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Generalized solutions of mixed problems for quasilinear hyperbolic systems of functional partial differential equations in the Schauder canonic form

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Abstract. Theorems of existence, uniqueness and continuous dependence on output data are proved concerning a.e. solutions of mixed problems for quasilinear hyperbolic systems of functional partial differential equations of the first order.

1. Introduction. We shall consider quasilinear hyperbolic systems in the following Schauder (or bicharacteristic) canonical form

$$(1)\quad \sum_{j=1}^{n} A_{ij}(x, y, u(x, y), (V_1 u)(x, y))\big[D_x u_j(x, y) + \lambda_i(x, y, u(x, y), (V_2 u)(x, y)) D_y u_j(x, y)\big] = f_i(x, y, u(x, y), (V_3 u)(x, y)),$$

$(x, y) \in I_{a_0}$, $i = 1, \ldots, n$, with the initial condition

$$(2)\qquad u(0, y) = \varphi(y), \quad y \in [0, b],$$

and the boundary conditions

$$(3)\qquad \begin{aligned} u_i(x, 0) &= h_{0i}(x, u(x, 0), (V_0 u)(x, 0)), \quad i \in J_0 = \{i\colon \operatorname{sgn} \lambda_i(0, 0, 0, 0) = 1\}, \\ u_i(x, b) &= h_{bi}(x, u(x, b), (V_0 u)(x, b)), \quad i \in J_b = \{i\colon \operatorname{sgn} \lambda_i(0, b, 0, 0) = -1\}, \end{aligned}$$

$x \in [0, a_0]$, where $I_x = \{(t, y)\colon 0 \leqslant t \leqslant x,\ 0 \leqslant y \leqslant b\}$, $\varphi(y) = (\varphi_1(y), \ldots, \varphi_n(y))$, $D_x = \partial/\partial x$, $D_y = \partial/\partial y$, $a_0, b, \Omega > 0$, are given constants, and V_s, $s = 0, 1, 2, 3$, are operators of the Volterra type. Whenever $A = [A_{ij}]$ is the identity matrix, systems (1) reduce to the first canonic (or diagonal) form [14].

Systems (1) contain as a particular case $(V_s u)(x, y) = u(\alpha_s(x, y), \beta_s(x, y))$ the systems of differential equations with a retarded argument [9], [10], which arise in the theory of the distribution of wealth [7].

Several kinds of integral-differential systems can be derived from systems (1) by specializing the operators V_s, $s = 0, 1, 2, 3$. For instance, problems arising from laser problem in Nonlinear Optics are also the special cases of problem (1)–(3) ($(V_s u)(x, y) = \int_{-\infty}^{y} K(y-t)u(x, t)dt$) [4].

In this paper, we consider the local existence, uniqueness and continuous dependence of generalized (in the sense almost everywhere) solutions of mixed problem (1)–(3) on output data.

Generalized solutions of quasilinear hyperbolic systems with the Cauchy and boundary conditions of the Cesàri type have been investigated by Bassanini [2], [3], Cesàri [5], [6], Kamont and Turo [9], [10], and Turo [12], [13].

Continuous generalized solutions (satisfying corresponding integral system) of mixed problems for hyperbolic systems have been considered by Filimonov [8], Myshkis and Filimonov [11], and Abolinia and Myshkis [1].

We shall prove, by means of the group property of characteristics and chain rule differentiation statements of real analysis, that continuous generalized solutions are generalized solutions (see the proof of Theorem 1).

2. Assumptions and lemmas. Let us introduce the norm $\|D\| = \max_{1 \le i \le n} \sum_{j=1}^{n} |d_{ij}|$ of the matrix $D = [d_{ij}]$, $i, j = 1, \ldots, n$. We denote by $|u|_n = \max_{1 \le i \le n} |u_i|$ the norm of u in $\mathbf{R}^n$.

