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L^p-MULTIPLIERS FOR THE LAGUERRE EXPANSIONS

BY

JOLANTA DŁUGOSZ (WROCŁAW)

0. Introduction. Let L be the homogeneous sublaplacian on the Heisenberg group H and let $E(\lambda)$ be the corresponding resolution of the identity, i.e.,

$$L = \int_0^\infty \lambda \, dE(\lambda).$$

For a bounded measurable function K on $\boldsymbol{R}_+$ we can define the operator

$$T_K = \int_0^\infty K(\lambda) \, dE(\lambda) \tag{0.1}$$

which is bounded on $L^2(H)$. If T_K is bounded on $L^p(H)$, then K will be called an *L^p-multiplier for the sublaplacian.* We know some sufficient conditions on K to be an L^p-multiplier. For example, the following theorem holds:

THEOREM A. *There exists an $N \in N$ such that if $K \in C^N(\boldsymbol{R}_+)$ and*

$$\sup_{\lambda > 0} |K^{(j)}(\lambda) \lambda^j| \leqslant B < \infty, \quad j = 0, 1, \ldots, N, \tag{0.2}$$

then the operator T_K defined by (0.1) *is bounded on $L^p(H)$,* $1 < p < \infty$, *and*

$$\|T_K\|_{L^p(H), L^p(H)} \leqslant C_p B.$$

We know several proofs of this theorem (see, e.g., [3], [5], [12]). It is also a corollary to multiplier theorems of [9] and [10]. The best N obtained by these authors is equal to $\frac{1}{2}Q + 2$, where Q is the homogeneous dimension of H.

In this paper we prove that every continuous L^p-multiplier for the sublaplacian on the Heisenberg group is an L^p-multiplier for the Laguerre expansions. Related results are in [2], [7], [8] and [12], where (among other things) authors interpret some theorems concerning nilpotent Lie groups in terms of eigenfunction expansions of differential operators on $\boldsymbol{R}^n$. For example, in [8] and [12] theorems concerning summability methods and

L^p-multipliers are obtained for the eigenfunction expansions of the operators on $\boldsymbol{R}$ of the form

$$(-1)^k \frac{d^{2k}}{dx^{2k}} + p(x),$$

where $p(x)$ is a positive polynomial and k is an arbitrary positive integer. In particular, for $k = 1$ we have the Hermite expansion. Proofs of these theorems use the observation that every such operator is an image by a unitary representation of some positive Rockland operator on a homogeneous group (cf. [1] and [8]). These results do not include the Laguerre expansions. We apply to this case a different approach based on eigen-expansions of the sublaplacian acting on the space of functions considered by Geller [6], and on analogy to the classical passage from L^p-multipliers on $\boldsymbol{R}^n$ to those on $\boldsymbol{T}^n$.

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1. Let H_n be the Heisenberg group with underlying manifold $\boldsymbol{R} \times \boldsymbol{C}^n$ and multiplication given by

$$(t, z)(t', z') = (t + t' + 2\operatorname{Im} z\overline{z'}, z + z'), \quad \text{where } z\overline{z'} = \sum_{j=1}^{n} z_j \overline{z'_j}.$$

We sometimes write the element (t, z) as $(t, x, y) \in \boldsymbol{R} \times \boldsymbol{R}^{2n}$, where $z_j = x_j + iy_j$, $j = 1, \ldots, n$.

Let L be the homogeneous sublaplacian on H_n, i.e.,

$$L = -\frac{1}{4} \sum_{j=1}^{n} (X_j^2 + Y_j^2),$$

where X_j (Y_j) is the element of the Lie algebra of H_n corresponding to the one-parameter subgroup $(0, \ldots, 0, t, 0, \ldots, 0)$, t in the x_j (y_j) position. Let $E(\lambda)$ be the spectral resolution of the identity corresponding to L, i.e.,

$$L = \int_0^\infty \lambda \, dE(\lambda).$$

For a bounded measurable function K on $\boldsymbol{R}_+$ we define the operator T_K by

$$T_K = \int_0^\infty K(\lambda) \, dE(\lambda). \tag{1.1}$$

T_K is always bounded on $L^2(H_n)$. Let $1 \leqslant p \leqslant \infty$. If T_K is bounded on $L^p(H_n)$, we say that the function K is an *L^p-multiplier for the sublaplacian.*

