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A generalization of a theorem of S. Gołąb and M. Kucharzewski

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Let M be an n-dimensional manifold and let p be a point of M. Any differentiable transformation of local coordinates $(x^a) \to (y^a)$ in a neighborhood of p involves a system of parameters such that

$$(1) \qquad A^a_\beta = \frac{\partial y^a}{\partial x^\beta}\bigg|_{(p)}, \ldots, A^a_{a_1\ldots a_r} = \frac{\partial^r y^a}{\partial x^{a_1}\ldots\partial x^{a_r}}\bigg|_{(p)},$$

where $\det(A^a_\beta) \neq 0$ and the $A^a_{a_1\ldots a_r}$ are symmetric with respect to the lower indices. The formula of derivation of composite functions implies the law of multiplication of the parameters (1). Thus

$$(2) \qquad \begin{aligned} C^a_\beta &= A^a_\tau B^\tau_\beta, \\ C^a_{\beta\gamma} &= A^a_\tau B^\tau_{\beta\gamma} + A^a_{\varrho\sigma} B^\varrho_\beta B^\sigma_\gamma, \\ &\ldots\ldots\ldots\ldots\ldots \end{aligned}$$

(see [2]), where $C^a_\beta, C^a_{\beta\gamma}, \ldots$ are parameters of the product. The set of all the elements

$$L = (A^a_\beta, A^a_{a_1a_2}, \ldots, A^a_{a_1\ldots a_r})$$

with the multiplication (2) forms a Lie group L^r_n called the *differential group* of order r $(1 \leqslant r < \infty)$. Its dimension

$$N = n\binom{n+r-1}{r}.$$

The structure of this group has been thoroughly investigated by Nijenhuis in [5].

The subgroup of all elements of the form $(A^a_\beta, 0, \ldots, 0)$ is isomorphic to the full linear group $GL(n, R)$ of all non-singular real $n\times n$ matrices; it will be denoted by L_0.

The subject of our investigation is the multiplicative functional matrix equation

$$(3) \qquad F(L_1L_2) = F(L_1)F(L_2),$$

where $L_1, L_2 \in L_n^r$ and F is a non-singular real matrix of order m, in the case $r = 2$ and $2 \leqslant m \leqslant n$.

In general case solving of this equation rembers possible the determination of all linear purely differential geometric objects of the type $[m, n, r]$, i.e. with m components and of the class r (see [2]).

In the special case $m = 1$ and $n \geqslant 2$ equation (3) was solved by Gołąb and Kucharzewski in [2]. They proved that any scalar function f defined on the group L_n^r $(n \geqslant 2)$ satisfying equation (3) depends only on the matrix parameters A_β^α, i.e. on elements of the subgroup L_0. Thus we can write $f(L) = h(A_\beta^\alpha)$ and h satisfies the equation

$$h(A_\beta^\alpha)h(B_\beta^\alpha) = h(A_\tau^\alpha B_\beta^\tau). \tag{4}$$

For $n = 2$ equation (4) was solved by Gołąb in [1] and then for arbitrary n by Kucharzewski in [3]. The solution is

$$h(A_\beta^\alpha) = \varphi\big(\det(A_\beta^\alpha)\big),$$

where φ is an arbitrary multiplicative function of real variable.

The purpose of this note is to generalize the result of S. Gołąb and M. Kucharzewski to arbitrary m, $2 \leqslant m \leqslant n$, with restriction to $r = 2$ (1).

THEOREM. *In the case $2 \leqslant m \leqslant n$ the general solution F of functional equation* (3) *may depend only on the matrix parameters A_β^α and satisfies the equation*

$$\Phi(A_\tau^\alpha B_\beta^\tau) = \Phi(A_\beta^\alpha)\Phi(B_\beta^\alpha). \tag{5}$$

Proof. The group L_n^2 consists of elements

$$L = (A_\beta^\alpha, A_{\beta\gamma}^\alpha), \quad \det(A_\beta^\alpha) \neq 0, \quad A_{\beta\gamma}^\alpha = A_{\gamma\beta}^\alpha, \quad \alpha, \beta, \gamma = 1, \ldots, n.$$

Notice the following identities:

$$(\delta_\beta^\alpha, A_{\beta\gamma}^\alpha)(\delta_\beta^\alpha, B_{\beta\gamma}^\alpha) = (\delta_\beta^\alpha, A_{\beta\gamma}^\alpha + B_{\beta\gamma}^\alpha), \tag{6}$$

$$(A_\beta^\alpha, A_{\beta\gamma}^\alpha) = (A_\beta^\alpha, 0)(\delta_\beta^\alpha, \bar{A}_\tau^\alpha A_{\beta\gamma}^\tau), \tag{7}$$

where $[\bar{A}_\beta^\alpha]$ is the inverse matrix to $[A_\beta^\alpha]$.