ASSUMPTION H_1. Suppose that

(i) $A\colon E_{a_0} \to \mathbf{R}^{n^2}$ is continuous, where $E_{a_0} = I_{a_0} \times \bar{\Omega} \times \bar{\Omega}$, $\bar{\Omega} = [-\Omega, \Omega]^n \subset \mathbf{R}^n$ and $\Omega > 0$ is given constant;

(ii) $\det A(x, y, u, v) \geqslant \mu > 0$ in E_{a_0} for some constant μ;

(iii) there are constants $M > 0$, $m_s \geqslant 0$, $s = 0, 1, 2, 3$, such that for all $(x, y, u, v), (\bar{x}, \bar{y}, \bar{u}, \bar{v}) \in E_{a_0}$, we have

$$\|A(x, y, u, v)\| \leqslant M,$$

$$\|A(x, y, u, v) - A(\bar{x}, \bar{y}, \bar{u}, \bar{v})\| \leqslant m_0|x-\bar{x}| + m_1|y-\bar{y}| + m_2|u-\bar{u}|_n + m_3|v-\bar{v}|_n.$$

Since $\det A(x, y, u, v) \geqslant \mu > 0$ in E_{a_0}, relations (iii) of H_1 yield analogous relations for the inverse matrix $A^{-1} = [A_{ij}^{-1}]$. Thus, there are constants $M' > 0$, $m'_s \geqslant 0$, $s = 0, 1, 2, 3$, such that for all $(x, y, u, v), (\bar{x}, \bar{y}, \bar{u}, \bar{v}) \in E_{a_0}$, we have

$$\|A^{-1}(x, y, u, v)\| \leqslant M',$$

$$\|A^{-1}(x, y, u, v) - A^{-1}(\bar{x}, \bar{y}, \bar{u}, \bar{v})\| \leqslant m'_0|x-\bar{x}| + m'_1|y-\bar{y}| + m'_2|u-\bar{u}|_n + m'_3|v-\bar{v}|_n.$$

ASSUMPTION H_2. Suppose that

(i) the functions $\operatorname{sgn}\lambda(\cdot, 0, \cdot, \cdot)$, $\operatorname{sgn}\lambda(\cdot, b, \cdot, \cdot)$: $E^1_{a_0} = [0, a_0] \times \bar{\Omega} \times \bar{\Omega} \to \boldsymbol{R}^n$ are constant in $E^1_{a_0}$;

(ii) $\lambda(\cdot, y, u, v)$: $[0, a_0] \to \boldsymbol{R}^n$ is measurable for every $(y, u, v) \in E = [0, b] \times \bar{\Omega} \times \bar{\Omega}$;

(iii) there are a constant $\Lambda > 0$ and integrable functions l_s: $[0, a_0] \to \boldsymbol{R}_+ = [0, +\infty)$, $s = 1, 2, 3$, such that for all (y, u, v), $(\bar{y}, \bar{u}, \bar{v}) \in E$, almost everywhere (a.e.) in $[0, a_0]$, we have

$$|\lambda(x, y, u, v)|_n \leqslant \Lambda,$$

$$(4) \quad |\lambda(x, y, u, v) - \lambda(x, \bar{y}, \bar{u}, \bar{v})|_n \leqslant l_1(x)|y - \bar{y}| + l_2(x)|u - \bar{u}|_n + l_3(x)|v - \bar{v}|_n;$$

(iv) there are constants $\varepsilon_0 \in (0, b)$ and $\Lambda_0 > 0$, such that $\lambda_i(x, y, u, v) \geqslant \Lambda_0$ for $i \in J_0$, $y \in [0, \varepsilon_0]$, $(x, u, v) \in E^1_{a_0}$, and $-\lambda_i(x, y, u, v) \geqslant \Lambda_0$ for $i \in J_b$, $y \in [b - \varepsilon_0, b]$, $(x, u, v) \in E^1_{a_0}$.

Remark 1. If in (4) $l_1 = \text{const}$, then assumption (iv) of H_2 can be replaced by the following: the functions $\lambda_i(x, 0, u, v)$, $i \in J_0$, $-\lambda_i(x, b, u, v)$, $i \in J_b$, are bounded from below in E_{a_0} by the positive constant, say $2\Lambda^1_0$. Indeed, the inequality $|\lambda_i(x, y, u, v) - \lambda_i(x, 0, u, v)| \leqslant l_1 y$ yields $-l_1 y + \lambda_i(x, 0, u, v) \leqslant \lambda_i(x, y, u, v)$. Hence, we have $-l_1 y + 2\Lambda^1_0 \leqslant \lambda_i(x, y, u, v)$. Finally, for $0 \leqslant y \leqslant \min\{\Lambda^1_0 l_1^{-1}, b\}$ we obtain $\lambda_i(x, y, u, v) \geqslant \Lambda^1_0$. Similar considerations apply to the interval $[b - \Lambda^1_0 l_1^{-1}, b]$. Therefore, in this case we can take $\varepsilon_0 = \min\{\Lambda^1_0 l_1^{-1}, b\}$, $\Lambda_0 = \Lambda^1_0$ in the assumption (iv) of H_2.

