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ON INFINITESIMAL DEFORMATIONS OF SUBMANIFOLDS OF A RIEMANNIAN MANIFOLD

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§ 0. Introduction

Recently infinitesimal deformations of submanifolds of a Riemannian manifold have been studied by K. Yano [6], [2], B. Y. Chen [2], S. Tachibana [5], R. A. Goldstein and P. I. Ryan [3] and other authors.

Let M^m be an m-dimensional compact orientable submanifold of an n-dimensional orientable Riemannian manifold M^n. In the present paper conditions have been found in which the submanifold M^m does not allow non-trivial infinitesimal isometric and non-trivial infinitesimal conformal deformations.

In § 1, we give some known results on the infinitesimal deformations of submanifolds and some definitions needed for the later discussions.

In § 2, we consider infinitesimal isometric deformations of the submanifold M^m. We find conditions in which M^m is rigid.

In § 3, we find conditions in which the submanifold M^m is rigid with respect to infinitesimal conformal deformations. All manifolds are assumed connected.

§ 1. Preliminaries

Let M^n be an n-dimensional connected Riemannian manifold covered by a system of coordinate neighbourhoods $\{U, x^i\}$ and let g_{ij}, Γ^k_{ij}, ∇_i, R^h_{ijk} and R_{ij} denote, respectively, the metric tensor, the Christoffel symbols, the operator of covariant differentiation with respect to Γ^k_{ij}, the curvature tensor and the Ricci tensor of M^n. The indices i, j, k, h assume the values $1, 2, \ldots, n$.

Let M^m be an m-dimensional connected Riemannian manifold, covered by a system of coordinate neighbourhoods $\{V, u^\alpha\}$ and let $g_{\alpha\beta}$, $\Gamma^\delta_{\alpha\beta}$, ∇_α, $R^\delta_{\alpha\beta\gamma}$ and $R_{\alpha\beta}$ denote the corresponding quantities of M^m. The indices α, β, γ, δ run over the range $\{1, 2, \ldots, m\}$.

We suppose that the manifold M^m is isometrically immersed in M^n by the immersion $r\colon M^m \to M^n$ and we identify $r(M^m)$ with M^m. We represent the immersion r by

$$x^i = x^i(u^\alpha) \tag{1.1}$$

and denote

$$B^i_\alpha = \frac{\partial x^i}{\partial u^\alpha}. \tag{1.2}$$

B^i_α are m linearly independent vectors of M^n tangent to M^m.

We denote by N^i_λ $(\lambda, \mu = m+1, m+2, \ldots, n)$ $n-m$ mutually orthogonal unit normals to M^m and by $D\colon I\times M^m \to M^n$, $I = (-\varepsilon, \varepsilon)$, $\varepsilon > 0$, an arbitrary deformation of M^m. Then the field z^i of the deformation D can be represented as

$$z^i = \xi^\alpha B^i_\alpha + \xi^\lambda N^i_\lambda, \tag{1.3}$$

where ξ^α and ξ^λ are, respectively, tangential and normal components of the field of deformation z^i.

We call a deformation D of the submanifold M^m *trivial*, when the field of the deformation z^i is identically equal to zero.

A deformation D of M^m is

(a) *infinitesimal isometric* (IID), if the components ξ^α and ξ^λ of the vector field of deformation z^i satisfy the following system of equations:

$$\nabla_\alpha \xi_\beta + \nabla_\beta \xi_\alpha - 2h_{\alpha\beta\lambda}\xi^\lambda = 0, \tag{1.4}$$

where $h^\lambda_{\alpha\beta}$ are the second fundamental tensors of M^m with respect to the normals N^i_λ; $h^\beta_{\alpha\lambda} = g^{\beta\delta}h_{\alpha\delta\lambda}$; $h_\lambda = h^\alpha_{\alpha\lambda} = g^{\alpha\beta}h_{\alpha\beta\lambda}$,

(b) *infinitesimal conformal* (*homothetic*) (ICD, (IHD)) if ξ^α and ξ^λ satisfy the system of equations:

$$\nabla_\alpha \xi_\beta + \nabla_\beta \xi_\alpha - 2h_{\alpha\beta\lambda}\xi^\lambda = 2\varrho g_{\alpha\beta}, \tag{1.5}$$

where ϱ is a certain function (constant) of u^α.

Let v^α be a vector field in M^m.