According to the decomposition (7) and to equation (3) we have

$$F(A_\beta^\alpha, A_{\beta\gamma}^\alpha) = F[(A_\beta^\alpha, 0)(\delta_n^\alpha, \bar{A}_\tau^\alpha A_{\beta\gamma}^\tau)] = F(A_\beta^\alpha, 0)F(\delta_\beta^\alpha, \bar{A}_\tau^\alpha A_{\beta\gamma}^\tau). \tag{8}$$

Write

$$G(A_\beta^\alpha) \overset{\text{df}}{=} F(A_\beta^\alpha, 0), \quad H(A_{\beta\gamma}^\alpha) \overset{\text{df}}{=} F(\delta_\beta^\alpha, A_{\beta\gamma}^\alpha); \tag{9}$$

we shall write shortly $G(A)$ and $H(X)$, where $A = [A_\beta^\alpha]$ and $X = (A_{\beta\gamma}^\alpha)$.

(1) The presented proof forms a part of author's PhD thesis.

In virtue of (8) and (9) we get

$$F(A^\alpha_\beta, A^\alpha_{\beta\gamma}) = G(A^\alpha_\beta)H(\bar{A}^\alpha_\tau A^\tau_{\beta\gamma}). \tag{10}$$

It can be written briefly

$$F(L) = G(A)H(\bar{A}^\alpha_\tau A^\tau_{\beta\gamma}).$$

As the matrix F is non-singular, so are the matrices G and H. Let $L_1 = (\delta^\alpha_\beta, A^\alpha_{\beta\gamma})$, $L_2 = (\delta^\alpha_\beta, B^\alpha_{\beta\gamma})$. Write

$$X = (A^\alpha_{\beta\gamma}), \quad Y = (B^\alpha_{\beta\gamma}).$$

According to (6) we have

$$L_1L_2 = (\delta^\alpha_\beta, A^\alpha_{\beta\gamma}+B^\alpha_{\beta\gamma}) = (E, X+Y).$$

Since

$$F(L_1) = H(X), \quad F(L_2) = H(Y)$$

we get from (3)

$$H(X+Y) = H(X)H(Y). \tag{11}$$

Put $L_3 = (\delta^\alpha_\beta, A^\alpha_{\beta\gamma})$, $L_4 = (B^\alpha_\beta, 0)$. Then

$$F(L_3L_4) = F(L_3)F(L_4) = H(A^\alpha_{\beta\gamma})G(B^\alpha_{\beta\gamma}) \tag{12}$$

holds. In view of $L_3L_4 = (B^\alpha_\beta, A^\alpha_{\lambda\mu}B^\lambda_\beta B^\mu_\gamma)$ and of identity (9) we get

$$L_3L_4 = (B^\alpha_\beta, 0)(\delta^\alpha_\beta, \bar{B}^\alpha_\tau A^\tau_{\lambda\mu}B^\lambda_\beta B^\mu_\gamma).$$

Hence

$$\begin{aligned} F(L_3L_4) &= F(B^\alpha_\beta, 0)F(\delta^\alpha_\beta, \bar{B}^\alpha_\tau A^\tau_{\lambda\mu}B^\lambda_\beta B^\mu_\gamma) \\ &= G(B^\alpha_\beta)H(\bar{B}^\alpha_\tau A^\tau_{\lambda\mu}B^\lambda_\beta B^\mu_\gamma). \end{aligned} \tag{13}$$

Comparing (12) and (13) one obtains

$$G(B^\alpha_\beta)H(\bar{B}^\alpha_\tau A^\tau_{\lambda\mu}B^\lambda_\beta B^\mu_\gamma) = H(A^\alpha_{\beta\gamma})G(B^\alpha_\beta). \tag{14}$$

On the other hand,

$$(A^\alpha_\beta, 0)(B^\alpha_\beta, 0) = (A^\alpha_\lambda B^\lambda_\beta, 0).$$

Hence $F(A^\alpha_\beta, 0)F(B^\alpha_\beta, 0) = F(A^\alpha_\lambda B^\lambda_\beta, 0)$ and using (9) we get

$$G(A^\alpha_\beta)G(B^\alpha_\beta) = G(A^\alpha_\lambda B^\lambda_\beta);$$

thus G satisfies equation (5).

The general solution of equation (5), in the case $m \leqslant n$, was given by Kucharzewski and Zajtz in [4]. It has one of the forms

1° $G(A) = \Phi(J)$, $J = \det A$;
2° $G(A) = C^{-1}\varphi(J)(A^T)^{-1}C$;
3° $G(A) = C^{-1}\varphi(J)A\cdot C$,

where C is a non-singular $n\times n$ constant matrix; $\Phi(J)$ is a matrix function of real argument $J\neq 0$ satisfying the equation

$$\Phi(J_1 1_2) = \Phi(J_1)\Phi(J_2), \tag{15}$$

and $\varphi(J)$ is a scalar multiplicative function.