ASSUMPTION H_3. Suppose that

(i) $f(\cdot, y, u, v)$: $[0, a_0] \to \boldsymbol{R}^n$ is measurable for every $(y, u, v) \in E$;

(ii) there are a constant $F > 0$, and integrable functions k_s: $[0, a_0] \to \boldsymbol{R}_+$, $s = 1, 2, 3$, such that for all (y, u, v), $(\bar{y}, \bar{u}, \bar{v}) \in E$, a.e. in $[0, a_0]$, we have

$$|f(x, y, u, v)|_n \leqslant F,$$

$$|f(x, y, u, v) - f(x, \bar{y}, \bar{u}, \bar{v})|_n \leqslant k_1(x)|y - \bar{y}| + k_2(x)|u - \bar{u}|_n + k_3(x)|v - \bar{v}|_n.$$

ASSUMPTION H_4. Suppose that

(i) the functions h_{0i}: $E^1_{a_0} \to \boldsymbol{R}$, $i \in J_0$, are independent of u_j, v_j, for $j \in J_0$, and the functions h_{bi}: $E^1_{a_0} \to \boldsymbol{R}$, $i \in J_b$, are independent of u_j, v_j, for $j \in J_b$;

(ii) there are constants $H_s \geqslant 0$, $s = 1, 2, 3$, such that for all (x, u, v), $(\bar{x}, \bar{u}, \bar{v}) \in E^1_{a_0}$, we have

$$|h_{0i}(x, u, v) - h_{0i}(\bar{x}, \bar{u}, \bar{v})| \leqslant H_1|x - \bar{x}| + H_2|u - \bar{u}|_n + H_3|v - \bar{u}|_n, \qquad i \in J_0,$$

$$|h_{bi}(x, u, v) - h_{bi}(\bar{x}, \bar{u}, \bar{v}) \leqslant H_1|x - \bar{x}| + H_2|u - \bar{u}|_n + H_3|v - \bar{v}|_n, \qquad i \in J_b,$$

(iii) the compatibility conditions

$$h_{0i}(0, \varphi(0), \varphi(0)) = \varphi_i(0), \quad i \in J_0, \quad h_{bi}(0, \varphi(b), \varphi(b)) = \varphi_i(b), \quad i \in J_b,$$

are satisfied;

(iv) there is a constant $\Phi \geqslant 0$ such that for all $y, \bar{y} \in [0, b]$ we have

$$|\varphi(y) - \varphi(\bar{y})|_n \leqslant \Phi |y - \bar{y}| \quad \text{and} \quad \max_{[0,b]} |\varphi(y)|_n = \Phi_1 < \Omega.$$

We denote by $D(a)$ $(0 < a \leqslant a_0)$ the set of all continuous functions u: $I_a \to \boldsymbol{R}^n$ Lipschitzian with respect to both variables. Let $B(a)$ be the set of all functions u, $u \in D(a)$ satisfying the conditions

$$|u(x, y)|_n \leqslant \Omega, \quad u(0, y) = \varphi(y).$$

We write $B(a, P, Q)$ to denote the set of all functions u, $u \in B(a)$ satisfying the following condition $|u(x, y) - u(\bar{x}, \bar{y})|_n \leqslant P|x - \bar{x}| + Q|y - \bar{y}|$ for all $(x, y), (\bar{x}, \bar{y}) \in I_a$, where we assume that $Q \geqslant \Phi$ which ensures that $B(a, P, Q)$ is not empty.

Let us consider in $B(a, P, Q)$ the following ball

$$B(a, P, Q, \omega) = \{u\colon u \in B(a, P, Q), \max_{I_a} |u(x, y) - \varphi(y)|_n \leqslant \omega\},$$

where $0 \leqslant \omega \leqslant \Omega - \Phi_1$.

