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Non-linear eigenvalue problems and bifurcations for differentiable multivalued maps

by RENATO GUZZARDI * (Rende, Italy)

0. Introduction. In this paper we study the solution set of an inclusion of the form $x \in F(\lambda, x)$ where $F(\lambda, x)$ is a multivalued α-contraction. In the proof we use a new concept of differentiability of multivalued maps introduced in [6]–[8], in Section 1 we collect the notations to be used in the sequel and the main properties of the fixed point index introduced in [5].

In Section 2 we introduce the concept of multivalued differentiability along P for multivalued maps, and in Section 3 we prove a fixed point theorem for this type of maps.

In Section 4 we prove that the solution set of an inclusion of the form $x \in F(\lambda, x)$ contains a non-empty closed connected subset which is unbounded in $\boldsymbol{R}_+ \times P$.

Section 5 is devoted to the study of the case when $F(\lambda, 0) = 0$ for every $\lambda \in \boldsymbol{R}_+$, and we proof that bifurcation from the line of trivial solutions $\boldsymbol{R}_+ \times \{0\}$ occurs.

Finally in Section 6 we discuss the situation where bifurcation from infinity takes place.

We would like to mentione that results of Section 3 to Section 6 contains as particular cases some results of H. Amann [3].

1. Notation and Preliminaries. Let E be a real Banach space. A subset P is called a *Cone* if $P+P \subset P$, $\boldsymbol{R}_+ P \subset P$, $P \cap (-P) = \{0\}$, $\bar{P} = P$, where $\boldsymbol{R}_+ := [0, +\infty)$ and $\bar{P}$ denotes the closure of P. Each cone P induces an ordering $\leqslant$ by setting $x \leqslant y$ iff $y - x \in P$. The relation $\leqslant$ is reflexive, transitive, antisymmetric and compatible with the linear structure and the topology of E. By an ordered Banach space (OBS), usually denoted by (E, P), we mean a Banach space E together with an ordering $\leqslant$ induced by a cone P, the positive cone of E. We write $x < y$ iff $y - x \in \dot{P} := P \setminus \{0\}$. The elements of $\dot{P}$ are called *positive*. The norm of an OBS is called *monotone* if $0 \leqslant x \leqslant y$

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implies $\|x\| \leqslant \|y\|$. The positive cone is called *total* if $\overline{P-P} = E$ and generating if $P-P = E$. In the sequel the cone P is considered total.

Given two real Banach spaces E and F, a multivalued map $T\colon A \subset E \multimap F$ is called *upper-semicontinuous* (u.s.c.) at $x \in A$ if for any neighbourhood U of $T(x)$ there exists a neighbourhood V of x such that $T(y) \subset U$ for any $y \in A \cap V$.

Denoting by $K(F)$ the family of all non-empty compact convex subset of E, the multivalued map $T\colon A \subset E \to K(F)$ is upper-semicontinuous on A if T is upper-semicontinuous at each point of A.

A multivalued u.s.c. map $T\colon E \to K(F)$ is called *homogeneous* if $T(tx) = tT(x)$ for any $x \in E$ and $t \in \boldsymbol{R}_+$. Let $L(E, F)$ denote the space of linear continuous mappings from E in to F; then a multivalued u.s.c. map $L\colon E \to K(F)$ is said to be *linear* from E into F, if there exists a maximal set $\mathscr{L} \subset L(E, F)$ such that

$$Lx := \mathscr{L}x = \bigcup_{l \in \mathscr{L}} lx \quad \text{for } x \in E.$$

Let (E, P) be an ordered Banach space with total positive cone P; then a multivalued map $T\colon P \to K(E)$ is *positive* if $T(P) \subset P$, where $T(P) := \bigcup_{x \in P} T(x)$.

For any $a \in E$, X and Y non-empty, $r > 0$, we denote:

$$d(a, X) := \inf\{\|a-x\|,\ x \in X\},$$

$$d(X, Y) := \inf\{\|x-y\|,\ x \in X \text{ and } y \in Y\},$$

$$d^*(X, Y) := \sup\{d(x, Y),\ x \in X\},$$

$$H(X, Y) := \max\{d^*(X, Y),\ d^*(Y, X)\},$$

$$|X| := d^*(X, \{0\}) = \sup\{\|x\|,\ x \in X\},$$

$$B(X, r) := \{x \in E \mid d(x, X) < r\}; \qquad B := \{x \in E \mid \|x\| \leqslant 1\};$$

$$S := \{x \in E,\ \|x\| = 1\}.$$

We recall that if X and Y are compact subset of E

$$d^*(X, Y) = \inf\{t > 0 \mid X \subset Y + tB\}$$

and consequently

$$H(X, Y) = \inf\{t > 0 \mid X \subset Y + tB,\ Y \subset X + tB\}.$$

We recall following known result.

Proposition 1.1. *Let $X, Y \subset E$ be bounded. Then for every $z \in E$ we have $d(z, X) \leqslant d(z, Y) + d^*(Y, X)$.*

Proof. For every $x \in X$ and $y \in Y$ we have

$$\|x-z\| \leqslant \|y-z\| + \|x-y\|$$

and

$$d(z, X) \leqslant \|y-z\| + d(y, X)$$
$$\leqslant \|y-z\| + d^*(Y, X) \leqslant d(z, Y) + d^*(Y, X). \quad \text{Q.E.D.}$$

Let E be real Banach space. If $Q \subset E$ is any bounded set we define the *measure of non-compactness* of Q, $\alpha(Q)$, to be the inf $\{\varepsilon > 0 | \ Q$ can be covered by a finite number of sets, each of which has diameter less than $\varepsilon\}$ (Kuratowski [9]).

