \circ g$ such that $p \notin k(\partial B \times I)$. By (7.7), we have

$$d(B, p, f\circ g) = d(B, p, h_1 \circ g).$$

Note that for $j \in J \setminus J^*$, we have $p \notin h_1(\bar{A}_j)$ and hence $d(A_j, p, h_1|_{\bar{A}_j}) = 0$. By (6.10), we have

$$\begin{aligned} d(B, p, f\circ g) &= d(B, p, h_1\circ g) \\ &= \sum_{j\in J} d(A_j, p, h_1|_{\bar{A}_j}) d(B, a_j, g) \\ &= \sum_{j\in J} d(A_j, p; f|_{\bar{A}_j}) d(B, a_j, g). \end{aligned}$$

Now let A, B be any open subsets of E. Let F, G be the compact maps corresponding to f, g respectively. Let

$$D_k = \{a \in A \setminus g(\partial B) : d(B, a, g) = k\} \quad \text{for} \quad k = \pm 1, \pm 2, \ldots$$

By (12.2), D_k is the union of some components of $A \setminus g(\partial B)$. Furthermore by (8.1), we have

$$\sum_{j\in J} d(A_j, p, f|_{\bar{A}_j}) d(B, a_j, g) = \sum_{k=\pm 1, \pm 2, \ldots} d(D_k, p, f|_{\bar{D}_k}) k.$$

Let $Y = \{b \in B: p \in f \circ g(b)\}$. Then Y is a compact subset of B. Let B^* be a bounded open subset of B with $Y \subset B^*$. Clearly $p \notin f \circ g(\bar{B} \setminus B^*)$ and by (8.2), we get

$$d(B, p, f \circ g) = d(B^*, p, f \circ g|_{\overline{B^*}}).$$

Let $X = \{a \in A: p \in f(a)\}$. Then $g(\overline{B^*}) \cup X$ is a compact subset of A. Let A^* be a bounded open subset of A with $g(\overline{B^*}) \cup X \subset A^*$ Clearly $p \notin f(\bar{A} \setminus A^*)$ and $X \subset A^* \setminus g(\bar{B} \setminus B^*)$. Let

$$D_k^* = \{a \in A^* \setminus g(\partial B^*): d(B^*, a, g|_{\overline{B^*}}) = k\} \quad \text{for} \quad k = \pm 1, \pm 2, \ldots$$

By (12.2), D_k^* is the union of some components of $A^* \setminus g(\partial B^*)$. By the first case, we have

$$d(B^*, p, f \circ g|_{\overline{B^*}}) = \sum_{k = \pm 1, \pm 2, \ldots} d(D_k^*, p, f|_{\overline{D_k^*}}) k.$$

Now if $a \in X$, then $a \in A^* \setminus g(\bar{B} \setminus B^*)$ and thus

$$d(B, a, g) = d(B^*, a, g|_{\overline{B^*}}).$$

Hence $X \cap D_k = X \cap D_k^*$. By (8.2), we have

$$\begin{aligned} d(D_k, p, f|_{\bar{D}_k}) &= d(D_k \cap D_k^*, p, f|_{\overline{D_k \cap D_k^*}}) \\ &= d(D_k^*, p, f|_{\overline{D_k^*}}). \end{aligned}$$

The proof is complete when we combine all the equalities.

Proof of (15.1). Let $Y = \{a \in \bar{A}: p \in f(a)\}$. Since $p \in E \setminus [f \circ g(\partial B) \cup \cup f(\partial A)]$, the compact set Y is disjoint from the closed set $A \cup g(\partial B)$. There exists an open convex symmetric neighborhood U of $o \in E$ such that $Y + 3U \subset A \setminus g(\partial B)$. The set $B^* = \{b \in B: g(b) \in Y + U\}$ is an open subset of B. Suppose that for some $b \in \bar{B}$, we have $p \in f \circ g(b)$. Then $g(b) \in Y$ and $b \notin \bar{B} \setminus B^*$ Thus $p \in E \setminus [f(\partial A) \cup f \circ g(\bar{B} \setminus B^*)]$ and by (8.2),