Assumption H_5. Suppose that

(i) V_s: $B(a_0, P, Q, \omega) \to B(a_0)$, $s = 0, 1, 2, 3$;

(ii) there are constants $p_k, q_k, c_1, d_1 \geqslant 0$, $k = 0, 1$, and integrable functions c_l, d_l: $[0, a_0] \to \boldsymbol{R}_+$, $l = 2, 3$, such that for every $u \in D(a_0)$, we have

$$\|(V_k u)(\cdot, y)\|_* \leqslant p_k \|u(\cdot, y)\|_* + q_k, \quad k = 0, 1, \; y \in [0, b],$$

$$\|(V_1 u)(x, \cdot)\|_* \leqslant c_1 \|u(x, \cdot)\|_* + d_1,$$

$$\|(V_l u)(x, \cdot)\|_* \leqslant c_l(x) \|u(x, \cdot)\|_* + d_l(x), \quad l = 2, 3, \text{ a.e. in } [0, a_0],$$

where

$$\|u(\cdot, y)\|_* = \sup_{x, \bar{x} \in [0, a_0]} \frac{|u(x, y) - u(\bar{x}, y)|_n}{|x - \bar{x}|},$$

$$\|u(x, \cdot)\|_* = \sup_{y, \bar{y} \in [0, b]} \frac{|u(x, y) - u(x, \bar{y})|_n}{|y - \bar{y}|};$$

(iii) there are constants $r_k \geqslant 0$, $k = 0, 1$, and integrable functions r_l: $[0, a_0] \to \boldsymbol{R}_+$, $l = 2, 3$, such that for all $u, v \in D(a_0)$, $x \in [0, a_0]$, we have

$$\|V_k u - V_k v\|_x \leqslant r_k \|u - v\|_x, \quad k = 0, 1,$$

$$\|V_l u - V_l v\|_x \leqslant r_l(x) \|u - v\|_x, \quad l = 2, 3,$$

where $\|u\|_x = \sup_{(t, y) \in I_x} |u(t, y)|_n$.

Remark 2. In particular, from assumption (iii) of H_5 it follows that V_s, $s = 0, 1, 2, 3$, are operators of the Volterra type.

Remark 3. Note that, for $u \in B(a, P, Q, \omega)$ we have $|u(x, y)|_n \leqslant \omega + \Phi_1 \leqslant \Omega$. Hence, for $u \in B(a, P, Q, \omega)$ the points $(x, y, u(x, y), (V_s u)(x, y))$, where $(x, y) \in I_a$, belong to E_{a_0}. Thus, for every $u \in B(a, P, Q, \omega)$ the corresponding family of characteristic is defined.

We consider, for $u \in D(a)$, the problem

$$\text{(5)}\qquad \begin{aligned} &D_t g(t; x, y) = \lambda_i\big(t, g(t; x, y), u(t, g(t; x, y)), (V_2 u)(t, g(t; x, y))\big), \quad \text{a.e. in } [0, a],\\ &g(x; x, y) = y. \end{aligned}$$

Because of assumptions (ii), (iii) of H_2, (ii) of H_5, and $u \in D(a)$, we conclude that the functions $\lambda_i(\cdot, \cdot, u(\cdot, \cdot), (V_2 u)(\cdot, \cdot))$: $I_a \to \boldsymbol{R}$, $i = 1, \ldots, n$ satisfy the Carathéodory conditions. Thus, for every $u \in D(a)$, there is a unique solution $g_i = g_i[u](t; x, y)$ of problem (5).

Remark 4. Note that, since $g_i = g_i[u](t; x, y)$ is the unique solution of problem (5), g_i satisfies the following groupal property [6]

$$\text{(6)}\qquad g_i(t'; t, g_i(t; x, y)) = g_i(t'; x, y), \qquad t, t' \in [0, x],\ (x, y) \in I_a.$$

We denote by $\tau_i(x, y, u)$ the smallest value of the argument x for which the solution $g_i = g_i[u](t; x, y)$ of problem (5) is defined. Then, the point $\big(\tau_i(x, y, u), g_i[u](\tau_i(x, y, u); x, y)\big)$ belongs to the boundary of I_a.

We introduce the following notations

$$I^u_{\varphi i} = \{(x, y)\colon (x, y) \in I_a,\ \tau_i(x, y, u) = 0\},$$

$$I^u_{0i} = \{(x, y)\colon (x, y) \in I_a,\ \tau_i(x, y, u) > 0,\ g_i[u](\tau_i(x, y, u); x, y) = 0\},$$

$$I^u_{bi} = \{(x, y)\colon (x, y) \in I_a,\ \tau_i(x, y, u) > 0,\ g_i[u](\tau_i(x, y, u); x, y) = b\}.$$

From now on we admit that the constants $\omega \in (0, \Omega - \Phi_1]$, $P \geqslant 0$, $Q \geqslant \Phi$, $a \in (0, a_0]$ are given.