(a) A vector field v^α is a *harmonic vector* if it satisfies

$$\nabla_\alpha v_\beta - \nabla_\beta v_\alpha = 0, \quad \nabla_\alpha v^\alpha = 0. \tag{1.6}$$

(b) A *Killing vector* is a vector field v^α which satisfies

$$\nabla_\alpha v_\beta + \nabla_\beta v_\alpha = 0. \tag{1.7}$$

(c) An *affine Killing vector* is a vector field v^α which satisfies

$$\nabla_\gamma \nabla_\beta v^\alpha + R_{\delta\gamma\beta}{}^\alpha v^\delta = 0. \tag{1.8}$$

(d) A *conformal Killing vector* is a vector field which satisfies

$$\nabla_\alpha v_\beta + \nabla_\beta v_\alpha = \frac{2}{m}(\nabla_\varepsilon v^\varepsilon) g_{\alpha\beta}. \tag{1.9}$$

§ 2. Infinitesimal isometric deformations

THEOREM 2.1. *Let M^m be a non-totally geodesic, compact orientable submanifold of an orientable Riemannian manifold M^n. If M^m satisfies the conditions:*

(a) *the matrix $A = (h_{\alpha\beta\lambda}h_{\mu}^{\alpha\beta})$ is positively definite,*

(b) *the Ricci tensor is positively definite,*

then M^m does not allow non-trivial infinitesimal isometric deformations for which the tangential component of the deformation vector is a harmonic vector.

Proof. Let us suppose that M^m allows non-trivial IID. Then the tangential ξ^α and the normal ξ^λ components of the field of deformation z^h do not vanish at the same time and satisfy equation (1.4).

For a compact orientable submanifold M^m the following integral formula is valid:

$$\int_{M^m} \{R_{\alpha\beta}\xi^\alpha\xi^\beta + \nabla^\beta\xi^\alpha\nabla_\alpha\xi_\beta - (\nabla_\alpha\xi^\alpha)^2\}\,dv = 0 \tag{2.1}$$

for any vector ξ^α in M^m [7]. Since the vector ξ^α is harmonic, we have

$$\nabla_\alpha\xi_\beta = \nabla_\beta\xi_\alpha, \quad \nabla_\alpha\xi^\alpha = 0. \tag{2.2}$$

We multiply (1.4) by $\nabla^\alpha\xi^\beta$ and by $h_{\mu}^{\alpha\beta}\xi^\mu$. We obtain

$$\nabla_\beta\xi_\alpha\cdot\nabla^\alpha\xi^\beta = h_{\alpha\beta\lambda}h_{\mu}^{\alpha\beta}\xi^\lambda\xi^\mu. \tag{2.3}$$

Equality (2.1) after applying (2.2) and (2.3) becomes:

$$\int_{M^m} R_{\alpha\beta}\xi^\alpha\xi^\beta dv = -\int_{M^m} h_{\alpha\beta\lambda}h_{\mu}^{\alpha\beta}\xi^\lambda\xi^\mu dv. \tag{2.4}$$

The above equality in view of conditions (a) and (b) of the theorem is valid only when ξ^α and ξ^λ are identically equal to zero. The theorem is thus proved.

THEOREM 2.2. *Let M^m be a non-totally geodesic compact orientable submanifold of an orientable Riemannian manifold M^n. If M^m satisfies the conditions:*

(a) *the matrix $A = (h_{\alpha\beta\lambda}h_{\mu}^{\alpha\beta})$ is positively definite,*

(b) *the Ricci tensor is negatively definite,*

then M^m does not allow non-trivial infinitesimal non-tangential and non-normal isometric deformations for which the tangential component of the deformation vector is a Killing vector.

Proof. If we assume that there exists a non-trivial, non-tangential and non-normal IID of M^m, then ξ^α and ξ^λ do not vanish and satisfy the equations (1.4).

The condition for a vector field ξ^α to be a Killing vector is given by (1.7). We multiply (1.7) by $g^{\alpha\beta}$ and we obtain

$$\nabla_\alpha\xi^\alpha = 0. \tag{2.5}$$

From (1.4), taking account of (1.7), we have

$$(\nabla_\alpha \xi_\beta + \nabla_\beta \xi_\alpha) h_\mu^{\alpha\beta} \xi^\mu = 2h_\mu^{\alpha\beta} h_{\alpha\beta\lambda} \xi^\lambda \xi^\mu = 0. \tag{2.6}$$

From condition (a) of the theorem it follows that (2.6) is valid iff $\xi^\lambda \equiv 0$.

The integral formula (2.1), in view of (1.7) and (2.5), becomes

$$\int_{M^n} R_{\alpha\beta} \xi^\alpha \xi^\beta \, dv = \int_{M^m} \nabla_\alpha \xi_\beta \cdot \nabla^\alpha \xi^\beta \, dv. \tag{2.7}$$

This equality, if account is taken of condition (b) of the theorem, is fulfilled only when ξ^α is identically equal to zero. The theorem is proved.