Let

$$l_j^k(v) = \left(\frac{j!}{(j+k)!}\right)^{1/2} v^{k/2} e^{-v/2} L_j^k(v), \quad v \in \boldsymbol{R}_+,$$

be the Laguerre function of type k, where $L_j^k(v)$ is the Laguerre polynomial

$$L_j^k(v) = \sum_{p=1}^{j} \binom{j+k}{j-p} \frac{(-v)^p}{p!}.$$

For $\boldsymbol{m} = (m_1, \ldots, m_n) \in \boldsymbol{Z}^n$ and $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n) \in \boldsymbol{N}^n$ put

$$l_{\boldsymbol{\alpha}}^{\boldsymbol{m}}(\boldsymbol{v}) = \sum_{j=1}^{n} l_{\alpha_j}^{|m_j|}(v_j), \qquad \text{where } \boldsymbol{v} = (v_1, \ldots, v_n) \in \boldsymbol{R}_+^n.$$

We define *L^p-multipliers for the Laguerre expansions* as follows:

Let F be a bounded function on N^n. For a function $\varphi \in L^p(\boldsymbol{R}_+^n)$ with the Laguerre expansion

$$\varphi \sim \sum_{\alpha \in N^n} (\varphi, l_\alpha^m) l_\alpha^m \tag{1.2}$$

we define $T_F \varphi$ by

$$T_F \varphi \sim \sum_{\alpha \in N^n} F(\alpha)(\varphi, l_\alpha^m) l_\alpha^m.$$

If T_F is a bounded operator on $L^p(\boldsymbol{R}_+^n)$, then F will be called an *L^p-multiplier for the Laguerre series.*

THEOREM. *Let $1 \leqslant p \leqslant \infty$. If K is a continuous function on $\boldsymbol{R}_+$ and if it is an L^p-multiplier for the sublaplacian on the Heisenberg group H_n, then for every $s > 0$ the operator $\tilde{T}_{K_s}$ defined for a $\varphi \in L^p(\boldsymbol{R}_+^n)$ with the Laguerre expansion* (1.2) *by*

$$\tilde{T}_{K_s} \varphi \sim \sum_{\alpha \in N^n} K\big(s(2|\alpha| + n)\big)(\varphi, l_\alpha^m) l_\alpha^m \tag{1.3}$$

is bounded on $L^p(\boldsymbol{R}_+^n)$ and

$$\|\tilde{T}_{K_s}\|_{L^p(\boldsymbol{R}_+^n), L^p(\boldsymbol{R}_+^n)} \leqslant \|T_K\|_{L^p(H_n), L^p(H_n)}.$$

As an application of the Theorem we present the following

COROLLARY. *Let n be a positive integer. If a function $K \in C^{n+3}(\boldsymbol{R}_+)$ and if it satisfies condition* (0.2), *then for every $s > 0$ the function $\{K\big(s(2|\alpha| + n)\big)\}_{\alpha \in N^n}$ is an L^p-multiplier for the Laguerre series of type $\boldsymbol{m}$ for all p, $1 < p < \infty$, and $\boldsymbol{m} \in N^n$. In addition, we have the following estimation for the norm of the corresponding operator $\tilde{T}_{K_s}$:*

$$\|\tilde{T}_{K_s}\|_{L^p(\boldsymbol{R}_+^n), L^p(\boldsymbol{R}_+^n)} \leqslant C_p B,$$

where the constant C_p does not depend on K, s and $\boldsymbol{m}$.