This measure of non-compactness satisfies a number of properties [9] among which are the following:

$$A \subset B \subset E \quad \text{implies} \quad \alpha(A) \leqslant \alpha(B),$$
$$0 \leqslant \alpha(A) < \delta(A), \quad \text{where } \delta(A) \text{ is the diameter of } A,$$
$$\alpha(A) = \alpha(\bar{A}) \quad \text{and} \quad \alpha(A \cup B) \leqslant \max\{\alpha(A), \alpha(B)\},$$
$$\alpha(A+B) \leqslant \alpha(A) + \alpha(B),$$
$$\alpha(c \circ A) = \alpha(A),$$

where $c \circ A$ denotes the closed convex hull of A.

Let T: $D \subset E \to K(E)$ be a u.s.c. mapping called *condensing* (resp. α-Lipschitz with constant $k \geqslant 0$) provided that $\alpha(T(Q)) < k\alpha(Q)$ for each $Q \subset D$ with $\alpha(Q) \neq 0$. In particular, if $k < 1$, then T is called α-contraction (Darbo [4]).

Let X and D be subset of E and set $D_X := D \cap X$. We denote by $\partial_X D$ the boundary and the closure of D_X relative to X.

If $X \subset E$ is closed and convex, $D \subset E$ is open and T: $\bar{D}_X \to K(X)$ is condensing and such that $x \notin T(x)$ if $x \in \partial_X D$, then it has been shown by Fitzpatrick and Petryshyn [5] that there exists an integer $i(T, D_X)$, the *fixed point index* of T on D_X, which has the following properties:

(P_1) SOLVABILITY. *If $i(T, D_X) \neq 0$, then T has a fixed point in D_X.*

(P_2) NORMALIZATION. *If $x_0 \in D_X$, then $i(\hat{x}_0, D_X) = 1$, where $\hat{x}_0$ denotes the mapping whose constant value is x_0.*

(P_3) ADDITIVITY. *$i(T, D_X) = i(T, D_{1X}) + i(T, D_{2X})$ for every pair of disjoint open subset D_1, D_2 of D such that T has no fixed point in $\bar{D} \setminus (D_1 \cup D_2)$.*

(P_4) HOMOTOPY. *If H: $[0, 1] \times \bar{D}_X \to K(X)$ is u.s.c. and such that $\alpha(H([0, 1] \times Q)) < \alpha(Q)$ for $Q \subset \bar{D}_X$ with $\alpha(Q) \neq 0$ and if $x \notin H(t, x)$ for $t \in [0, 1]$ and $x \in \partial_X D$, then $i(H(t, \cdot), D_X)$ is independent of $t \in [0, 1]$.*

These properties were established in [5] in the more general setting where E is a Frechét space.

Let us use properties (P_1)–(P_2) to deduce some further properties of the fixed point index.

(P$_5$) Excision. *Let $V \subset U$ be open and $x \notin T(x)$ on $\bar{D} \setminus V$. Then $i(T, D_X) = i(T, V_X)$.*

Proof. The additivity gives $i(T, D_X) = i(T, D_X) + i(T, \emptyset)$ hence $i(T, \emptyset) = 0$. Again by additivity we get $i(T, D_X) = i(T, V_X) + i(T, \emptyset) = i(T, V_X)$ Q.E.D.

Let $\Lambda \subset \mathbf{R}$ be an arbitrary interval and let A be a subset of $\Lambda \times E$. Then for every $\lambda \in \Lambda$, we denote by A_λ the slice at λ, that is

$$A_\lambda := \{x \in E | \ (\lambda, x) \in A\}.$$

Observe that A_λ is open in E if A is open in $\Lambda \times E$.

(P$_6$) General homotopy invariance. *Let $U \subset \mathbf{R} \times X$ be bounded and open. H: $U \to K(X)$ u.s.c. condensing and such that $x \notin H(t, x)$ for every $(t, x) \in \partial U$. Then $i(H(t, \cdot), U_t)$ is independent of $t \in [0, 1]$.*

Proof. Note that the index is well defined since $\partial(U_t) \subset (\partial U)_t$. From [5], Proposition 3.1, it follows that $I - H$ is proper and the set

$$\Phi := \{(t, x) \in U | \ 0 \in x - H(t, x)\}$$

is a compact, possibly empty, subset of U. Hence $\varepsilon := d(\Sigma, \partial U) > 0$. For $t \in [0, 1]$ define :

$$V_t := \{x \in U_t | \ d((t, x), \partial U) > \tfrac{1}{3}\varepsilon\}, \qquad I_t := \{t' \in [0, 1] | \ |t, t'| < \tfrac{1}{3}\varepsilon\}.$$

It is not difficult to check that

$$\Phi \cap (I_t \times E) \subset I_T \times V_t \subset U.$$

Now, by excision, for $t' \in I_t$ we obtain

$$i(H(t', \cdot), U_{t'}) = i(H(t', \cdot), V_t).$$

By the homotopy invariance the last index is independent of $t' \in I_t$. Since $[0, 1]$ can be covered by finitely many I_t*'s* the statement follows. Q.E.D.