Now we shall deal with equation (14) substituting there successively the solutions 1°, 2° and 3°.

Case 1°. Taking for A an unimodular diagonal matrix

$$\begin{bmatrix} \varrho_1 & & \\ & \ddots & \\ & & \varrho_n \end{bmatrix}, \qquad \varrho_1, \ldots, \varrho_n = 1,$$

i.e. $J=1$, we get $G(A)=E$, the unit matrix.

In fact, $\Phi(1)\Phi(1) = \Phi(1.1)$ and Φ being non-singular, it implies $\Phi(1) = E$.

Then equation (14) takes the form

$$H\left(\frac{1}{\varrho_a} A^a_{\beta\gamma}\varrho_\beta\varrho_\gamma\right) = H(A^a_{\beta\gamma}). \tag{16}$$

Let us fix the indices a, β, γ and choose ϱ's such that

$$\frac{\varrho_\beta\varrho_\gamma}{\varrho_a} = 2, \qquad \varrho_1, \ldots, \varrho_n = 1. \tag{17}$$

For $n\geqslant 2$ equations (17) always have a solution $\varrho_1, \ldots, \varrho_n$ because the left-hand sides of (17) cannot be identical.

We assume that the function $H(X^\lambda_{\mu\nu})$ depends only on the independent variables $X^\lambda_{\mu\nu}$, where $\mu\leqslant\nu$. Thus we can consider such arguments only for which $X^a_{\beta\gamma}\neq 0$, with fixed a, β, γ, and the others $X^\lambda_{\mu\nu}$ $(\mu\leqslant\nu)$ equal to 0.

Write

$$H(X_0) = H(0, \ldots, 0, X^a_{\beta\gamma}, 0, \ldots, 0).$$

By (16) and (17) we obtain for the corresponding ϱ's $H(2X_0) = H(X_0)$. But in view of (11) $H(2X_0) = H^2(X_0)$, and consequently

$$H^2(X_0) = H(X_0).$$

Since H is a non-singular matrix, it implies $H(X_0) = E$. Thus we proved that for any fixed a, β, γ

$$H(0, \ldots, 0, X^a_{\beta\gamma}, 0, \ldots, 0) = E \tag{18}$$

holds. For $n\geqslant 2$ the general sequence of arguments

$$(X^1_{11}, X^1_{12}, \ldots, X^n_{nn})$$

can be written as the sum

$$(X^1_{11}, 0, \ldots, 0) + (0, X^1_{12}, 0, \ldots, 0) + \ldots + (0, \ldots, 0, X^n_{nn}).$$

By (11)

$$H(X_{11}^1, \ldots, X_{nn}^n) = H(X_{11}^1, 0, \ldots, 0) \ldots H(0, \ldots, 0, X_{nn}^n);$$

in view of (18) every factor of the right-hand product is the unit matrix, so we get

$$H(X_{11}^1, \ldots, X_{nn}^n) = E.$$

Cases 2° and 3°. Substituting $A_0 = \varrho E$ we obtain

$$G(A_0) = \varphi(\varrho^n)\frac{1}{\varrho}E \quad \text{or} \quad G(A_0) = \varphi(\varrho^n)\varrho E,$$

respectively. Thus both the $G(A_0)$ are scalar matrices, and consequently commuting with H in equality (14). Multiplying (14) by $G^{-1}(A_0)$ we get

$$H(\varrho A_{\beta\gamma}^a) = H(A_{\beta\gamma}^a).$$

Therefore $H(X)$ is a homogeneous function of order 0. Putting $\varrho = 2$ we obtain $H(2A_{\beta\gamma}^a) = H(A_{\beta\gamma}^a)$ and, as previously, it implies

$$H(A_{\beta\gamma}^a) = E.$$

In all the cases H being unit matrix we get finally from (10)

$$F(A_\beta^a, A_{\beta\gamma}^a) = G(A_\beta^a).$$

This finishes the proof.

Our theorem can be applied to characterization of the geometric objects of type $[m, n, 2]$, $m \leqslant n$, with linear homogeneous transformation rule. Then the following corollary holds:

COROLLARY. *There are no geometric objects with linear homogeneous transformation rule whose component number $m \geqslant 2$ is not greater than the dimension of manifold and which are essentially of the second class, i.e. whose transformation rule depends non-trivially on the partial derivatives of second order $\partial^2 y^a/\partial x^\beta \partial x^\gamma$.*

References

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