Put

$$L_1 = L_1(a) = \exp\Big(\int_0^a [l_1(t) + l_2(t) Q + l_3(t)(c_2(t) Q + d_2(t))]\, dt\Big),$$

$$L_2 = L_2(a) = \int_0^a (l_2(t) + l_3(t) r_2(t))\, dt.$$

Lemma 1 [14]. *Suppose that Assumptions* (ii), (iii) *of* H_2 *and* H_5 *are satisfied, and for* $u, v \in B(a, P, Q, \omega)$ *the solutions* $g_i[u]$, $g_i[v]$ *of problem* (5) *are defined on the interval* $[\alpha, \beta] \subset [0, a]$. *Then, for all* $(x, y), (\bar{x}, \bar{y}) \in I_a$, *the* *following inequality*

$$|g_i[u](t; x, y) - g_i[v](t; \bar{x}, \bar{y})| \leqslant L_1(\Lambda|x-\bar{x}| + |y-\bar{y}| + L_2\|u-\bar{u}\|_a), \qquad t \in [\alpha, \beta],$$

holds.

LEMMA 2 [14]. *If Assumptions* H_2 *and* H_5 *are satisfied and* a, $0 < a \leqslant a_0$, *is sufficiently small so that* $\Lambda a \leqslant \varepsilon_0$, *where* ε_0 *is given in* (iv) *of* H_2. *Then, for all* $(x, y), (x, \bar{y}) \in \bar{I}^u_{0i}$ *or* $(x, y), (x, \bar{y}) \in \bar{I}^u_{bi}$ (*where the bar means closure of the set*) *and* $u \in B(a, P, Q, \omega)$, *we have*

$$|\tau_i(x, y, u) - \tau_i(x, \bar{y}, u)| \leqslant \Lambda_0^{-1} L_1 |y-\bar{y}|,$$

and for $(x, y) \in \bar{I}^u_{oi} \cap \bar{I}^v_{0i}$ *or* $(x, y) \in \bar{I}^u_{bi} \cap \bar{I}^v_{bi}$, *and* $u, v \in B(a, P, Q, \omega)$, *we have*

$$|\tau_i(x, y, u) - \tau_i(x, y, v)| \leqslant \Lambda_0^{-1} L_1 L_2 \|u-v\|_a, \qquad i = 1, \ldots, n.$$

3. Operator S **and its properties.** Now we consider in $D(a)$ the operator S defined by

$$(7) \qquad (Su)_i(x, y) = \sum_{j=1}^{n} A_{ij}^{-1}\big(x, y, u(x, y), (V_1 u)(x, y)\big)[(\tilde{R}u)_j(x, y) + (\tilde{T}u)_j(x, y) + (Zu)_j(x, y)], \qquad i = 1, \ldots, n,$$

where

$$(8) \qquad (\tilde{R}u)_j(x, y) = \begin{cases} \sum_{k=1}^{n} A^u_{jk}[0, g_j]\,\varphi_k\big(g_j(0; x, y)\big), & (x, y) \in I^u_{\varphi j}, \\ \sum_{k \in J_0} A^u_{jk}[\tau_j, 0]\,h^u_{0k}[\tau_j] + \sum_{k \notin J_0} A^u_{jk}[\tau_j, 0]\,u_k[\tau_j, 0], & (x, y) \in I^u_{0j}, \\ \sum_{k \in J_b} A^u_{jk}[\tau_j, b]\,h^u_{bk}[\tau_j] + \sum_{k \notin J_b} A^u_{jk}[\tau_j, b]\,u_k[\tau_j, b], & (x, y) \in I^u_{bj}, \end{cases}$$

$$(9) \qquad (\tilde{T}u)_j(x, y) = \int_{\tau_j}^{x} \sum_{k=1}^{n} \frac{d}{dt} A^u_{jk}[t, g_j]\,u_k[t, g_j]\,dt,$$

$$(10) \qquad (Zu)_j(x, y) = \int_{\tau_j}^{x} f^u_j[t, g_j]\,dt, \qquad j = 1, \ldots, n.$$

