ANNALES
POLONICI MATHEMATICI
XXIV (1971)

A remark on the general theory of summability

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The theory of evaluating sequences by matrix founded Mazur and Orlicz [3] has been generalized by Włodarski [6] for continuous methods.

Basing on the ideas contained in the papers mentioned above, Persson [5] has developed the theory of evaluation of functions defined on topological spaces which are locally bounded Baire functions.

Persson's approach, however, is inconvenient for the spaces R_+^n (R_+ is the space of real non-negative numbers) or N^k (N is the discrete space of natural numbers) e.g. according to Persson $a = \lim a_{m.n.}$ means $a = \lim_{m\to+\infty} a_{m.n.}$ uniformly with respect to $n = 1, 2, 3, \ldots$ The assumption of boundedness is also unnatural for functions defined in R_+.

In this paper I show that for functions the topology in the domain in which they are defined is in fact unessential. We know that for the existence of the limit of a function it suffices that its field be a directed set or that a filter (semifilter) be defined in its domain. The topology is necessary only in the set of values of a function. The corresponding definitions and theorems are recalled in section 0. For proofs and details see paper [1]. In this paper I make use of semifilters, notions of the theory of integral an measure according to [2] and of the theory of linear metric spaces according to [4].

I wish to express my many thanks to Professor L. Włodarski for his precious remarks concerning this paper.

0. Filters and convergence.

0.1. Let X be a set and Φ a non-empty family of its subsets. Φ is called a *filter* if

(F_1) $A \in \Phi$ and $A \subset A_1 \subset X$ implies $A_1 \in \Phi$;

(F_2) $A, B \in \Phi$ implies $A \cap B \in \Phi$;

(F_3) $\emptyset \notin \Phi$ ($\emptyset$ the empty set).

A non-empty family Π of subsets of the set X is called a *semifilter* if Π satisfies (F_3) and if

(F_2') $A, B \in \Pi$ implies that there exists a $C \in \Pi$ such that $C \subset A \cap B$.

It can easily be verified that the family

$$\Phi_{\Pi} = \{A \subset X : \text{there exists a } B \in \Pi \text{ such that } B \subset A\}$$

is a filter.

Every semifilter Π such that $\Phi = \Phi_{\Pi}$ is called a *basis of the filter* Φ.

0.2. Let $\langle X, r\rangle$ be a topological space and Φ a filter of subsets of X. We say that x_0 is the *limit of* Φ (or Φ converges to x_0) if every neighbourhood of x_0 is included in Φ. A semifilter of subsets of X is said to *converge to* $x_0 \in X$ if the filter Φ_{Π} defined in 0.1 converges to x_0.

It can easily be verified that a semifilter Π converges to x_0 if and only if any neighbourhood of x_0 includes an element of the semifilter.

0.3. Let X be a set (Y, τ) a topological space, f a function defined on X whose values belong to Y, and Π a semifilter of subsets of X. The family $f(\Pi) = \{f(A)\}_{A \in \Pi}$ is a semifilter of subsets of the set Y. The function f is said to *possess the limit* y with respect to Π if the semifilter $f(\Pi)$ converges to Y in $\langle Y, \tau\rangle$. We denote it by $y = \lim_{\Pi} f$.

By a proper choice of X and Π one may obtain the required kind of convergence. To illustrate this, we consider the following examples:

Let $X = N^k$ and let $\langle Y, \tau\rangle = \langle R, \tau\rangle$ be the space of real numbers. It follows that a function f defined on X with the values from Y is a k-fold sequence. Take for Π the familly of sets of the form $\{(n_1, \ldots, n_k) : n_i \geqslant m \text{ for } i = 1, 2, \ldots, k\}$. It follows that $\lim_{\Pi} f$, if it exists, is an ordinary limit of the sequence $f(n_1, \ldots, n_k)$ as $n_1 \to \infty, \ldots, n_k \to \infty$.

We denote by $F(X)$ the set of all functions with complex values defined on X.