$$d(B, p, f \circ g) = d(B^*, p, f \circ g|_{\overline{B^*}}).$$

Also we have

$$g(B^*) + U \subset \overline{Y + U} + U \subset Y + 3U \subset A.$$

Let $D_k = \{a \in A \setminus g(\partial B): d(B, a, g) = k\}$ for $k = \pm 1, \pm 2, \ldots$ By (12.2), D_k is the union of some components of $A \setminus g(\partial B)$. By (8.1), we have

$$\sum_{j \in J} d(A_j, p, f|_{\bar{A}_j}) d(B, a_j, g) = \sum_{k = \pm 1, \pm 2, \ldots} d(D_k, p, f|_{\bar{D}_k}) k.$$

Since p does bot belong to the closed set $f(\partial A) \cup f \circ g(\bar{B} \setminus B^*)$, there exists an open convex symmetric neighborhood of $o \in E$ such that $(p + V) \cap \cap [f(A) \cup f \circ g(\bar{B} \setminus B^*)] = \emptyset$. Let Q be the closure of $F(\bar{A})$ where F is the compact map corresponding to f. Let ξ be a continuous single-valued map defined on the compact set Q into some finite dimensional vector subspace E_1 of E such that $\xi(z)-z\in V$ for all $z\in Q$. We may assume $p\in E_1$. For each $a\in\bar{A}$, define $f^*(a) = a-\overline{\text{co}}\,\xi\circ F(a)$. Observe that $\partial D_k\subset\partial A\cup\cup g(\partial B)$. Hence $(p+V)\cap f(\partial D_k)=\emptyset$ and by (7.8) (a), we have

$$d(D_k, p, f|_{\bar{D}_k}) = d(D_k, p, f^*|_{\bar{D}_k}).$$

Since $p\notin f^*\circ g(\bar{B}\setminus B^*)$, the compact set $X = \{a\in\bar{A}: p\in f^*(a)\}$ is disjoint from the closed set $g(\bar{B}\setminus B^*)$. There exists an open convex symmetric neighborhood W of $o\in E$ such that $(X+W)\cap g(\bar{B}\setminus B^*)=\emptyset$ and $W\subset U$. Let G be the compact map corresponding to g and Q^* the closure of $G(\bar{B})$. Let ϱ be a continuous single-valued map defined on the compact set Q^* into some finite dimensional vector subspace E_2 such that $\varrho(z)-z\in W$ for all $z\in Q^*$ We may assume $E_1\subset E_2$. Define $g^*(b) = b-\varrho\circ G(b)$ for $b\in\bar{B}$. By (12.2), the set

$$D_k^* = \{a\in A\setminus g^*(\partial B^*):\ d(B^*, a, g^*|_{\overline{B^*}}) = k\} \quad \text{for} \quad k=\pm 1, \pm 2, \ldots$$

is the union of some components of $A\setminus g^*(\partial B^*)$. For each $x\in X$, we have

$$d(B, x, g) = d(B^*, x, g|_{\overline{B^*}}) = d(B^*, x, g^*|_{\overline{B^*}}).$$

In other words,

$$X\cap D_k = X\cap D_k^*,$$

i.e.,

$$p\notin f^*(\bar{D}_k\setminus D_k^*)\cup f^*(\overline{D_k^*}\setminus D_k).$$

By (8.2),

$$d(D_k, p, f^*|_{\bar{D}_k}) = d(D_k\cap D_k^*, p, f^*|_{\overline{D_k\cap D_k^*}}) = d(D_k^*, p, f^*|_{\overline{D_k^*}}).$$

Define $h(b,t) = b-(1-t)G(b)-t\varrho\circ G(b)$ for $(b,t)\in\overline{B^*}\times I$. Then $h(b,t)\subset g(b)+W\subset g(\overline{B^*})+U\subset A$. Therefore $f^*\circ H$ is a well-defined composite map. Since

$$X\cap h(\partial B^*\times I)\subset X\cap[g(\partial B^*)+W] \subset (X+W)\cap g(\bar{B}\setminus B^*)=\emptyset,$$

we have $p\notin f^*\circ h(\partial B^*\times I)$. Consequently $f^*\circ h$ is a homotopy for $f^*\circ g\simeq f^*\circ g^*$ in $C(\partial\overline{B^*}, B^*, p)$. Hence

$$d(B^*, p, f^*\circ g|_{\overline{B^*}}) = d(B^*, p, f^*\circ g^*|_{\overline{B^*}}).$$