2. In this section we investigate the action of L on spaces of functions of the form

$$f(t, z) = \exp\big(i \sum_{j=1}^{n} m_j \theta_j\big) f_0(t, r_1, \ldots, r_n), \tag{2.1}$$

where $\boldsymbol{m} = (m_1, \ldots, m_n) \in \boldsymbol{Z}^n$ and $z_j = r_j \exp(i\theta_j)$, $j = 1, \ldots, n$. These spaces were considered by Geller [6].

For a multi-index $\boldsymbol{m} = (m_1, \ldots, m_n) \in \boldsymbol{Z}^n$ we use the notation

$$\boldsymbol{m}^+ = (m_1^+, \ldots, m_n^+) \quad \text{and} \quad \boldsymbol{m}^- = (m_1^-, \ldots, m_n^-),$$

where $m_j^+ = \max(m_j, 0)$ and $m_j^- = \max(-m_j, 0)$, $j = 1, \ldots, n$.

LEMMA 1. *The functions*

$$\varphi_{\alpha,\lambda}^{\boldsymbol{m}}(t, z) = \exp\Big(i \sum_{j=1}^{n} m_j \theta_j\Big) l_{\alpha}^{\boldsymbol{m}}(2|\lambda| \boldsymbol{r}^2) e^{-i\lambda t}, \tag{2.2}$$

where $\boldsymbol{r} = (r_1^2, \ldots, r_n^2)$, *are eigenfunctions of* L *on* H_n *with the corresponding eigenvalues* $|\lambda|(2|\boldsymbol{\alpha} + \boldsymbol{m}^-| + n)$ *for* $\lambda < 0$ *and* $|\lambda|(2|\boldsymbol{\alpha} + \boldsymbol{m}^+| + n)$ *for* $\lambda > 0$.

Proof. It is routine to verify this writing L in polar coordinates

$$x_j = r_j \cos\theta_j,$$
$$y_j = r_j \sin\theta_j,$$

$j = 1, \ldots, n$, and using the second order differential equation satisfied by $L_j^k(v)$ (see [4]):

$$v \frac{d^2}{dv^2} L_j^k(v) + (k + 1 - v) \frac{d}{dv} L_j^k(v) + j L_j^k(v) = 0.$$

Let $L_{\boldsymbol{m}}^2(H_n)$ be the subspace of $L^2(H_n)$ of functions of the form (2.1). Denote by $\mathscr{F}_1(f_0)$ the Fourier transform of $f_0 \in L^2(H_n)$ with respect to the central variable, i.e.,

$$\mathscr{F}_1(f_0)(\lambda, r_1, \ldots, r_n) = \int_{-\infty}^{\infty} e^{i\lambda t} f_0(t, r_1, \ldots, r_n)\, dt \tag{2.3}$$

for $f_0 \in L^1 \cap L^2(H_n)$. For $\boldsymbol{m} \in \boldsymbol{Z}^n$ and $\boldsymbol{\alpha} \in \boldsymbol{N}^n$ let (cf. [6])

$$R_{\alpha}^{\boldsymbol{m}}(\lambda, f) = (2\pi)^n \int_0^{\infty} \ldots \int_0^{\infty} (\mathscr{F}_1 f_0)(\lambda, r_1, \ldots, r_n) \times \prod_{j=1}^{n} l_{\alpha_j}^{|m_j|}(2|\lambda| r_j^2) r_j\, dr_j. \tag{2.4}$$

We have for $f \in L_{\boldsymbol{m}}^2(H_n)$

$$(\mathscr{F}_1 f_0)(\lambda, r_1, \ldots, r_n) = \left(\frac{2|\lambda|}{\pi}\right)^n \sum_{\alpha \in \boldsymbol{N}^n} R_{\alpha}^{\boldsymbol{m}}(\lambda, f)\, l_{\alpha}^{\boldsymbol{m}}(2|\lambda| \boldsymbol{r}^2) \tag{2.5}$$