1. Methods of evaluating. In [5] the method of evaluating functions has been defined as a distributive, i.e. additive and homogeneous mapping M of $D(M) \subset B(X)$ into $B(T)$, where X and Y are locally compact and σ-compact spaces and $B(X)$, $B(T)$ denote the familly of all Baire functions bounded on compact sets and defined on X and T, respectively. The M limit of $f \in D(M)$, if it exists, is $\lim_{\Pi} Mf$, where Π is a semifilter of complementaires of the compact subsets of T. More generally,

1.1. Definition. The method of evaluating functions defined in the set X is a *triplet* $M = \langle T, \Pi, \hat{M}\rangle$, where T is an arbitrary set, Π a semifilter of subsets of the set T and $\hat{M}$ a distributive mapping of $D(M) \subset F(X)$ *into* $F(T)$.

Let Y be a linear subspace of $F(X)$. $D(M_l : Y)$ denotes the set of functions f such that $f \in Y \cap D(M)$ and $\lim_{\Pi} \hat{M}F$ exists.

We denote this limit by ΛMf.

It is easily seen that this definition contains the methods defined in [5] and that $D(M_l)$ of [5] is $D\big(M_l : B(X)\big)$ in the sense of our argument.

2. Let P be a semifilter of subsets of X and Y a linear subspace of $F(X)$. We say that M is *P-permanent* for Y if $f \in Y$ and the existence of $\lim_P f$ imply $f \in D(M_l: Y)$ and $\Lambda Mf = \lim_P f$.

From 2 we obtain Persson's definition by putting for P complementaries of compact subsets of X.

3. Let K be a class of methods of evaluating functions defined on X and M a P-permanent method for Y belonging to K. We say that M is *P, Y-perfect* in K if for every P-permanent for Y method L belonging to the class $K: D(M_l: Y) \subset D(L_l: Y)$ implies $\Lambda Mf = \Lambda Lf$ for every $f \in D(M_l: Y)$.

We see that in the case of evaluating sequences the above definition is equivalent to the usual one with K as the class of matrix methods, $Y = F(N)$ and $P = \{[n, \infty) \cap N\}_{n \in N}$.

4. We see from the following proposition that the restriction of the notion of perfectness to some class K is essential.

4.1. *Let K be the class of all methods of evaluating functions defined on X and let P, Y be given. If the method M is P, Y-perfect in K, then $f \in D(M_l: Y)$ if and only if $f \in Y$ and* $\lim_P f$ *exists.*

Proof. Suppose that $f_0 \in D(M_l: Y)$ and $\lim_P f_0$ does not exist. Let ξ be any distributive functional defined on $D(M_l: Y)$ and such that $\xi(f) = 0$ if $\lim_P f$ exists and $\xi(f_0) = 1$.

Let $L = \langle T, \Pi, \hat{L} \rangle$, where T and P are defined as in M and $(\hat{L}f)(t) = (Mf)(t) + \xi(f)$.

Then $D(L_l: Y) = D(M_l: Y)$, L is P-permanent and $\Lambda Lf_0 \neq \Lambda Mf_0$ contrary to the assumption that M is perfect in K.

5. Let Y be a linear subspace of $F(X)$ and let $\{\|\cdot\|_K\}_{K \in N}$ be a countable family of seminorms on Y. We say that $f = g$ if $\|f - g\|_K = 0$ for every $K \in N$.

$\langle Y, \tau \rangle$ is called a B_0 *space* if the topology τ is generated by the metric

$$\varrho(f, g) = \sum_{k=1}^{\infty} 2^{-k} \|f - g\|_K (1 + \|f - g\|_K)^{-1}$$

and if the metric space $\langle T, \varrho \rangle$ is complete (see [4]). In what follows we consider methods $M = \langle T, \Pi, \hat{M} \rangle$ such that $\|f - g\|_K = 0$ for every $K \in N$ implies $(\hat{M}f)(t) = (\hat{M}g)(t)$ for every $t \in T$.

5.1. Definition. Let $\langle Y, \tau \rangle$ be a B_0 space such that $Y \subset F(X)$. We say that a method $M = \langle T, \Pi, \hat{M} \rangle$ is *topological* in $\langle Y, \tau \rangle$ if

5.1.1. $f \in D(M_l: Y)$ implies $\|f\|_{M,\infty} = \sup_{t \in T} |(\hat{M}f)(t)| < \infty$.