The fact $(p+V)\cap f\circ g(\partial B^*)=\emptyset$ shows

$$d(B^*, p, f\circ g|_{\overline{B^*}}) = d(B^*, p, f^*\circ g|_{\overline{B^*}}).$$

By (7.5), we have

$$d(B^*, p, f^*\circ g^*|_{\overline{B^*}}) = d_2(B^*\cap E_2, p, f^*\circ g^*|_{\overline{B^*\cap E_2}}),$$

$$d(D_k^*, p, f^*|_{\overline{D_k^*}}) = d_2(D_k^*\cap E_2, p, f^*|_{\overline{D_k^*\cap E_2}}),$$

$$d(B^*, a, g^*|_{\overline{B^*}}) = d_2(B^*\cap E_2, a, g^*|_{\overline{B^*\cap E_2}}) \quad \text{for} \quad a\in D_k^*\cap E_2.$$

Observe that

$$D_k^*\cap E_2 = \{a\in A\cap E_2\setminus g^*[\partial_2(B^*\cap E_2)]:\ d_2(B^*\cap E_2, a, g^*|_{\overline{B^*\cap E_2}}) = k\}.$$

By (15.3), we have

$$d_2(B^*\cap E_2, p, f^*\circ g^*|_{\overline{B^*\cap E_2}}) = \sum_{k=\pm 1, \pm 2, \dots} d_2(D_k^*\cap E_2, p, f^*|_{\overline{D_k^*\cap E_2}})\,k.$$

The proof is complete when we combine all equalities as follows:

$$\begin{aligned} d(B, p, f\circ g) &= d(B^*, p, f\circ g|_{\overline{B^*}}) \\ &= d(B^*, p, f^*\circ g|_{\overline{B^*}}) \\ &= d(B^*, p, f^*\circ g^*|_{\overline{B^*}}) \\ &= d_2(B^*\cap E_2, p, f^*\circ g^*|_{\overline{B^*\cap E_2}}) \\ &= \sum_k d_2(D_k^*\cap E_2, p, f^*|_{\overline{D_k^*\cap E_2}})\,k \\ &= \sum_k d(D_k^*, p, f^*|_{\overline{D_k^*}})\,k \\ &= \sum_k d(D_k, p, f^*|_{\overline{D_k}})\,k \\ &= \sum_k d(D_k, p, f|_{\overline{D_k}})\,k \\ &= \sum_{j\in J} d(A_j, p, f|_{\overline{A_j}})\,d(B, a_j, g). \end{aligned}$$

CHAPTER III

EXTENSION OF SOME CLASSICAL RESULTS TO SET-VALUED MAPS

§ 16. Fixed point theorems and fixed point indices.

(16.1) THEOREM. *Let A be an open neighborhood, not necessarily convex, of the origin in a separated locally convex space E, $\mathscr{K}E$ the family of all non-empty compact convex subsets of E and for each $x \in \partial A$, $L_x = \{ax: a > 1\}$ the open half-ray at x. If $F: \bar{A} \to \mathscr{K}E$ is a set-valued compact map such that $L_x \cap F(x) = \emptyset$ for all $x \in \partial A$, then F has at least one fixed point $a \in \bar{A}$, i.e., $a \in F(a)$.*

Proof. Without loss of generality, we may assume $x \notin F(x)$ for all $x \in \partial A$. Define $f(x) = x - F(x)$ for all $x \in \bar{A}$. Then f is a set-valued compact field in $C(\partial A, \bar{A}, o)$. Since $L_x \cap F(x) = \emptyset$ for all $x \in \partial A$, by (9.2), we have $d(A, o, f) = 1$. By (10.1), there exists $a \in A$ such that $o \in f(a)$, i.e., $a \in F(a)$. This completes the proof.