(the series being convergent in $L^2(\boldsymbol{R}_+^n)$ for a.e. λ). For $f \in \mathscr{S}(H_n)$ of the form (2.1) we have

$$f_0(t, r_1, \ldots, r_n) = \left(\frac{2}{\pi}\right)^n \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\lambda t} \sum_{\alpha \in \boldsymbol{N}^n} R_{\alpha}^{\boldsymbol{m}}(\lambda, f)\, l_{\alpha}^{\boldsymbol{m}}(2|\lambda| \boldsymbol{r}^2) |\lambda|^n\, d\lambda \tag{2.6}$$

and for $f \in L^2_{\boldsymbol{m}}(H_n)$

$$2\pi \|f\|^2_{L^2(H_n)} = \left(\frac{2}{\pi}\right)^n \int_{-\infty}^{\infty} \sum_{\alpha \in N^n} |R^{\boldsymbol{m}}_\alpha(\lambda, f)|^2 |\lambda|^n d\lambda. \tag{2.7}$$

LEMMA 2. *The spaces $L^2_{\boldsymbol{m}}(H_n)$ are preserved by the operators T_K of the form* (1.1) *and for $f \in L^2_{\boldsymbol{m}}(H_n)$ we have*

$$R^{\boldsymbol{m}}_\alpha(\lambda, T_K f) = K^{\boldsymbol{m}}_\alpha(\lambda) R^{\boldsymbol{m}}_\alpha(\lambda, f) \text{ a.e.}, \tag{2.8}$$

where

$$K^{\boldsymbol{m}}_\alpha(\lambda) = \begin{cases} K\big(|\lambda|(2|\alpha + \boldsymbol{m}^-| + n)\big) & \text{for } \lambda < 0, \\ K\big(|\lambda|(2|\alpha + \boldsymbol{m}^+| + n)\big) & \text{for } \lambda > 0. \end{cases} \tag{2.9}$$

Proof. Assume first that T_K is a convolution operator by an L^1 radial function (i.e., a function satisfying (2.1) with $\boldsymbol{m} = (0, \ldots, 0)$). It is obvious that T_K preserves the spaces $L^2_{\boldsymbol{m}}(H_n)$. From Lemma 1 we have

$$T_K \varphi^{\boldsymbol{m}}_{\alpha,\lambda} = K^{\boldsymbol{m}}_\alpha(\lambda) \varphi^{\boldsymbol{m}}_{\alpha,\lambda}.$$

Hence from the formula (2.6) it follows that if

$$\int_{-\infty}^{\infty} \sum_{\alpha \in N^n} |R^{\boldsymbol{m}}_\alpha(\lambda, f)| |\lambda|^n d\lambda < \infty, \tag{2.10}$$

then

$$\begin{aligned} T_K f(t, z_1, \ldots, z_n) = \exp\Big(i \sum_{j=1}^n m_j \theta_j\Big) \left(\frac{2}{\pi}\right)^n \frac{1}{2\pi} \\ \times \int_{-\infty}^{\infty} e^{-i\lambda t} \sum_{\alpha \in N^n} K^{\boldsymbol{m}}_\alpha(\lambda) R^{\boldsymbol{m}}_\alpha(\lambda, f) l^{\boldsymbol{m}}_\alpha(2|\lambda| r^2) |\lambda|^n d\lambda. \end{aligned} \tag{2.11}$$

Since the formula (2.10) is satisfied for functions in $\mathscr{S}(H_n)$, it holds for a dense subset of $L^2_{\boldsymbol{m}}(H_n)$. Thus the formula (2.8) holds for all $f \in L^2_{\boldsymbol{m}}(H_n)$ in virtue of the fact that the map $f \to R^{\boldsymbol{m}}_\alpha(\lambda, f)$ from $L^2_{\boldsymbol{m}}(H_n)$ to $L^2(\boldsymbol{R}, |\lambda|^n d\lambda)$ is continuous (cf. formula (2.7)).

Now, if $K \in L^\infty(\boldsymbol{R}_+)$ is arbitrary, we can find a sequence of functions K_j in $L^\infty(\boldsymbol{R}_+)$ such that T_{K_j} are convolutions by L^1 radial functions, T_{K_j} tends to T_K in strong operator topology, and $K_j(\lambda)$ tends to $K(\lambda)$ for almost all λ. (To do this we can use, e.g., [7], Theorem 1.12.)