2. Differential for multivalued maps. In [6] we introduce a definiton of differentiability for multivalued maps, T, acting between Banach spaces. In [7] we extended this concept to the case when T is single-valued and the domain is a cone. In this section we give the above definition for multivalued maps defined in a cone. Let (E, P) be an ordered Banach space with total positive cone and F: $P \to K(E)$. A homogeneous map u.s.c. Tx_0: $E \to K(E)$ is called an *upper-H-Differential* of F at $x_0 \in P$ along P if there exists $\delta > 0$ such that

$$F(x_0 + h) \subset F(x_0) + Tx_0(h) + R(x_0, h) \quad \text{whenever } \|h\| < \delta.$$

Here $R(x_0, \cdot)$ is a multivalued map from P into E such that $|R(x_0, h)| = o(\|h\|)$ as $h \to 0$.

A homogeneous map T_∞: $E \to K(E)$ is called an *upper-H-Differential* of F at infinity along P if there exist $\delta > 0$ such that

$$F(x) \subset T_\infty(x) + R(x) \quad \text{whenever } \|x\| > \delta.$$

Here R is a multivalued map from P into E such that $|R(x)| = o(\|x\|)$ as $\|x\| \to +\infty$.

For $m = x_0, \infty$ a map F is said be *H-Differentiable* at m along P, if there exist an upper-H-differential T'_m: $E \to K(E)$ of F at m along P such that for any upper-H-differential T_m of F at m along P, we have $T'_m(h) \subset T_m(h)$ for all $h \in P$. In this case the multivalued map T'_m is called the *H-Differential* of T at m along P. T'_m is called the *Differential* (resp. *upper-Differential*) of F at m along P if T'_m is a linear multivalued map.

Assume that F is α-Lipschitz with constant k; then the multivalued map T'_m is called (*upper-*) *αH-Differential* (resp. (*upper*) *α-Differential*) of F at m along P, if is an (upper-) H-Differential (resp. (upper-) Differential) of F at m along P and it is α-Lipschitz with constant $h \leqslant k$.

Let F: $\boldsymbol{R} \times P \to K(E)$ for every $\lambda \in \boldsymbol{R}$ we denote with $\frac{\partial}{\partial x_2} F(\lambda, x_0)$ (called upper-H-Differential respect the second variable) a upper-H-Differential of $F(\lambda, \cdot)$ at x_0 along P. In the sequel we denote by (E, P) an OBS with total cone P. Given $\varrho > 0$ we set $P_\varrho := B \cap P$. The closure $\bar{P}_\varrho$ of P_ϱ in P coincides with $\varrho \bar{B} \cap P$. Hence we have that the boundary S^+_ϱ coincides with $\varrho S \cap P$. Finally we put $S^+ = S^+_1$.

3. Fixed point of differentiable multivalued maps. In this section we obtain results regarding the existence of positive fixed points for multivalued maps by imposing conditions for the upper-Differentials at 0 and at ∞.

LEMMA 3.1. *Let (E, P) be an* OBS, *ϱ a positive number and let F: $\bar{P}_\varrho \to k(P)$ be an α-contraction.*

(a) *Suppose that $F(0) = 0$ and assume that there exists a positive upper-αH-Differential F'_0 at 0 along P such that $\lambda x \notin F'_0(x)$ for every $x > 0$, and every $\lambda \geqslant 1$.*

Then there exists $\varrho_0 > 0$ such that for every $\varrho \in (0, \varrho_0]$

$$i(F, P_\varrho) = 1.$$

(b) *Suppose that $F(0) = 0$ and assume that there exists a positive upper-α-Differential F'_0 at 0 along P such that $x \notin F'_0 x$ for every $x > 0$, and for some $\lambda > 1$, there is an element $x > 0$ such that $\lambda x \in F'_0 x$. Then there exists $\varrho_0 > 0$ such that for every $\varrho \in (0, \varrho_0]$.*

$$i(F, P_\varrho) = 0.$$

Proof. The homogeneous map F'_0 is an α-contraction and $(I - F'_0)(S^+)$

is closed. Consequently, since $0 \notin (I - F'_m)(S^+)$ there exists $\gamma_0 > 0$ such that for all $x \in P$

(1) $$d(x, F'_0(x)) \geqslant \gamma_0 \|x\|.$$

According to the assumptions, there exists $\gamma_0 > 0$ such that $F(x) \subset F'_0(x) + B(0, \gamma_0\|x\|)$ for every $\|x\| < \gamma_0$ and

(2) $$d^*(F(x), F'_0(x)) < \gamma_0 \|x\|.$$

We claim that the homotopy: $(1-\lambda)F'_0(x) + \lambda F(x)$ is u.s.c. α-contractive and has no fixed points on S^+_ϱ, where $\varrho \in (0, \varrho_0]$. Indeed, suppose that there exists $\bar{x} \in S^+_\varrho$ such that $\bar{x} \in (1-\lambda)F'_0(\bar{x}) + \lambda F(\bar{x})$. From (2) it follows that $(1-\lambda)F'_0(x) + \lambda F(\bar{x}) \subset B_{(F'_0(\bar{x}), \gamma_0\|x\|)}$ and $\bar{x} \in B_{(F'_0(\bar{x}), \gamma_0\|\bar{x}\|)}$, $d(\bar{x}_0, F'_0(\bar{x})) < \gamma_0\|\bar{x}\|$, contradicting (1). Therefore

$$i(F, P_\varrho) = i(F'_0, P_\varrho).$$

In case (a) the homotopy $\beta \cdot F'_0(x)$ is u.s.c. α-contraction and fixed point free on S^+, since $x \notin \beta F'_0(x)$ for every $x > 0$ and for $\beta \in [0, 1]$ according to our assumptions hence

$$i(F, P_\varrho) = i(F'_0, P_\varrho) = i(0, P_\varrho) = 1.$$

In case (b) from the linearity of F'_0, there exists $l'_m \in L(E, E)$ and for some $\lambda_0 > 1$ there is an element $h_0 > 0$ such that $\lambda_0 h_0 = l'_0 h_0$. We claim that the equation