(16.2) COROLLARY. *Let Y be the closed unit ball in a normed space E and $F: Y \to E$ a single-valued compact map. Then each of the following conditions is su,ficient for F to have a fixed point in Y.*

Inward map: For each $x \in \partial Y$, there exists $\lambda > 0$ such that

$$(1-\lambda)x + \lambda F(x) \in Y,$$

Rothé:

$$F(\partial Y) \subset Y,$$

Altman:

$$\|F(x)\|^2 - \|x\|^2 \leqslant \|x - F(x)\|^2 \quad \text{for} \quad x \in \partial Y.$$

(16.3) THEOREM. *Let A be an open convex symmetric neighborhood of the origin in a separated locally convex space E and $\mathscr{K}E$ the family of all non-empty compact convex subsets of E. If $F: Y \to \mathscr{K}E$ is a set-valued compact map such that no point $x \in \partial A$ belongs to the convex bull of $F(x) \cup [-F(-x)]$, then F has at least one fixed point in $\bar{A}$.*

Proof. Without loss of generality, we may assume $x \notin F(x)$ for all $x \in \partial A$. Define $f(x) = x - F(x)$ for $x \in \bar{A}$. Then f is a set-valued compact field in $C(\partial A, \bar{A}, o)$. Clearly from the given condition of F, we have $f(x) \cap \lambda f(-x) = \emptyset$ for $x \in \partial A$ and $\lambda > 0$. By (9.3), $d(A, o, f)$ is an odd integer. By (10.1) there exists $a \in A$ such that $o \in f(a)$, i.e., $a \in F(a)$.

(16.4) *Fixed point indices.* Let A be an open set in a separated locally convex space E, $p \in E$ and f a set-valued compact field in $C(\partial A, \bar{A}, p)$. A point $a \in A$ is called an *isolated p-point* of f if there exists a neighborhood V of a such that a is the only point x in $V \cap A$ with $p \in f(x)$. In this case, by (8.2), the integer $d(V \cap A, p, f|_{\overline{V \cap A}})$ is independent of the choice of V and this integer is called the *fixed point index* $i(a)$ of f at a. Suppose that the set $M = \{x \in A\colon\ p \in f(x)\}$ contains only isolated p-points of f. Let Q be the closure of $F(\bar{A})$ where F is the compact map corresponding to f. Then M is a discrete subset of the compact set $p+Q$ and hence M is finite. By (8.1), we have

$$d(A, p, f) = \sum \{i(a)\colon\ a \in M\}$$

when we choose pairwise disjoint neighborhoods V of $a \in M$. If E is finite dimensional, it can be proved that every set-valued compact field in $C(\partial A, \bar{A}, p)$ is homotopic to a single-valued compact field g in $C(\partial A, \bar{A}, p)$ such that $\{x \in A\colon\ p \in g(x)\}$ consists of only isolated p-points. When f is a single-valued compact field in Banach space, see Krasnoselskii [21].

§ 17. Extension of Borsuk's sweeping theorem.

(17.1) THEOREM. *Let Y be a closed subset of separated locally convex space E, $\mathcal{K}E$ the family of all non-empty compact convex subsets of E and h: $Y \times I \to \mathcal{K}E$ an I-compact field such that h_0 is the single-valued identity map on Y. If a, b belong to different components of $E \setminus Y$ but the same component of $E \setminus h(Y, 1)$, then $a \in h(Y \times I)$ or $b \in h(Y \times I)$.*

Proof. Let A be the component of $E \setminus Y$ with $a \in A$. Define $\varphi_a(x) = x - a$ and $\varphi_b(x) = x - b$ for $x \in \bar{A}$. Since E is locally connected, there exists a continuous single-valued map s: $I \to E \setminus h(Y, 1)$ such that $s(0) = a$, $s(1) = b$. Define for each $(x, t) \in \bar{A} \times I$,

$$w_t(x) = h(x, 1) - s(t),$$

$$\bar{w}_t(x) = h(x, t) - a,$$

$$\bar{\bar{w}}_t(x) = h(x, t) - b.$$

Suppose that none of a, b belongs to $h(Y \times I)$. Then we have

$$\varphi_a = \bar{w}_0 \simeq \bar{w}_1 = w_0 \simeq w_1 = \bar{\bar{w}}_1 \simeq \bar{\bar{w}}_0 = \varphi_b \quad \text{in} \quad C(\partial A, \bar{A}, o).$$