5.1.2. $D(M_l: Y)$ is a B_0 space in a topology not weaker than the topology generated by the seminorms of Y and the seminorm $\|\cdot\|_{M,\infty}$.

5.2. We see that a "topological method" in the sense of [5] is topological in the meaning of 5.1 in $B(X)$ for a σ-compact and locally compact space X, where $B(X)$ is considered with the seminorms $\sup_{t\in Z}|f(x)|$ and Z is a compact set included in X.

This class of methods will be denoted by BT.

5.3.1. Let μ be a σ-finite positive measure on the set X, i.e. $X = \bigcup_{n=1}^{\infty} X_n$ and $\mu(X_n) < \infty$ for every n. $L(\mu)$ denotes the family of functions $f \in F(X)$ such that for every measured $E \subset X$: if $\mu(E) < \infty$, then $\|f\|_E^* = \int_E |f(x)|\mu(dx)$. Putting, in 5.1, $Y = L(\mu)$ with the topology generated by the seminorms $\|f\|_E^*$ we obtain the class $L(\mu)T$.

5.4. $X = N$ (evaluation of sequences), it is easily seen that the class BT is equal to the class $L(\mu)T$ with $\mu(E) = \mathrm{Card}(E)$ for matrix methods.

For $X = R_+$ the methods used in practice belong to both classes considered above. In particular this is the case with the methods of class Z defined in 5.5.

5.5. Let λ be the Lebesgue measure on R_+ and let $\overline{M}(t, x)$ be a contiunous function in R_+^2. For $f \in L(\lambda)$; let $(\hat{M}f)(t) = \int_{[0,t]} \overline{M}(t, x)f(x)dx$; by Π we denote the family of intervals $[a, \infty)$ for $a > 0$.

The class of methods thus defined is called the *class* Z.

5.5.1. PROPOSITION. *The method M from Z belongs to BT and $L(\mu)$ and* $\Lambda Mf = \lim_{t\to\infty} \int_{[0,t]} \overline{M}(t, x)f(x)dx$.

Proof. The topology in $D(M_1: B(X))$ is defined by seminorms $\|f\|_n = \sup_{X\leqslant n} |f(x)|$, $n = 1, 2, 3, \ldots$, and $\sup_t |\int_{[0,t]} \overline{M}(t, x)f(x)dx| = \|f\|_{M,\infty}$.

In view of the continuity of $\overline{M}$ the mapping: $t \to \int_{[0,t]} \overline{M}(t, x)\, f(x)dx$ is continuous as a function of t for every $f \in L(\lambda) \supset B(R_+)$. Then for every f from $Y = L(\lambda)$ or $= B(R_+)$ we have $\|f\|_{M,\infty} < \infty$ if $f \in D(M_l: Y)$. In $D(M_l: L(X))$ we substitute $\|f\|_n^* = \int_{[0,n]} |f(x)|dx$ instead of $\|f\|_n$.

It is easily seen that with these seminorms and $Y = B(R_+)$ or $Y = L(\lambda)$ $D(M_l: Y)$ are B_0 spaces. The rest of the proposition is obvious.

6. In the definition of the topological in Y method M we have said "in topology not weaker than..." In view of the following proposition the topology in $D(M_l: Y)$ has been defined uniquely.

6.1. *Let M and S be methods of the type B_0 topological in $\langle Y, \tau\rangle$ and $D(M_l: Y) \subset D(S_l: Y)$. Then the topology induced on $D(M_l: Y)$ by* *the topology given in $D(S_l: Y)$ is not stronger than the topology given in $D(M_l: Y)$.*

Proof (similar to [5]). Let $I: f \to f$ be an injection of $D(M_l: Y)$ in $D(S_l: Y)$. Let $\lim f_n = f$ in $D(M_l: Y)$ and $\lim If_n = \lim f_n = g$ in $D(S_l: Y)$. Since neither topology is weeker than the topology of $\langle Y, \tau \rangle$ we have also $\lim f_n = f$ and $\lim f_n = g$ in $\langle Y, \tau \rangle$; so $f = g$.