(3) $$x = l'_0 x + \gamma h_0$$

has no positive solutions for every real $\gamma > 0$. In fact, suppose that there exists $\bar{x}_0 > 0$ and $\alpha > 0$ such that $\bar{x}_0 = l'_0 \bar{x}_0 + \alpha h_0$. Let $\tau_0 \geqslant 0$ be such that $\bar{x}_0 = l'_0 \bar{x} + \alpha h_0$. Let $\tau_0 \geqslant 0$ be such that $\bar{x}_0 \ngeqslant \tau h_0$ for all $\tau > \tau_0$. Then $\bar{x}_0 = l'_0 \bar{x}_0 + \alpha h_0 \geqslant l_0 \tau_0 h_0 + \alpha h_0 = \tau_0 \lambda_0 h_0 + \alpha h_0 > (\tau_0 + \alpha) h_0$ which contradicts the maximality of τ_m. Hence for every real $\gamma > 0$ and $\varrho > 0$

$$i(l'_0 + \gamma h_0, P_\varrho) = 0.$$

Now, the homotopy $(1-t) l'_0 x + t(l'_0 x + \gamma h_0)$ is u.s.c. α-contractive and has no fixed points on S^+_ϱ. Indeed, suppose that there exists $\bar{x} \in S^+_\varrho$ such that $\bar{x} = (1-t) l'_0 \bar{x} + t(l'_0 + \gamma h_0) = l'_0 \bar{x} + t\gamma h_m$ and equation (3) has positive solutions in contrast with the previsions result. Hence $i(l'_0, P_\varrho) = i(l'_0 + \gamma h_0, P_\varrho)$. Obviously, F'_m is a positive upper α-Differential of l'_0 and there exists ϱ_0 such that for all $\varrho \in (0, \varrho_0]$

$$0 = i(l', P_\varrho) = i(F'_0, P_\varrho) = i(F, P_\varrho). \quad \text{Q.E.D.}$$

LEMMA 3.2. *Let (E, P) be an OBS and let $F\colon P \to K(P)$ be an α-contraction.*

(a) *Suppose that there exists a positive αH-Differential F'_∞ at ∞ along P*

such that $\lambda x \notin F'_\infty(x)$ *for every* $x > 0$ *and every* $\lambda > 1$. *Then there exists* $\varrho_0 > 0$ *such that for every* $\varrho > \varrho_\infty$

$$i(F, P_\varrho) = 1.$$

(b) *Suppose that there exists a positive upper-α-Differential* F'_∞ *at* ∞ *along* P *such that* $x \notin F'_\infty x$ *for every* $x > 0$, *and for some* $\lambda > 1$, *there is an element* $x > 0$ *such that* $\lambda x \in F'_\infty x$. *Then there exists* $\varrho_0 > 0$ *such that for every* $\varrho \geqslant \varrho_0$

$$i(F, P_\varrho) = 0.$$

The proof is, with obvious modifications, the same as that of Lemma 3.1.

THEOREM 3.1. *Let* (E, P) *be an* OBS *and let* $F: \boldsymbol{R}_+ \times P \to K(P)$ *be a map such that* $F(\lambda, \cdot)$: $P \to K(P)$ *is α-contractive for every* $\lambda \in \boldsymbol{R}_+$ *and* $F(\lambda, 0) = 0$ *for all* $\lambda \in \boldsymbol{R}_+$, *suppose that for* $m = 0, \infty$ *there exists a multivalued linear map* F'_m: $E \to K(E)$ *and a map* r_m: $\boldsymbol{R}_+ \times P \to K(P)$, *such that*

$$F(\lambda, x) \subset \lambda T_m x + r_m(\lambda, x), \qquad (\lambda, x) \in \boldsymbol{R}_+ \times P$$

with $|r_m(\lambda, x)| = O(\|x\|)$ *as* $\|x\| \to m$ *in* P *for all* $\lambda \in \boldsymbol{R}_+$. *Finally, suppose that* T'_m *has exactly one positive eigenvalue* μ_m *with a positive eigenvector. Then for every* $\lambda \in \boldsymbol{R}_+$ *satisfying*

$$\min\{1/\mu_0, 1/\mu_\infty\} < \lambda < \max\{1/\mu_0, 1/\mu_\infty\}$$

the map $F(\lambda, \cdot)$ *possesses a positive fixed point.*

Proof. From Theorem 3.1 and Lemma 3.2 it follows that there exists two real numbers ϱ and δ with $0 < \varrho < \delta$ such that $i(F(\lambda, \cdot), P) = 1$ and $i(F(\lambda, \cdot), P) = 0$ or $i(F(\lambda, \cdot), P) = 0$ and $i(F(\lambda, \cdot), P) = 1$. Hence, in both cases, from the Additivity property of the fixed point index it follows that

$$i(F, P_\delta \setminus \bar{P}_\varrho) = i(F, P_\delta) - i(F, P_\varrho) \neq 0.$$

From the solution property of the fixed point index there exists x such that $\varrho < \|x\| < \delta$ and $x \in F(\lambda, x)$. Q.E.D.