By (7.7),

$$d(A, o, \varphi_a) = d(A, o, \varphi_b).$$

By (9.1) and (12.1),

$$d(A, o, \varphi_a) = d(A, a, 1_{\bar{A}}) = 1.$$

Since $b \notin A$, by (10.1),

$$d(A, o, \varphi_b) = 0.$$

This contradiction completes the proof.

When h is a single-valued map in Banach space, see Granas [14].

§ 18. Extension of Borsuk-Ulam's theorem.

(18.1) THEOREM. *Let A be an open convex symmetric neighborhood of the origin in a separated locally convex space E, $\mathscr{K}E$ the family of all non-empty compact convex subsets of E and $f\colon \bar{A} \to \mathscr{K}E$ a set-valued compact field. If there exists a vector subspace M of E with $f(\partial A) \subset M$ and $A \not\subset M$, then there exists $x \in \partial A$ such that $f(x) \cap f(-x) \neq \emptyset$.*

Proof. Suppose that for each $x \in \partial A$, we have $f(x) \cap f(-x) = \emptyset$. Define $g(x) = \frac{1}{2}[f(x) - f(-x)]$ for $x \in \bar{A}$. Then g is a set-valued antipodal compact field in $C(\partial A, \bar{A}, o)$. By (9.3), $d(A, o, g)$ is an odd integer. Let $b \in \partial A \setminus M$ and $L = \{\lambda b\colon \lambda > 0\}$. Then we have $g(\partial A) \cap L \subset M \cap L = \emptyset$. This contradiction to (10.2) completes the proof.

When f is a single-valued compact field defined on the unit ball of a Banach space, see Granas [14].

§ 19. Extension of Brouwer's invariance of domains.

(19.1) LEMMA. *Let Y be a subset of a separated locally convex space E and $\mathscr{K}E$ the family of all non-empty compact convex subsets of E. Then $p \in Y$ is an interior point of Y iff there exist an open convex symmetric neighborhood A of $o \in E$ and a set-valued compact field f in $C(\partial A, \bar{A}, p)$ such that $f(\bar{A}) \subset Y$ and $d(A, p, f) \neq 0$.*

Proof. If p is an interior point of Y, there exists an open convex symmetric neighborhood A of $o \in E$ such that $p + \bar{A} \subset Y$. Define $f(x) = x + p$ for $x \in \bar{A}$. Then clearly f is a compact field in $C(\partial A, \bar{A}, p)$ and $f(\bar{A}) \subset Y$. By (12.1) and (9.1), we have

$$d(A, p, f) = d(A, o, 1_{\bar{A}}) = 1.$$

Conversely suppose that A is an open convex symmetric neighborhood of $o \in E$ and f a set-valued compact field in $C(\partial A, \bar{A}, p)$ such that $d(A, p, f) \neq 0$ and $f(\bar{A}) \subset Y$. Let V be a convex neighborhood of $o \in E$ such that $(p + V) \cap f(\partial A) = \emptyset$. Let $g(x) = f(x) - v$ for all $x \in \bar{A}$ and $v \in V$. Then by (7.8) (a), we have

$$d(A, p, g) = d(A, p, f) \neq 0.$$

By (10.1) we obtain $p \in g(A)$, i.e., $p + v \in f(A) \subset Y$. Since v is an arbitrary point of V, we have $p + V \subset Y$ and p is an interior point of Y.

For acyclic set-valued maps in finite dimensional vector-spaces, see Granas and Jaworowski [15].