COROLLARY. *If M^m satisfies the conditions of Theorem* 2.2, *then M^m does not allow non-trivial non-tangential and non-normal* IID *for which the tangential component of the deformation vector field is an affine Killing vector field.*

THEOREM 2.3. *Let M^m be a non-totally geodesic compact orientable submanifold of an orientable Riemannian manifold M^n. If M^m satisfies the conditions*:

(a) *the matrix* $A = \left(h_{\alpha\beta\lambda} h_\mu^{\alpha\beta} - \frac{1}{m} h_\lambda h_\mu\right)$ *is positively definite,*

(b) *the Ricci tensor is negatively definite,*

then M^m does not allow non-trivial infinitesimal non-tangential and non-normal isometric deformations for which the tangential component of the deformation vector is a conformal Killing vector.

Proof. From the transvection of (1.4) with $g^{\alpha\beta}$ we obtain

$$\nabla_\alpha \xi^\alpha = h_{\alpha\lambda}^\alpha \xi^\lambda. \tag{2.8}$$

From (1.4), (1.9) and (2.8) we have

$$\frac{1}{m} h_\lambda \xi^\lambda \cdot g_{\alpha\beta} = h_{\alpha\beta\lambda} \xi^\lambda, \tag{2.9}$$

and then

$$\left(h_{\alpha\beta\lambda} h_\mu^{\alpha\beta} - \frac{1}{m} h_\lambda h_\mu\right) \xi^\lambda \xi^\mu = 0. \tag{2.10}$$

Further the proof is analogous to that of Theorem 2.2.

PROPOSITION 2.1. *A compact orientable submanifold M^m of an orientable Riemannian manifold M^n does not allow non-trivial tangential* IID *if the Ricci tensor is negatively definite.*

THEOREM 2.4. *If a compact orientable submanifold M^m of an orientable Riemannian manifold M^n satisfies the conditions*:

(a) *the matrix* $A = (h_\lambda h_\mu - 2h_{\alpha\beta\lambda} h_\mu^{\alpha\beta})$ *is positively definite,*

(b) *the Ricci tensor is negatively definite,*

then M^m does not allow non-trivial IID.

Proof. We multiply (1.4) by $\nabla^\alpha \xi^\beta$ and by $h^{\alpha\beta}_\mu \xi^\mu$. We obtain

$$\nabla^\alpha \xi^\beta \cdot \nabla_\beta \xi_\alpha = -\nabla_\alpha \xi_\beta \cdot \nabla^\alpha \xi^\beta + 2h_{\alpha\beta\lambda} h^{\alpha\beta}_\mu \xi^\lambda \xi^\mu. \tag{2.11}$$

Equality (2.1) together with (2.8) and (2.11) gives:

$$\int_{M^m} R_{\alpha\beta} \xi^\alpha \xi^\beta \, dv = \int_{M^m} \{(h_\lambda h_\mu - 2h_{\alpha\beta\lambda} h^{\alpha\beta}_\mu) \xi^\lambda \xi^\mu + \nabla_\alpha \xi_\beta \cdot \nabla^\alpha \xi^\beta\} \, dv. \tag{2.12}$$

The above equality, in view of the conditions (a) and (b) of the theorem, is valid only when ξ^α and ξ^λ are identically equal to zero. The proof is complete.

§ 3. Infinitesimal conformal deformations

THEOREM 3.1. *Let M^m be a non-totally geodesic compact orientable submanifold of an orientable Riemannian manifold M^n. If M^m satisfies the conditions*:

(a) *the matrix* $A = \left(-\dfrac{1}{m} h_\lambda h_\mu + h_{\alpha\beta\lambda} h^{\alpha\beta}_\mu\right)$ *is positively definite,*

(b) *the Ricci tensor is positively definite,*

then M^m does not allow non-trivial ICD *for which the tangential component of the deformation vector is a harmonic vector.*

Proof. We multiply (1.5) by $g^{\alpha\beta}$ and determine the function ϱ

$$\varrho = \frac{1}{m} (\nabla_\alpha \xi^\alpha - h_\lambda \xi^\lambda). \tag{3.1}$$

Then we can represent equation (1.5) as follows:

$$\nabla_\alpha \xi_\beta + \nabla_\beta \xi_\alpha - 2h_{\alpha\beta\lambda} \xi^\lambda = \frac{2}{m} (\nabla_\varepsilon \xi^\varepsilon - h_\lambda \xi^\lambda) g_{\alpha\beta}. \tag{3.2}$$