4. Global of positive fixed points. Let (E, P) be an OBS such that $P \neq \{0\}$ and let F: $\boldsymbol{R}_+ \times P \to K(P)$ be α-contractive. In this section we study inclusions of the from:

$$x \in F(\lambda, x). \tag{1}$$

In other words, we study fixed point of one-parameter families of multivalued maps. In what follows we denote by Σ the solution set of inclusion (1) that is,

$$\Sigma := \{(\lambda, x) \in \boldsymbol{R}_+ \times P \mid x \in F(\lambda, x)\}$$

and we set

$$\Lambda := \{\lambda \in \mathbf{R}_+ |\ x \in P \text{ and } (\lambda, x) \in \Sigma\}.$$

Recall that a non-empty closed connected subset of a topological space X is called a *subcontinuum of* X. Finally, denote by Σ^+ the set of positive solutions of inclusion (1), Σ^+ consist of the union of $\Sigma \cap (\dot{\mathbf{R}}_+ \times \dot{P})$ and the set $\{(\lambda, 0) \in \dot{\mathbf{R}}_+ \times \{0\} |\ \lambda \text{ is a bifurcation point}\}$.

THEOREM 4.1. *Let* (E, P) *be an* OBS *and let* $F: \mathbf{R}_+ \times P \to K(P)$ *be* α-*contractive. Suppose that* $F(0, 0) = 0$ *and that zero is the only fixed point of* $F(0, \cdot)$. *Moreover, assume that there exists a positive number* ϱ *such that* $\delta x \notin F(0, x)$ *for every* $x \in S_\varrho^+$ *and every* $\delta \geq 1$. *Then the solution set* Σ *contains an unbounded subcontinuum containing* $(0, 0)$.

Proof. The proof is with slight modifications the same as that of Theorem 17.1 in [3]. We give it here for the sake of completeness.

Put $Q_\mu = [0, \mu] \times \bar{P}_\mu$ for every $\mu > 0$. Let C the component of Σ containing $(0, 0)$ and suppose that C is bounded. Then there exists a number $\mu > 0$ such that $C \cap \partial Q_\mu = \emptyset$.

Set $D := \partial Q_\mu \cap \Sigma$. We have $D \cap C = \emptyset$ and D and C are closed subset of the compact metric space $X := \Sigma \cap Q_\mu$. Then there exist two disjoint compact sets K_1 and K_2 such that $C \subset K_1$ and $D \subset K_2$ with $X = K_1 \cup K_2$. Since Q_μ is a metric space, there exists an open set U of Q_μ with $K_1 \subset U$ and $\bar{U} \cap (K_2 \cap \partial Q_\mu) = \emptyset$.

Consequently, U is a bounded open subset of $[0, \mu] \times P$ such that $x \notin F(\lambda, x)$ for all $x \in \partial U$ and all $\lambda \in [0, \mu]$. Hence, denoting by U_λ the slice of U at $\lambda \in [0, \mu]$, from property (P_6) of the fixed point index, we obtain

$$i(F(0, \cdot), U_0) = i(F(\mu, \cdot), U_\mu).$$

But $U_\mu = \emptyset$ and therefore

$$i(F(0, \cdot), U_0) = i(F(\mu, \cdot), U_\mu) = 0.$$

By the excision property of the fixed point index,

$$i(F(0, \cdot), U_0) = i(F(0, \cdot), P_\varrho).$$

Now, the homotopy $\tau \to \tau F(0, \cdot)$, $\tau \in [0, 1]$ is u.s.c., α-contractive and has no fixed points in P for every $\tau \in [0, 1]$. From the homotopy property of the fixed point index it follows that

$$i(F(0, \cdot), P_\varrho) = i(0, P_\varrho) = 1.$$

This contradiction proves the assertion. Q.E.D.

COROLLARY 4.1. *Let* (E, P) *be an* OBS *and let* $F: \mathbf{R}_+ \times P \to K(P)$ *be* α-*contractive. Suppose that* $F(0, 0) = 0$ *and that zero is the only fixed point of* $F(0, \cdot)$. *Assume that there exists a positive number* ϱ *such that* $\delta x \notin F(0, x)$ *for every* $x \in S_\varrho^+$ *and every* $\delta \geq 1$. *Moreover,* $\bar{\lambda} > 0$ *such that* $F(\bar{\lambda}, \cdot)$ *has non-fixed* *points. Then* $\Sigma \cap ([0, \bar{\lambda}] \times P)$ *contains an unbounded subcontinuum emanating from* $(0, 0)$.

Proof. From Theorem 4.1 it follows that Σ contains an unbounded subcontinuum C emanating from $(0, 0)$. Denote by P_1: $\boldsymbol{R}_+ \times P \to \boldsymbol{R}_+$ the projection defined by $P_1(\lambda, x) = \lambda$, since P_1 is continuous, $P_1(C)$ is a connected subset of $\boldsymbol{R}_+$ hence an interval containing 0. Since $P_1(C) \subset \Lambda$ and $\bar{\lambda} \notin \Lambda$, it follows that $\Sigma \cap ([0, \bar{\lambda}] \times P)$ contains an unbounded subcontinuum emanating from $(0, 0)$. Q.E.D.

