SUBDIRECT PRODUCTS OF CHAINS

BY

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1. Introduction. Let K be a chain and let $\mathscr{C}_K$ be the class of (distributive) lattices which are subdirect products of copies of K. In particular, $\mathscr{C}_n$, $n \geqslant 2$, will denote the class of lattices which are subdirect products of n-element chains. In [1], Anderson and Blair have given necessary and sufficient conditions for a lattice to belong to $\mathscr{C}_K$. Using their results, Balbes and Dwinger [2] have shown that $L \epsilon \mathscr{C}_3$ if and only if L does not have a non-degenerate Boolean algebra as a direct factor.

In the present paper* we will be concerned more generally with subdirect products of chains which are not necessarily finite and not necessarily isomorphic. It seems impossible to generalize the result quoted above for $\mathscr{C}_3$ to the class $\mathscr{C}_K$, where K is a chain of more than 3 elements. Indeed, the conditions stated in [1] seem to be the best possible ones in the case K has more than 3 elements.

Our objective in this paper is to exhibit some important classes of lattices whose members belong to $\mathscr{C}_K$, where K is an arbitrary chain. For example, the free distributive lattice on a set S of generators belongs to $\mathscr{C}_K$, where K is a chain whose cardinality is that of S (see Corollary 6). More generally, the free product of a set $\{L_s\}_{s \epsilon S}$ of lattices belongs to $\mathscr{C}_K$, where $|K| = |S|$ (see Theorem 5). We will, in particularly, focus our attention on the class $\mathscr{C}_\infty$ of lattices which are subdirect products of (infinite) chains without extreme elements. Besides the infinite free products of distributive lattices, every distributive lattice without maximal or minimal prime ideals, every homogeneous distributive lattice (homogeneous in the sense of Section 4) and every non-trivial l-group belong to $\mathscr{C}_\infty$ (see Theorems 8, 10 and 11). We note that if $L \epsilon \mathscr{C}_\infty$, then L has obviously no relatively complemented elements, but the converse is not true. Indeed, if Z denotes the chain of integers, then $Z \times 2$ has no relatively complemented elements but $Z \times 2 \notin \mathscr{C}_3$, and thus $Z \times 2 \notin \mathscr{C}_\infty$.

We will always, when using the term lattice, mean the lattice to be distributive. The category in which we work is the category of distributive lattices and lattice homomorphisms. A prime ideal of a lattice is defined as usual but, for the sake of convenience, we will consider the lattice itself also as a prime ideal. A proper prime ideal I is *maximal* (*minimal*) if it is maximal (minimal) in the partially ordered set of proper prime ideals. The smallest element of a partially ordered set, if it exists, will be denoted by the symbol 0 and we will use, without danger of confusion, the same symbol for the smallest element of different partially ordered sets under consideration. The same applies to the symbol 1 for the largest element. The symbol n will always denote an integer $n \geqslant 1$ and at the same time the chain of integers $\{0, 1, \ldots, n-1\}$. The dual of an ordinal a will be denoted by a^* and the ordinal sum of the chains K and K' by $K \oplus K'$. Finally, ω is the chain of positive integers.

We will require the following definition (cf. [1]):

Definition 1. Let L be a lattice and let K be a chain. A chain C of prime ideals of L (ordered by the set inclusion) is a *K-chain* if (i) C is isomorphic to K; (ii) for each $a \in L$, there exists a smallest $P(a) \in C$ such that $a \in P(a)$; (iii) for each $P \in C$, there exists an $a \in L$ such that $P(a) = P$.

Our main tool will be the following theorem which is a slight generalization of Theorem 1 in [1] and, therefore, we omit the proof.

THEOREM 2. *Let L be a lattice, $|L| \geqslant 2$, and let $\mathcal{K}$ be a class of chains. Then L is a subdirect product of members of $\mathcal{K}$ if and only if, for each $a, b \in L$, $a < b$, there exists a $K \in \mathcal{K}$ and a K-chain of prime ideals in L such that a and b are in distinct prime ideals.*

2. Free products and free lattices.
We start this section with Lemmas 3 and 4.

LEMMA 3. *Suppose L is the free product of a set $\{L_s\}_{s \in S}$ of lattices. Let, for each $s \in S$, $\mathcal{P}_s$ denote the partially ordered set of prime ideals of L_s to which is adjoined the void subset of L_s, and let $\mathcal{P}$ denote the partially ordered set of prime ideals of L to which is adjoined the void subset of L. Then $\mathcal{P}$ and $\underset{s \in S}{\mathsf{X}} \mathcal{P}_s$ are isomorphic.*

Proof. Define

$$\Phi: \mathcal{P} \to \underset{s \in S}{\mathsf{X}} \mathcal{P}_s$$

by

$$\big(\Phi(P)\big)_s = P \cap L_s \quad \text{for each } s \in S \text{ and } P \in \mathcal{P}.$$

Obviously, $P \subseteq P'$ implies $\Phi(P) \subseteq \Phi(P')$.