3. Proof of the Theorem. To simplify the notation we present the proof for the case $n = 1$ only, the modifications for $n > 1$ being obvious. The corresponding group H_1 will be denoted by H.

Assume first that $1 < p < \infty$. Let m be a non-negative integer and let

$s > 0$. Consider the functions f and g on H of the forms

$$f(t, z) = f_0(|z|)\,e^{im\theta}e^{ist}, \qquad g(t, z) = g_0(|z|)\,e^{im\theta}e^{ist}, \tag{3.1}$$

where $z = |z|\,e^{i\theta}$ and

$$f_0(|z|) \in L^2 \cap L^p(C), \qquad g_0(|z|) \in L^2 \cap L^q(C), \qquad 1/p + 1/q = 1.$$

For $\delta > 0$ define the function

$$w_\delta(t) = \exp(-\pi\delta t^2), \qquad t \in \boldsymbol{R}.$$

We have

$$f(t, z)\,w_{\varepsilon\beta}(t) \in L^2 \cap L^2(H), \qquad g(t, z)\,w_{\varepsilon\gamma}(t) \in L^2 \cap L^q(H)$$

for ε, β, $\gamma > 0$. Now, by the assumption, the operator T_K defined by (1.1) is bounded on $L^p(H)$. Consequently,

$$\Big|\int_H T_K(fw_{\varepsilon\beta})\,\bar{g}w_{\varepsilon\gamma}\Big| \leqslant \|T_K\|_{L^p(H),L^p(H)}\,\|f_0\|_{L^p(C)}\,\|g_0\|_{L^q(C)}\,\|w_{\varepsilon\beta}\|_{L^p(\boldsymbol{R})}\,\|w_{\varepsilon\gamma}\|_{L^q(\boldsymbol{R})}. \tag{3.2}$$

Put $\beta = 1/p$, $\gamma = 1/q$ and multiply both sides of (3.2) by $\varepsilon^{1/2}$. On the right-hand side we obtain the expression

$$\Big(\varepsilon^{1/2}\int_{\boldsymbol{R}}\exp(-\pi\varepsilon t^2\,dt)\Big)^{1/p}\Big(\varepsilon^{1/2}\int_{\boldsymbol{R}}\exp(-\pi\varepsilon t^2)\,dt\Big)^{1/q} = 1.$$

On the left-hand side we have, by the Plancherel formula (cf. (2.7)),

$$\varepsilon^{1/2}\int_H T_K(fw_{\varepsilon\beta})\,\bar{g}w_{\varepsilon\gamma} = \varepsilon^{1/2}\frac{1}{2\pi}\frac{2}{\pi}\sum_{\alpha\in N}\int_{-\infty}^{\infty} R_\alpha^m(\lambda, T_K(fw_{\varepsilon\beta}))\,\overline{R_\alpha^m(\lambda, gw_{\varepsilon\gamma})}\,|\lambda|\,d\lambda \tag{3.3}$$

(we have changed the order of integration and summation, which is permissible for functions from $L^2(H)$). Now, by the definition of R_α^m (cf. (2.4))

$$\begin{aligned} R_\alpha^m(\lambda, gw_{\varepsilon\gamma}) &= 2\pi\int_{-\infty}^{\infty} e^{i\lambda t}\exp(-\pi\varepsilon\gamma t^2)\,e^{ist}\,dt\int_0^{\infty} g_0(r)\,l_\alpha^m(2|\lambda|\,r^2)\,r\,dr \\ &= 2\pi(\varepsilon\gamma)^{-1/2}\exp(-|\lambda+s|^2/4\pi\varepsilon\gamma)\int_0^{\infty} g_0(r)\,l_\alpha^m(2|\lambda|\,r^2)\,r\,dr. \end{aligned}$$

By Lemma 2 we know that

$$R^m(\lambda, T_K(fw_{\varepsilon\beta})) = K_\alpha^m(\lambda)\,R_\alpha^m(\lambda, fw_{\varepsilon\beta}).$$