COROLLARY 4.2. *Let* (E, P) *be an* OBS *and let* F: $\boldsymbol{R}_+ \times P \to K(P)$ *be* α-*contractive. Suppose that* $F(0, 0) = 0$ *and that zero is the only fixed point of* $F(0, \cdot)$. *Assume that there exists a positive number* ϱ *such that* $\delta x \notin F(0, x)$ *for every* $x \in S_\varrho^+$ *and every* $\delta \geq 1$. *Moreover, suppose that* $F(\lambda, x) \neq x$ *for every* $\lambda > 0$ *and every* $x \in S_\varrho^+$. *Then* $\Lambda = \boldsymbol{R}_+$.

Proof. Denote by p_2: $\boldsymbol{R}_+ \times P \to P$ the natural projection. Let C denote the unbounded subcontinuum of Σ emanating from $(0, 0)$. Then $p_2(C)$ is a connected subset of P which does not intersect S_ϱ^+. Hence, from the unboundedness of C, we have that $\Lambda = \boldsymbol{R}_+$. Q.E.D.

5. Bifurcation from the trivial solution. Let (E, P) be an OBS and let F: $\boldsymbol{R}_+ \times P \to K(P)$ be α-contractive and such that $F(\cdot, 0) = 0$. Then $\lambda_0 \in \boldsymbol{R}_+$ is called a *bifurcation point* for the inclusion

$$x \in F(\lambda, x).$$

With respect to the trivial solution, if for every neighbourhood U of $(\lambda_0, 0)$ in $\boldsymbol{R}_+ \times P$ there exists a point $(\lambda, x) \in U$ with $x \in F(\lambda, x)$ and $x > 0$.

Theorems 5.1 and 5.2 contain necessary conditions for $\lambda_0 \in \boldsymbol{R}_+$ to be a bifurcation point. Theorems 5.3 and 5.4 contain sufficient conditions for the existence of solutions which bifurcate from the trivial solutions.

THEOREM 5.1. *Let* (E, P) *be an* OBS *and let* F: $\boldsymbol{R}_+ \times P \to K(P)$ *be an* α-*contractive map such that* $F(\cdot, 0) = 0$. *Suppose that* $\lambda_0 \in \boldsymbol{R}_+$ *is a bifurcation point such that there exists a positive upper-*α*H-Differential* $\frac{\partial}{\partial x_2} F(\lambda_0, 0)$. *If the map* $F(\cdot, 0)\|x\|^{-1}$: $\boldsymbol{R}_+ \to K(P)$ *is continuous at* λ_0, *uniformly on null sequences in* $\dot{P}$, *then* 1 *is an eigenvalue of* $\frac{\partial}{\partial x_2} F(\lambda_0, 0)$ *with a positive eigenvector.*

Proof. The assumptions imply the existence of a sequence $((\lambda_n, x_n))$ in $(\boldsymbol{R}_+ \times \dot{P}) \cap \Sigma$ which converges to $(\lambda_0, 0)$. Hence, letting $y_n := x_n/\|x_n\| \in S^+$ we have

$$d\left(y_n, \frac{\partial}{\partial x_2} F(\lambda_0, 0) y_n\right) = d\left(\frac{x_n}{\|x_n\|}, \frac{\partial}{\partial x_2} F(\lambda_0, 0) \frac{x_n}{\|x_n\|}\right)$$
$$\leq d\left(\frac{x_n}{\|x_n\|}, \frac{F(\lambda_0, x_n)}{\|x_n\|}\right) + d^*\left(\frac{F(\lambda_0, x_n)}{\|x_n\|}, \frac{\partial}{\partial x_2} F(\lambda_0, 0) \frac{x_n}{\|x_n\|}\right).$$

However, by hypothesis we have

$$F(\lambda_0, x_n) \subset F(\lambda_0, 0)+\frac{\partial}{\partial x_2}F(\lambda_0, 0)x_n+O_{(\lambda_0,0)}(x_n)$$

and

$$d^*\left(\frac{F(\lambda_0, x_n)}{\|x_n\|}, \left(\frac{\partial}{\partial x_2}F(\lambda_0, 0)x_n\right)\frac{1}{\|x_n\|}\right)\to 0 \quad \text{as } n\to+\infty.$$

Since $x_n/\|x_n\|\in F(\lambda_n, x_n)/\|x_n\|$ and

$$\lim_n H\left(\frac{F(\lambda_n, x_n)}{\|x_n\|}, \frac{F(\lambda_0, x_n)}{\|x_n\|}\right)=0.$$