Suppose, conversely, $\Phi(P) \subseteq \Phi(P')$. Let $h: L \to 2$ and $h': L \to 2$ be the homomorphisms such that $h^{-1}\{0\} = P$ and $h'^{-1}\{0\} = P'$. If $a \in P$,

then a can be represented by

$$a = \prod A_0 + \prod A_1 + \ldots + \prod A_{n-1},$$

where, for each i, $i < n$, A_i is a non-void finite subset of $\bigcup_{s \in S} L_s$.

Since $h(a) = 0$, we have $h(\prod A_i) = 0$ for $i < n$, and thus each A_i contains an element which is in P and, therefore, by hypotheses, $h'(\prod A_i) = 0$ for $i < n$. It follows that $h'(a) = 0$, and thus $a \in P'$.

Finally, to show that Φ is onto, notice that if $Q \in \underset{s \in S}{\mathsf{X}} \mathscr{P}_s$, then there exists a (unique) $P \in \mathscr{P}$ such that $\Phi(P) = Q$.

LEMMA 4. *Let $A_0, A_1, \ldots, A_{n-1}$, $n \geqslant 2$, be partially ordered sets each with a smallest and a largest element. If*

$$A = \underset{i < n}{\mathsf{X}} A_i \quad and \quad x \in A, \ x \neq 0, 1,$$

then x is contained in a chain $0 \neq y_0 < y_1 < \ldots < y_{n-1} = 1$.

Proof. Since $x \neq 0$, we may assume that there exists an integer k, $0 \leqslant k \leqslant n-1$, such that $x(i) \neq 0$ if and only if $0 \leqslant i \leqslant k$.

If $k < n-1$, define y_j for $0 \leqslant j \leqslant n-2$ by

$$y_j(i) = \begin{cases} 0 & \text{for } j < i, \\ x(i) & \text{for } j \geqslant i \text{ and } k \geqslant i, \\ 1 & \text{for } j \geqslant i > k. \end{cases}$$

If $k = n-1$, define y_j for $0 \leqslant j \leqslant n-2$ by

$$y_j(i) = \begin{cases} x(i) & \text{for } 0 \leqslant i \leqslant j+1, \\ 0 & \text{for } j+1 < i. \end{cases}$$

It is easy to see that in either case $0 \neq y_0 < y_1 \ldots < y_{n-2} < 1$ and that, for $k < n-1$, $y_k = x$ and, for $k = n-1$, $y_{n-2} = x$.

We now state the following theorem:

THEOREM 5. *Let L be the free product of a set $\{L_s\}_{s \in S}$ of lattices, $|S| \geqslant 2$. If K is a chain of cardinality $|S|$, then $L \in \mathscr{C}_K$.*

Proof. Suppose, first, that S is finite; thus L is the free product of $L_0, L_1, \ldots, L_{n-1}$, $n \geqslant 2$. Let $a, b \in L$, $a < b$, and let Q be a proper prime ideal of L such that $a \in Q, b \notin Q$. Since $Q \neq \varnothing$, $Q \neq L$, it follows from Lemmas 3 and 4 that Q is contained in an n-chain $P_0 \subset P_1 \subset \ldots \subset P_{n-1} = L$ of prime ideals. It follows from Theorem 2 that $L \in \mathscr{C}_n$.

Next, suppose that S is infinite. It suffices to show that, for $a, b \in L$, $a < b$, there exists a homomorphism $h \colon L \to K$ which is onto and such that $h(a) \neq h(b)$. a and b are contained in a sublattice L' of L which is generated by $\bigcup_{s \in S_1} L_s$, where S_1 is a non-void finite subset of S. Let $g \colon L' \to 2$ be a homo-

morphism from L' onto a 2-element subchain of K such that $g(a) \neq g(b)$. Since $|K| = |S|$ and S is infinite, g can be extended to a homomorphism $h \colon L \to 2$ which is the desired homomorphism.

If L is free on a set S of generators, then L is the coproduct of a set of cardinality $|S|$ of 1-element lattices. Hence Theorem 5 yields the following corollary:

COROLLARY 6. *Let L be free on a set S, $|S| \geqslant 2$. Then $L \in \mathscr{C}_K$, where* $|K| = |S|$.