Thus, the expression (3.3) is equal to

(3.4) $$\frac{1}{2\pi}\frac{2}{\pi}(2\pi)^2 \sum_{\alpha\in N} \int_0^\infty K_\alpha^m(\lambda) \int_0^\infty f_0(r)\, l_\alpha^m(2|\lambda| r^2)\, r dr$$
$$\times \int_0^\infty \overline{g_0(r)\, l_\alpha^m(2|\lambda| r^2)\, r dr}\, (\varepsilon\beta\gamma)^{-1/2} \exp(-|\lambda+s|^2/4\pi\varepsilon\beta\gamma)\, |\lambda|\, d\lambda$$

(we recall that $1/\beta + 1/\gamma = 1/\beta\gamma$, since $\beta = 1/p$, $\gamma = 1/q$).

Observe that the integral with respect to λ in the above expression is, in fact, the convolution estimated at the point $-s$ of some function of λ continuous for $\lambda \neq 0$ with the Gauss–Weierstrass kernel

$$W_{\varepsilon\beta\gamma}(\lambda) = \frac{1}{2\pi}(\varepsilon\beta\gamma)^{-1/2} \exp(-|\lambda|^2/4\pi\varepsilon\beta\gamma).$$

Observe also that for $f_0, g_0 \in L^2(\boldsymbol{R}_+, rdr)$ the absolute value of the remainder of the series is small independently of $\varepsilon > 0$. Therefore, letting $\varepsilon \to 0$ in (3.4), we can change the order of $\lim_{\varepsilon\to 0}$ and $\sum_{\alpha\in N}$. Thus, in virtue of the inequality (3.2) and the whole consideration, we obtain finally

(3.5) $$\Big|8\pi \sum_{\alpha\in N} K_\alpha^m(-s) \int_0^\infty f_0(r)\, l_\alpha^m(2sr^2)\, r dr$$
$$\times \int_0^\infty \overline{g_0(r)\, l_\alpha^m(2sr^2)\, r dr}\, s\Big| \leqslant \|T_K\|_{L^p(H),L^p(H)} \|f_0\|_{L^p(C)} \|g_0\|_{L^q(C)}.$$

Define the functions φ and ψ on $\boldsymbol{R}_+$ as follows:

$$\varphi(u) = f_0\left(\sqrt{\frac{u}{2s}}\right) \quad \text{and} \quad \psi(u) = g_0\left(\sqrt{\frac{u}{2s}}\right).$$

We have

(3.6) $$\|f_0\|_{L^p(C)} = \Big(2\pi \int_0^\infty |f_0(r)|^p r dr\Big)^{1/p}$$
$$= \left(\frac{\pi}{2s}\int_0^\infty |\varphi(u)|^p du\right)^{1/p} = \left(\frac{\pi}{2s}\right)^{1/p} \|\varphi\|_{L^p(\boldsymbol{R}_+)}$$

and similarly for g_0. Changing the variable in the same way on the left-hand side of (3.5), putting $K_\alpha^m(-s) = K(s(2\alpha+1))$ (cf. Lemma 2), and then dividing both sides by $\pi/2s$ we obtain

$$\Big|\sum_{\alpha\in N} K(s(2\alpha+1))(\varphi, l_\alpha^m)(\bar{\psi}, l_\alpha^m)\Big| \leqslant \|T_K\|_{L^p(H),L^p(H)} \|\varphi\|_{L^p(\boldsymbol{R}_+)} \|\psi\|_{L^q(\boldsymbol{R}_+)}$$

for $\varphi = L^2 \cap L^p(\boldsymbol{R}_+)$, $\psi \in L^2 \cap L^q(\boldsymbol{R}_+)$. Now the Theorem follows for $1 < p < \infty$ by the usual density arguments.