It follows that

$$\lim_n d\left(\frac{x_n}{\|x_n\|}, \frac{F(\lambda_0, x_n)}{\|x_n\|}\right)=0$$

and

$$\lim_n d\left(y_n, \frac{\partial}{\partial x_2}F(\lambda_0, 0)y_n\right)=0,$$

i.e.,

$$0\in\left(I-\frac{\partial}{\partial x_2}F(\lambda_0, 0)\right)(S^+).$$

But

$$\frac{\partial}{\partial x_2}F(\lambda_0, 0),$$

is α-contractive and, consequently, the set $\left(I-\frac{\partial}{\partial x_2}F(\lambda_0, 0)\right)(S^+)$ is closed. It follows that there exists $x>0$ such that

$$x\in\frac{\partial}{\partial x_2}F(\lambda_0, 0)x. \qquad \text{Q.E.D.}$$

THEOREM 5.2. *Let (E, P) be an OBS and let F: $\boldsymbol{R}_+\times P\to K(P)$ be an α-contractive map such that $F(\cdot, 0)=0$. Assume that there exists a homogeneous α-contraction T_0: $E\to K(E)$ satisfying $T_0(P)\subset P$ and a multivalued map R_0: $\boldsymbol{R}_+\times P\multimap P$ such that*

$$F(\lambda, x)\subset\lambda T_0(x)+R_0(\lambda, x) \quad \textit{for all } (\lambda, x)\in\boldsymbol{R}_+\times P.$$

Moreover, assume that the map $R_0(\cdot, x)\|x\|^{-1}$: $\boldsymbol{R}_+\multimap P$ is continuous uni-

formly on null sequences in $\dot{P}$ *and* $R_0(\lambda, x) = o(\|x\|)$ *for all* $\lambda \in \boldsymbol{R}_+$ *as* $\|x\| \to 0$. *Suppose that* $\lambda_0 \in \boldsymbol{R}_+$ *is a bifurcation point. Then* $\lambda_0 > 0$ *and* λ_0^{-1} *is an eigenvalue of* T_0 *with a positive eigenvector. In particular* $T_0 \neq 0$.

Proof. The hypotheses imply that $\lambda_0 T_0$ is a positive upper-αH-Differential with respect to the second variable at 0. Consequently, the map $F(\cdot, x)\|x\|^{-1}$: $\boldsymbol{R}_+ \multimap P$ is continuous at λ_0, uniformly on null sequences in $\dot{P}$. From Theorem 18.1 there exists $x > 0$ such that $x \in \lambda_0 T_0 x$ and this implies $\lambda_0 \neq 0$, the map $T_0 \neq 0$ and λ_0^{-1} is an eigenvalue of T_0. Q.E.D.

THEOREM 5.3. *Let* (E, P) *be an* OBS *and let* F: $\boldsymbol{R}_+ \times P \to K(P)$ *be an* α-*contractive map such that* $F(\cdot, 0) = 0$ *and* $F(0, \cdot) = 0$. *Assume that there exists a multivalued linear* α-*contraction* T_0: $E \to K(E)$ *such that* $T_0(P) \subset P$ *and a multivalued map* R_0: $\boldsymbol{R}_+ \times P \multimap P$ *satisfying*

$$F(\lambda, x) \subset \lambda T_0 x + R_0(\lambda, x) \quad \text{for all } (\lambda, x) \in \boldsymbol{R}_+ \times P.$$

Moreover, assume that the map $R_0(\cdot, x)$: $\boldsymbol{R}_+ \to P$ *is continuous uniformly on null sequences in* $\dot{P}$ *and* $R_0(\cdot, x) = o(\|x\|)$ *as* $x \to 0$. *Suppose that there exists* $\lambda_0, \lambda_1 \in \boldsymbol{R}_+$ *such that* $\delta^+(T_0) \subset [\lambda_0, \lambda_1]$, *where* $\delta^+(T_0)$ *is the non-empty set of all positive eigenvalue of* T_0 *with a positive eigenvector. Then bifurcation from the line of trivial solutions occurs, and* Σ^+ *contains an unbounded subcontinuum emanating from one of these eigenvalues.*

Proof. Denote by C the component of $\Sigma^+ \cup ([0, \lambda_0^{-1}] \times \{0\})$ containing $[0, \lambda_0^{-1}] \times \{0\}$, we have that C is an unbounded subcontinuum emanating from one of these eigenvalues. In fact, suppose that C is bounded, then there exists a positive number $\mu > \lambda_0^{-1}$, and $\mu^{-1} \notin \delta^+(T_0)$, such that

$$\big(([0, \mu] \times S_\mu^+) \cup (\{\mu\} \times \bar{P}_\mu)\big) \cap C = \emptyset.$$

Let $C_1 = C \cup ([0, \mu] \times \{0\})$ and let ε be a positive number such that $\varepsilon < \min\{\mu, \delta_0, \delta_\mu\}$, where δ_λ denotes the number ϱ_0 of Lemma 3.1 with respect to the map $F(\lambda, \cdot)$. Put

$$D = \big(\{0\} \times (\bar{P}_\mu \setminus P_\varepsilon)\big) \cup ([0, \mu] \times S_\mu^+) \cup \big(\{\mu\} \times (\bar{P}_\mu \setminus P_\varepsilon)\big)$$

we have $D \cap C_1 = \emptyset$. Then, as in the proof of Theorem 4.1 we can find an open subset U of $[0, \mu] \times P$ with $\Sigma \cap \partial U = \emptyset$, $C_1 \subset U$, and $\bar{U} \cap D = \emptyset$. Hence the general homotopy invariance property of the fixed point index imply that:

$$i\big(F(0, \cdot), U_0\big) = i\big(F(\mu, \cdot), U_\mu\big).$$

The excision property of the fixed point index yields

$$i\big(F(0, \cdot), P_\varepsilon\big) = i\big(F(0, \cdot), U_0\big) \quad \text{and} \quad i\big(F(\mu, \cdot), U_\mu\big) = i\big(F(\mu, \cdot), P_\varepsilon\big).$$