3. Lattices without maximal (minimal) proper prime ideals. In this section we consider the class of lattices without maximal (minimal) proper prime ideals and we show that they are subdirect products of infinite chains. Precisely, we have the following theorem:

THEOREM 7. *Suppose L, $|L| \geqslant 2$, is a lattice which has no maximal (minimal) proper prime ideals. Then L is a subdirect product of (dual) limit ordinals.*

Proof. Suppose L has no maximal proper prime ideals. By Theorem 2, it suffices to show that if P is a proper prime ideal of L, then P is contained in a γ-chain of prime ideals, where γ is a limit ordinal. Let δ be an ordinal such that $|\delta| > |L|$. Define, for each $a < \delta$, a prime ideal P of L as follows:

Let $P_0 = P$. Suppose a is an ordinal such that $0 < a < \delta$ and such that P_β has been defined for $\beta < a$ and such that $\beta_1 \leqslant \beta_2 < a$ implies $P_{\beta_1} \subseteq P_{\beta_2}$. If

$$\bigcup_{\beta < a} P_\beta \neq L,$$

then $\bigcup_{\beta < a} P_\beta$ is a proper prime ideal and we define P_a to be a proper prime ideal properly containing $\bigcup_{\beta < a} P_\beta$, which exists by hypothesis. In the case of

$$\bigcup_{\beta < a} P_\beta = L,$$

let $P_a = L$.

It is easy to see that there exists an ordinal, and thus a smallest ordinal $\gamma < \delta$ such that $P_\gamma = L$. Also, it easily follows that γ is a limit ordinal, and thus

$$\bigcup_{a < \gamma} P_a = L \quad \text{and} \quad P_a \neq L \quad \text{for } a < \gamma.$$

Also, if $a_1, a_2 < \gamma$, then $P_{a_1} \subseteq P_{a_2}$ if and only if $a_1 \leqslant a_2$. Consider the chain $C = \{P_a\}_{a < \gamma}$. We claim that C is a γ-chain of prime ideals. Certainly, C and γ are isomorphic. Suppose $a \in L$; then, since

$$\bigcup_{a < \gamma} P_a = L,$$

there is a smallest a, say $a_0 < \gamma$, such that $a \in P_{a_0}$ and P_{a_0} is the smallest element of C containing a.

Finally, in order to show that if $a < \gamma$, there is an $a \in L$ such that P_a is the smallest member of C containing a, pick, if $a = 0$, for a any element of P_0, and, if $a > 0$, pick for a an element of

$$P_a \sim \bigcup_{\beta < a} P_\beta.$$

It follows that C is an a-chain containing $P = P_0$ as a (first) element.

The case where L has no minimal proper prime ideals follows from a dual argument.

We state the following theorem without the proof:

THEOREM 8. *Suppose L, $|L| \geqslant 2$, is a lattice in which every proper prime ideal is neither maximal nor minimal. Then L is a subdirect product of chains of the type $a^* \oplus \beta$, where a and β are limit ordinals.*

Remark. If $L \in \mathscr{C}_K$, where K is a chain of more than 2 elements, then it is not necessary that L has no proper prime ideals which are both maximal and minimal. Feinstein [4] has constructed, for each $n \geqslant 3$, a lattice L which belongs to $\mathscr{C}_n$ and which has a proper prime ideal which is both maximal and minimal. On the other hand, it is also shown in [4] that if $L \in \mathscr{C}_n$, $n \geqslant 3$, and L is finite, then L has no proper prime ideals which are both minimal and maximal.

4. Homogeneous lattices.

Definition 9. A lattice L is *homogeneous* if, for each a, $b \in L$, there exists an automorphism of L such that $f(a) = b$ and such that if $a < b$, then $x < f(x)$ for each $x \in L$, and if a and b are incomparable, then x and $f(x)$ are incomparable for each $x \in L$.

We note that this notion of homogeneity is a stronger one than that in the sense of having a transitive group of automorphisms. (Berman [3] has shown that there are non-distributive lattices which have a transitive group of automorphisms but which are not homogeneous in the sense of Definition 9. It is also shown in [3] that a lattice which is homogeneous in the sense of Definition 9 must be distributive.)