The case $p = 1$ is even simpler. Let $f \in L^1 \cap L^2(H)$ be of the form

$$f(t, z) = e^{im\theta} f_0(|z|) h(t),$$

where $z = |z| e^{i\theta}$ and m is a non-negative integer. Since T_K is an L^1-multiplier and it preserves the space $L^2_m(H)$, so $T_K f \in L^1 \cap L^2_m(H)$. Thus

$$T_K f(t, z) = e^{im\theta} (T_K f)_0(t, |z|)$$

and

$$(3.7)\qquad \mathscr{F}_1 (T_K f)_0(\lambda, r) = \frac{2|\lambda|}{\pi} \sum_{\alpha \in N} R^m_\alpha(\lambda, T_K f)\, l^m_\alpha(2|\lambda| r^2)$$

$$= \frac{2|\lambda|}{\pi} \sum_{\alpha \in N} K^m_\alpha(\lambda) R^m_\alpha(\lambda, f)\, l^m_\alpha(2|\lambda| r^2)$$

$$= \frac{2|\lambda|}{\pi} 2\pi \sum_{\alpha \in N} K^m_\alpha(\lambda) \int_{-\infty}^{\infty} h(t) e^{i\lambda t} dt \int_0^\infty f_0(r_1)\, l^m_\alpha(2|\lambda| r_1^2) r_1\, dr_1 \times l^m_\alpha(2|\lambda| r^2)$$

by (2.5), Lemma 2 and (2.4). Now, for any $\lambda \in \boldsymbol{R}$,

$$(3.8)\qquad \int_0^\infty |\mathscr{F}_1 (T_K f)_0(\lambda, r)|\, r\, dr \leqslant \int_0^\infty \int_{-\infty}^{\infty} |T_K f(t, r)|\, dt\, r\, dr$$

$$= \frac{1}{2\pi} \|T_K f\|_{L^1(H)} \leqslant \frac{1}{2\pi} \|T_K\|_{L^1(H), L^1(H)} \|f_0\|_{L^1(\boldsymbol{C})} \|h\|_{L^1(\boldsymbol{R})}.$$

Let $s > 0$ and let $h(t) = \exp(-\pi t^2) e^{ist}$. We have

$$\int_{-\infty}^{\infty} h(t) e^{i\lambda t}\, dt = \exp(-|\lambda + s|^2/4\pi) \quad \text{and} \quad \|h\|_{L^1(\boldsymbol{R})} = 1.$$

Thus for $\lambda = -s$, in vritue of (3.7), the inequality (3.8) takes the form

$$4s \int_0^\infty \Big| \sum_{\alpha \in N} K\big(s(2\alpha + 1)\big) \int_0^\infty f_0(r_1)\, l^m_\alpha(2sr_1^2) r_1\, dr_1\, l^m_\alpha(2sr^2)\, r\, dr \Big|$$

$$\leqslant \frac{1}{2\pi} \|T_K\|_{L^1(H), L^1(H)} \|f_0\|_{L^1(\boldsymbol{C})}.$$

Define as for $p > 1$ the function

$$\varphi(u) = f_0\left(\sqrt{\frac{u}{2s}}\right).$$

By considerations analogous to the previous ones, the last inequality is equivalent to

$$\Big\|\sum_{\alpha\in N} K\big(s(2\alpha+1)\big)(\varphi,\, l_\alpha^m)\, l_\alpha^m\Big\|_{L^1(R_+)} \leqslant \|T_K\|_{L^1(H),L^1(H)}\|\varphi\|_{L^1(R_+)}$$

for $\varphi\in L^1\cap L^2(\boldsymbol{R}_+)$. This proves the case $p=1$.

If T_K is an L^∞-multiplier for the sublaplacian on H, then it is an L^1-multiplier. Then from the case $p=1$ we infer that the operator $\tilde{T}_{K_s}$ defined by (1.3) is bounded on $L^1(\boldsymbol{R}_+)$ and, consequently, it is bounded on $L^\infty(\boldsymbol{R}_+)$. Thus the Theorem is established.

Remark. The idea of the proof is taken from [11], pp. 260–263, the proof of Theorem 3.8.

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