From the normalization property we have

$$1 = i(0, P_\varepsilon) = i\big(F(0, \cdot), P_\varepsilon\big) = i\big(F(\mu, \cdot), P_\varepsilon\big). \tag{1}$$

Since μT_0 is a positive upper-α-Differential with respect the second variable at 0, it follows, from the assumptions, that there exist $\lambda \in [\lambda_0, \lambda_1]$, $\lambda \in \mu x \in \mu T_0 x$ for some $x > 0$, $\lambda, \mu > 1$, and 1 is not an eigenvalue with a positive eigenvector. From Lemma 3.1 we obtain $i(F(\mu, \cdot), P_\varepsilon) = 0$ in contrast with (1). Q.E.D.

COROLLARY 5.1. *Let (E, P) be an OBS and let F: $\boldsymbol{R}_+ \times P \to K(P)$ be an α-contractive map such that $F(\cdot, 0) = 0$ and $F(0, \cdot) = 0$. Assume that there exists a multivalued linear α-contraction T_0: $E \to K(E)$ such that $T_0(P) \subset P$ and a multivalued map R_0: $\boldsymbol{R}_+ \times P \multimap P$ such that*

$$F(\lambda, x) \subset \lambda T_0 x + R_0(\lambda, x), \quad (\lambda, x) \in \boldsymbol{R}_+ \times P.$$

Moreover, the map $R_0(\cdot, x)$: $\boldsymbol{R}_+ \to P$ is continuous uniformly on null sequences in $\dot{P}$ and $R(\cdot, x) = o(\|x\|)$ as $x \to 0$. Suppose that T_0 possesses exactly one positive eigenvalue μ_0 with a positive eigenvector., Then μ_0^{-1} is the unique bifurcation point. Moreover, Σ^+ (the set of positive solutions) contains an unbounded component C such that $C \cap (\boldsymbol{R}_+ \times \{0\}) = (\mu_0^{-1}, 0)$.

The proof follows trivially from Theorem 5.3. Q.E.D.

6. Bifurcation from the infinity. Let (E, P) be an OBS and let F: $\boldsymbol{R}_+ \times P \to K(P)$ be an α-contractive map. In what follows we say that the solution set Σ meets (λ_∞, ∞) for some $\lambda_\infty \in \boldsymbol{R}_+$, if $\Sigma \cap ((\lambda_\infty - \varepsilon, \lambda_\infty + \varepsilon) \times P \setminus \bar{P}_{1/\varepsilon}) \neq \emptyset$ for every $\varepsilon > 0$, λ_∞ is also said to be a *bifurcation point from infinity* for the inclusion $x \in F(\lambda, x)$. Theorems 6.1 and 6.2 contain necessary conditions for $\lambda_\infty \in \boldsymbol{R}_+$ to be a bifurcation point in the case when the map $F(\lambda, \cdot)$ has a positive upper-αH-Differential at ∞ along P.

These two theorems are analogues of Theorems 5.1 and 5.2.

THEOREM 6.1. *Let (E, P) be an OBS and let F: $\boldsymbol{R}_+ \times P \to K(P)$ be an α-contractive map. Suppose that there exists a sequence $((\lambda_j, x_j))$ in Σ such that $\lambda_j \to \lambda_0 \in \boldsymbol{R}_+$ and $\|x_j\| \to \infty$. Moreover, suppose that there exists a positive upper-αH-Differential $\frac{\partial}{\partial x_2} F(\lambda_0, \infty)$ and that the map $F(\cdot, x)\|x\|^{-1}$: $\boldsymbol{R}_+ \to K(P)$ is continuous at λ_0, uniformly on unbounded sequence in $\dot{P}$. Then 1 is an eigenvalue of $\frac{\partial}{\partial x_2} F(\lambda_0, \infty)$ to which corresponds a positive eigenvector.*

THEOREM 6.2. *Let (E, P) be an OBS and let F: $\boldsymbol{R}_+ \times P \to K(P)$ be an α-contractive map such that there exists a homogeneous α-contraction T_∞: $E \to K(E)$ such that $T_\infty(P) \subset P$ and a multivalued map R_∞: $\boldsymbol{R}_+ \times P \to P$ such that*

$$F(\lambda, x) \subset \lambda T_\infty(x) + R_\infty(\lambda, x) \quad \textit{for all } (\lambda, x) \in \boldsymbol{R}_+ \times P.$$

Moreover, assume that the map $R_\infty(\cdot, x)\|x\|^{-1}$: $\boldsymbol{R}_+ \to P$ is continuous uniformly on un bounded sequence in $\dot{P}$ and $R_\infty(\lambda, x) = o(\|x\|)$ as $\|x\| \to +\infty$ for every $\lambda \in \boldsymbol{R}_+$. Suppose finally that λ_∞ is a bifurcation point from infinity. Then

$\lambda_\infty > 0$ and λ_∞^{-1} are eigenvalues of T_∞ having positive eigenvectors. In particular, $T_\infty \neq 0$.

The proof of Theorems 6.1 and 6.2 are, with the obvious modifications, the same as those of Theorems 5.1 and 5.2.

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DIPARTIMENTO DI MATEMATICA
UNIVERSITÀ DELLA CALABRIA, ITALY

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