From equality (3.2), taking account of (1.6), we obtain

$$\nabla_\alpha \xi_\beta \cdot \nabla^\beta \xi^\alpha = \left(h_{\alpha\beta\lambda} h^{\alpha\beta}_\mu - \frac{1}{m} h_\lambda h_\mu\right) \xi^\lambda \xi^\mu. \tag{3.3}$$

The integral formula (2.1) in view of (3.3) and (1.6) becomes:

$$\int_{M^m} R_{\alpha\beta} \xi^\alpha \xi^\beta \, dv = \int_{M^m} \left(\frac{1}{m} h_\lambda h_\mu - h_{\alpha\beta\lambda} h^{\alpha\beta}_\mu\right) \xi^\lambda \xi^\mu \, dv \tag{3.4}$$

and hence we complete the proof, taking account of the assumptions.

PROPOSITION 3.1. *A compact orientable submanifold M^m of an orientable Riemannian manifold M^n does not allow non-trivial tangential* ICD *if the Ricci tensor is negatively definite.*

THEOREM 3.2. *Let M^m be a non-totally geodesic compact orientable submanifold of an orientable Riemannian manifold M^n. If M^m satisfies the conditions*:

(a) *the matrix* $A = \left(h_{\alpha\beta\lambda}h_\mu^{\alpha\beta} - \frac{1}{m}h_\lambda h_\mu\right)$ *is positively definite,*

(b) *the Ricci tensor is negatively definite,*

then M^m *does not allow non-trivial, non-tangential and non-normal* ICD *for which the tangential component of the deformation vector is a Killing vector.*

Proof. Let us suppose that ξ^α and ξ^λ do not vanish. Equality (3.2) in view of (1.7) becomes:

$$h_{\alpha\beta\lambda}\xi^\lambda = \frac{1}{m}h_\lambda\xi^\lambda g_{\alpha\beta}. \tag{3.5}$$

Multiplying (3.5) by $h_\mu^{\alpha\beta}\xi^\mu$, we obtain

$$\left(h_{\alpha\beta\lambda}h_\mu^{\alpha\beta} - \frac{1}{m}h_\lambda h_\mu\right)\xi^\lambda\xi^\mu = 0. \tag{3.6}$$

Further the proof is analogous to that of Theorem 2.2.

THEOREM 3.3. *A compact orientable submanifold* M^m *of an orientable Riemannian manifold* M^n *does not allow non-trivial* ICD *if the following conditions are fulfilled:*

(a) *the matrix* $A = \left(\frac{1}{m}h_\lambda h_\mu - h_{\alpha\beta\lambda}h_\mu^{\alpha\beta}\right)$ *is positively definite,*

(b) *the Ricci tensor is negatively definite.*

Proof. If we assume that there exists a non-trivial ICD of M^m, then ξ^α and ξ^λ do not vanish simultaneously and satisfy equality (3.2). From equality (3.2) we obtain the following equalities:

$$h_\mu^{\alpha\beta}\xi^\mu \cdot \nabla_\alpha\xi_\beta = h_{\alpha\beta\lambda}h_\mu^{\alpha\beta}\xi^\lambda\xi^\mu + \frac{1}{m}(\nabla_\varepsilon\xi^\varepsilon - h_\lambda\xi^\lambda)\cdot\nabla_\alpha\xi^\alpha, \tag{3.7}$$

$$\nabla_\alpha\xi_\beta\cdot\nabla^\beta\xi^\alpha = 2\left(h_{\alpha\beta\lambda}h_\mu^{\alpha\beta} - \frac{1}{m}h_\lambda h_\mu\right)\xi^\lambda\xi^\mu + \frac{2}{m}(\nabla_\alpha\xi^\alpha)^2 - \nabla_\alpha\xi_\beta\cdot\nabla^\alpha\xi^\beta. \tag{3.8}$$

The integral formula (2.1) in view of (3.8) becomes:

$$\int_{M^m}\left\{\frac{m-2}{m}(\nabla_\alpha\xi^\alpha)^2 + \nabla_\beta\xi_\alpha\cdot\nabla^\beta\xi^\alpha\right\}dv = \int_{M^m}\left\{R_{\alpha\beta}\xi^\alpha\xi^\beta + 2\left(h_{\alpha\beta\lambda}h_\mu^{\alpha\beta} - \frac{h_\lambda h_\mu}{m}\right)\xi^\lambda\xi^\mu\right\}dv. \tag{3.9}$$

From the conditions of the theorem it follows that equality (3.9) is fulfilled only when ξ^α and ξ^λ are identically equal to zero. The theorem is proved.

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