Homogeneous lattices are subdirect products of infinite chains and we have the following theorem:

THEOREM 10. *Let L be homogeneous, $|L| \geqslant 2$. Then L is a subdirect product of (infinite) chains without extreme elements.*

Proof. Let P be a proper prime ideal of L. Pick $a \in P$ and $b \notin P$ such that $a < b$. By homogeneity of L, there exists an automorphism f of L such that $f(a) = b$ and such that $x < f(x)$ for each $x \in L$. Obviously, $f[P]$ is a proper prime ideal of L. If $x \in P$, then, since $x < f(x)$, we have $x \in f[P]$. Also, $b \in f[P] \sim P$, whence $P \subset f[P]$. Again, $f^{-1}[P]$ is a proper prime ideal and $a \notin f^{-1}[P]$, whence $f^{-1}[P] \subset P$. It follows that P is neither maximal nor minimal and the application of Theorem 8 completes the proof.

5. The class $\mathscr{C}_\infty$. In the previous sections we have exhibited several classes of lattices which are subdirect products of infinite chains, in fact of infinite chains without extreme elements. In this section we will produce one more class of lattices of this type and, in addition, we will investigate the relationship between the various classes discussed in this paper. First, we introduce the following notation:

$\mathscr{C}_\infty$ is the class of subdirect product of infinite chains without extreme elements;

$\mathscr{C}_M$ is the class of lattices L, $|L| \geqslant 2$, in which every proper prime ideal is neither maximal nor minimal;

$\mathscr{C}_H$ is the class of homogeneous lattices L, $|L| \geqslant 2$;

$\mathscr{C}_L$ is the class of non-trivial l-groups (thus of those lattices L, $|L| \geqslant 2$, which admit a group structure).

We now have the following theorem:

THEOREM 11.

$$\mathscr{C}_L \subseteq \mathscr{C}_H \subset \mathscr{C}_M \subset \mathscr{C}_\infty \subset \bigcap_{n=2}^{\infty} \mathscr{C}_n \subset \ldots \subset \mathscr{C}_{n+1} \subset \mathscr{C}_n \subset \ldots \subset \mathscr{C}_3 \subset \mathscr{C}_2.$$

Proof. It is obvious that $\mathscr{C}_{n+1} \subset \mathscr{C}_n$ for each $n \geqslant 2$ and that $\bigcap_{n=2}^{\infty} \mathscr{C}_n \subseteq \mathscr{C}_n$ for each $n \geqslant 2$. Also, $\omega \oplus 1 \in \mathscr{C}_n$ for each $n \geqslant 2$ but $\omega \oplus 1 \notin \mathscr{C}_\infty$, whence

$$\mathscr{C}_\infty \subset \bigcap_{n=2}^{\infty} \mathscr{C}_n \quad (^1).$$

The inclusion $\mathscr{C}_M \subseteq \mathscr{C}_\infty$ follows from Theorem 8. On the other hand, if L is an infinite free lattice, then $L \notin \mathscr{C}_M$ (cf. Theorem 5) but $L \in \mathscr{C}_\infty$ (see Corollary 6); thus $\mathscr{C}_M \subset \mathscr{C}_\infty$. Next, it follows from the proof of Theorem 10 that $\mathscr{C}_H \subseteq \mathscr{C}_M$ and, in addition, since, obviously, $Z \oplus Q \in \mathscr{C}_M \sim \mathscr{C}_H$ (Q are rationals), we infer that $\mathscr{C}_H \subset \mathscr{C}_M$.

Finally, in order to show that $\mathscr{C}_L \subseteq \mathscr{C}_H$, let $L \in \mathscr{C}_L$ and suppose $a, b \in L$. The map $f\colon L \to L$ defined by $f(x) = b \circ a^{-1} \circ x$ (o is the group multiplication) is a (lattice) automorphism and it is easy to see that f satisfies the conditions of Definition 9. This completes the proof of the theorem.

Remark. The fact that $\mathscr{C}_L \subseteq \mathscr{C}_\infty$ can also be derived from the proof of Holland's [5] theorem that every l-group is a sub-l-group of automorphisms of a chain. It is an open question whether $\mathscr{C}_L \subset \mathscr{C}_H$. (P 887)

REFERENCES

[1] F. W. Anderson and R. L. Blair, *Representations of distributive lattices as lattices of functions*, Mathematische Annalen 143 (1961), p. 187-211.

(^{1}) This example was provided by Professor G. Grätzer (oral communication).

[2] R. Balbes and Ph. Dwinger, *A note on subdirect products of finite chains*, Algebra Universalis 1 (1971), p. 125-127.

[3] J. Berman, *Homogeneous lattices and lattice ordered groups*, Colloquium Mathematicum (to appear).

[4] D. C. Feinstein, Thesis, University of Illinois at Chicago Circle, 1972.

[5] C. Holland, *The lattice ordered group of automorphisms of an ordered set*, The Michigan Mathematical Journal 10 (1963), p. 399-408.

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