Let B be an open subset of a separated locally convex space E and $\mathscr{K}E$ the family of all non-empty compact convex subsets of E. Parallel to Altman [3], a set-valued map $g\colon B \to \mathscr{K}E$ is called a *local boundary map* if for each $b \in B$, there exists an open convex symmetric neighborhood A of $o \in E$ such that $b + \bar{A} \subset B$ and for every $x_1, x_2 \in b + \bar{A}$ with $g(x_1) \cap g(x_2) \neq \emptyset$, we have $x_1 - x_2 \notin \partial A$. Note that every single-valued injection is a local boundary map.

(19.2). THEOREM. *Let B an open set in a separated locally convex space E, $\mathscr{K}E$ the family of all non-empty compact convex subsets of E and $g\colon B \to \mathscr{K}E$ a set-valued compact field on B. If g is a local boundary map then $g(B)$ is open in E.*

Proof. Let $b \in B$ and let A be chosen in the definition of local boundary map. Take $p \in g(b)$. Define

$$f(y) = g(b+y) \quad \text{for} \quad y \in \bar{A},$$

and

$$h(y, t) = f\left(\frac{y}{1+t}\right) - f\left(\frac{-ty}{1+t}\right) \quad \text{for} \quad y \in \bar{A},\ t \in I.$$

Since f is a local boundary map, f is a set-valued compact field in $C(\partial A, \bar{A}, p)$ and h is a homotopy in $C(\partial A, \bar{A}, o)$. By (7.8) (a) and (9.3), $d(A, p, f) = d(A, o, h_0) = d(A, o, h_1) \neq 0$. By (19.1), p is an interior point of $g(B)$. This completes the proof.

Let B be an open subset of a separated locally convex space E and $\mathscr{K}E$ the family of all non-empty compact convex subsets of E. A set-valued map $g\colon B \to \mathscr{K}E$ is called a *locally non-opposite map* if for each $b \in B$ there exists an open convex symmetric neighborhood A of $o \in E$ such that $b + \bar{A} \subset B$ and for all $\lambda \geqslant 0$ and $x \in b + \partial A$, we have $\lambda(b-x) \notin g(x) - g(b)$. The following is a variant of our (19.2).

(19.3) THEOREM. *Let B be an open set in a separated locally convex space E, $\mathscr{K}E$ the family of all non-empty compact convex subsets of E and $g\colon B \to \mathscr{K}E$ a set-valued compact field. If g is a locally non-opposite map then $g(B)$ is open in E.*

Proof. Let $b \in B$ and $p \in g(b)$. Let A be chosen in the definition of locally non-opposite map. Define $f(x) = g(x+b)$ and $f^*(x) = f(x) - p$ for $x \in \bar{A}$. Then f is a set-valued compact field in $C(\partial A, \bar{A}, p)$. Also f^* is a set-valued compact field in $C(\partial A, \bar{A}, o)$ with $-\lambda x \notin f^*(x)$ for all $\lambda > 0$ and $x \in \partial A$. By (9.2), we have

$$d(A, p, f) = d(A, o, f^*) = 1 \neq 0.$$

By (19.1), p is an interior point of $g(B)$. This completes the proof.

(19.4) COROLLARY. *Let $\mathscr{K}E$ be the family of all non-empty compact convex subsets of a locally convex space E and $f\colon E \to \mathscr{K}E$ a set-valued compact field. If f is a local boundary map or a locally non-opposite map, then f is surjective, i.e., $f(E) = E$.*

Proof. By (3.2), (19.2) and (19.3), the set $f(E)$ is both open and closed in the connected space E. Hence we have $f(E) = E$.

REFERENCES

[1] M. Altman, *A fixed point theorem in Banach spaces*, Bull. Acad. Polon. Sci. 2 (1957), pp. 89–92.

[2] — *An extension to locally convex spaces of Borsuk's Theorem on antipodes*, Bull. Acad. Polon. Sci. 6 (1958), pp. 293–295.

[3] — *Continuous transformations of open sets in locally convex spaces*, Bull. Acad. Polon. Sci. 6 (1958), pp. 297–301.

[4] C. Berge, *Topological spaces*, MacMillan, New York 1963.

[5] M. S. Berger and M. S. Berger, *Perspectives in non-linearity*, Benjamin, New York 1968.

[6] L. Bers, *Topology*, New York University, Courant Institute of Mathematics (1956).