By the close graph theorem we conclude that I is continuous, which ends the proof.

6.2. Proposition. *Let M be a method topological in $\langle Y, \tau \rangle$ and let $Y \subset F(X)$. For every linear, i.e. distributive and continuous, function ξ on $D(M_l: Y)$ there exists a method S topological in $\langle Y, \tau \rangle$ and such that $D(M_l: Y) \subset D(S_l: Y)$ and $\xi(f) = \Lambda Sf$ for any $f \in D(M_l: Y)$.*

Proof (as in [5]). Let $M = \langle T, \Pi, M \rangle$. We put $(\hat{S}f)(t) = (\hat{M}f)(t) + \xi(f) - \Lambda Mf$ for $f \in D(M_l: Y)$ and $t \in T$. The method $S = \langle T, \Pi, \hat{S} \rangle$ has the required property.

7. The propositions from Section 6 and the Banach–Hahn theorem on prolongation of functionals give the following

7.1. Proposition. *The P, Y-perfectness of a method topological in $\langle Y, \tau \rangle$ is equivalent to each of the following*

7.1.1. *$\{f \in D(M_l: Y) : \lim_P f$ exists$\}$ is dense in $D(M_l: Y)$.*

7.1.2. *For every linear functional ξ on $D(M_l: Y)$: if $\xi(f) = 0$ for every f such that $\lim_P f$ exists implies $\xi = 0$.*

We omit the easy proof.

8. Let $T = X = R_+$ and $\Pi = P = \{[a, \infty)\}_{a>0}$.

8.1. Proposition. *For $M \in Z$: the P, $L(\lambda)$-perfectness in the class $L(\lambda)T$ implies the P, $B(R_+)$-perfectness in the class BT.*

Proof. We have: $\|f\|_k^* \leqslant \|f\|_k C_k$ for every $f \in B(R_+)$, some C_k and for $k = 1, 2, \ldots$

Moreover, $B(R_+)$ is dense in $L(\lambda)$ in the $L(\lambda)$ topology, so if a set E is dense in $B(R_+)$ it is also dense in $L(\lambda)$, 7.1.1, gives the theorem.

8.2. Proposition. *For $M \in Z$: the P, $L(\lambda)$-perfectness in $L(\lambda)T$ implies the P, $B(R_+)$-perfectness in BT.*

Proof. Let ξ be a linear functional on $D(M_1: B(R_+))$ such that $\xi(f) = 0$ if $\lim_P f$ exists.

By the theorems proved in [4] we have $\xi = \xi_1 + \xi_2$, where ξ_1 is continuous with respect to $\|\cdot\|_n$ for some n and ξ_2 is continuous with respect to $\|\cdot\|_{M,\infty}$.

Let us remark that ξ_2 is of the form

$$\xi_2(f) = \int\limits_{[0,\infty]} \Big(\int\limits_{[0,t]} M(t, x) f(x)\, dx \Big) \mu(dt)\, dx,$$

for some measure μ. Let us put $\xi_2 = \xi_{2;1} + \xi_{2;2}$, where $\xi_{2;j}(f) = \xi_2(f^{(j)})$, $j = 1, 2$, and

$$f^{(1)}(x) = \begin{cases} f(x) & \text{for } x \leqslant n, \\ 0 & \text{for } x > n, \end{cases}$$

and

$$f^{(2)} = f - f^{(1)}.$$

Thus $\xi(f) = \xi(f^{(1)}) + \xi_{2,2}(f)$ because $\lim_P f^{(1)} = 0$, $\xi_{2,2}$ may be considered as a linear functional on $D(M: L(\lambda))$ and by the density of $D(M_l: B(R_+))$ in $D(M_l: L(\lambda))$ we conclude that $\xi_{2,2}(f) = 0$ if $\lim_P f$ exists and $f \in D(M_l: L(\lambda))$. By the perfectness of M in $L(\lambda)T$ we have $\xi_{2,2} = 0$, so in $D(M_l: B(R_+))$ we also have $\xi_{2,2} = 0$ and by 7.1.2 M is perfect in BT.

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Reçu par la Rédaction le 12. 2. 1968