[7] K. Borsuk, *Drei Sätze über die n-dimensionale Euklidische Sphäre*, Fund. Math. 21 (1933), pp. 177–190.

[8] F. E. Browder, *Non-linear operators and non-linear equations of evolution in Banach spaces*, Symposium on Non-linear Functional Analysis, Amer. Math. Soc. 1968.

[9] J. Cronin, *Fixed points and topological degree in non-linear analysis*, Amer. Math. Soc. Survey Series, Vol. 11, Providence 1964.

[10] J. Dugundji, *An extension of Tietze's theorem*, Pacific J. Math. 1 (1951), pp. 353–367.

[11] K. Fan, *Fixed point and minimax theorems in locally convex topological linear spaces*, Proc. Nat. Acad. Sci., U.S. 38 (1952), pp. 121–126.

[12] A. Granas, *Theorems on antipodes and theorems on fixed points for a certain class of multivalued mappings in Banach spaces*, Bull. Acad. Polon. Sci. 8 (1959), pp. 271–275.

[13] — *Sur la notion de degré topologique pour une certaine classe de transformations multivalentes dans les espaces de Banach*, Bull. Acad. Polon. Sci. 7 (1959), pp. 191–194.

[14] — *The theory of compact fields and some of its applications to the topology of functional spaces, I*, Rozprawy Mat. 30 (1962), pp. 1–89.

[15] — and J. W. Jaworowski, *Some theorems on multivalued mappings of subsets of the Euclidean spaces*, Bull. Acad. Polon. Sci. 8 (1959), pp. 277–283.

[16] I. J. Glicksberg, *A further generalization of the Kakutani fixed point theorem with applications to Nash equilibrium points*, Proc. Amer. Math. Soc. 3 (1952), pp. 170–174.

[17] S. T. Hu, *Homotopy theory*, Academic Press, New York 1959.

[18] J. W. Jaworowski, *Theorems on antipodes for multivalued mappings and a fixed point theorem*, Bull. Acad. Polon. Sci. 4 (1956), pp. 187–192.

[19] S. Kakutani, *A generalization of Brouwer's fixed point theorem*, Duke Math. J. 8 (1941), pp. 457–459.

[20] V. L. Klee, *Leray-Schauder theory without local convexity*, Math. Ann. 141 (1960), pp. 286–296.

[21] M. A. Krasnoselskii, *Topological methods in the theory of non-linear integral equations*, English translation, MacMillan, New York 1964.

[22] J. Leray, *La théorie des points fixes et ses applications en analyse*, Proc. Int. Congress Math., Cambridge, Mass. 2 (1950), pp. 202–208.

[23] — and J. Schauder, *Topologie et équations fonctionelles*, Ann. Sci. Ecole Norm. Super. (3) 51 (1934), pp. 45–78.

[24] J. H. Michael, *Completely continuous movements in topological vector spaces*, Proc. Glasgow Math. Ass. 3 (1957), pp. 135–141.

[25] M. Nagumo, *Degree of mapping in convex linear topological spaces*, Amer. J. Math. 73 (1951), pp. 497–511.

[26] E. H. Rothe, *Theory of topological order in some linear topological spaces*, Iowa State College J. of Sci. 13 (1939), pp. 373–390.

[27] J. T. Schwartz, *Non-linear functional analysis*, Gordon and Beach, New York 1969.

[28] A. Cellina and A. Lasota, *A new approach to the definition of topological degree for multi-valued mappings*, Lincei-Rend. Sci. Mat. e Nat. 47 (1969), pp. 434–440.

[29] Ju. G. Borisovic, B. D. Gelman, E. Muhamadiev and V. V. Obuhovskii, *On the relation of multi-valued vector fields*, Dokl. Akad. Nauk. SSSR, 187 (1969), pp. 956–959.

[30] T. W. Ma, *Non-singular set-valued compact fields in locally convex spaces* (to appear in Fund. Math.).

[31] M. Hukuhara, *Sur l'application semi-continue dont la valeur est un compact convexe*, Funkcialaj Ekvacioj, 10 (1967), pp. 43–66.