and

$$A^u_{jk}[t, g_j] = A_{jk}\Big(t, g_j(t; x, y), u\big(t, g_j(t; x, y)\big), (V_1 u)\big(t, g_j(t; x, y)\big)\Big),$$

$$u_k[t, g_j] = u_k\big(t, g_j(t; x, y)\big),$$

$$h^u_{0k}[\tau_j] = h_{0k}\big(\tau_j(x, y, u'), u(\tau_j(x, y, u), 0), (V_0 u)(\tau_i(x, y, u), 0)\big),$$

$$h_{bk}^u[\tau_j] = h_{bk}(\tau_j(x, y, u), u(\tau_j(x, y, u), b), (V_0 u)(\tau_j(x, y, u), b)),$$
$$\tau_j = \tau_j(x, y, u),$$
$$f_j^u[t, g_j] = f_j(t, g_j(t; x, y), u(t, g_j(t; x, y)), (V_3 u)(t, g_j(t; x, y))).$$

We can rewrite (7) in the equivalent form

$$(11)\qquad (Su)_i(x, y) = (Ru)_i(x, y) + \sum_{j=1}^{n} A_{ij}^{-1}(x, y, u(x, y), (V_1 u)(x, y))\cdot$$
$$\cdot[(Tu)_j(x, y) + (Zu)_j(x, y)],\ i = 1, \ldots, n,$$

where

$$(12)\qquad (Ru)_i(x, y) = \begin{cases} \varphi_i(g_i(0; x, y)), & (x, y)\in I_{\varphi i}^u, \\ h_{0i}^u[\tau_i], & (x, y)\in I_{0i}^u, \\ h_{bi}^u[\tau_i], & (x, y)\in I_{bi}^u, \end{cases}$$

$$(13)\qquad (Tu)_j(x, y)$$
$$= \begin{cases} \int_0^x \sum_{k=1}^{n} \frac{d}{dt} A_{jk}^u[t, g_j]\,[u_k[t, g_j] - \varphi_k(g_j(0; x, y))]\,dt, & (x, y)\in I_{\varphi j}^u, \\ \int_{\tau_j}^x \Big\{\sum_{k\in J_0} \frac{d}{dt} A_{jk}^u[t, g_j]\,[u_k[t, g_j] - h_{0k}^u[\tau_j]] + \\ \quad + \sum_{k\notin J_0} \frac{d}{dt} A_{jk}^u[t, g_j]\,[u_k[t, g_j] - u_k[\tau_j, 0]]\Big\}\,dt, & (x, y)\in I_{0j}^u, \\ \int_{\tau_j}^x \Big\{\sum_{k\in J_b} \frac{d}{dt} A_{jk}^u[t, g_j]\,[u_k[t, g_j] - h_{bk}^u[\tau_j]] + \\ \quad + \sum_{k\notin J_b} \frac{d}{dt} A_{jk}^u[t, g_j]\,[u_k[t, g_j] - u_k[\tau_j, b]]\Big\}\,dt, & (x, y)\in I_{bj}^u, \end{cases}$$

Z is defined by (10).

Note that $A_{jk}^u[t, g_j]$ and $u_k[t, g_j]$ are absolutely continuous in t as superpositions of the Lipschitzian functions.

By force of usual chain rule differentiation statements of real analysis ([6]), we have a.e.

$$\left|\frac{d}{dt} A_{jk}^u[t, g_j]\right| \leqslant E + D\Lambda, \qquad \left|\frac{d}{dt} u_k[t, g_j]\right| \leqslant P + Q\Lambda,$$

where $E = m_0 + m_2 P + m_3(p_1 P + q_1)$, $D = m_1 + m_2 Q + m_3(c_1 Q + d_1)$. Hence

$$(14)\qquad |(Tu)_j(x, y)| \leqslant Ga, \quad |(Zu)_j(x, y)| \leqslant Fa,\ j = 1, \ldots, n,$$

where $G = n(E + D\Lambda)(P + Q\Lambda)\,a$.

From now on we make the assumptions: $2\Lambda a < b$ which yields $\bar{I}^u_{0i} \cap \bar{I}^v_{bi} = \emptyset$, and the assumption $\Lambda a \leqslant \varepsilon_0$ which guarantees that the inequalities $\lambda_i(x, y, u, v) \geqslant \Lambda_0$ and $-\lambda_i(x, y, u, v) \geqslant \Lambda_0$ are satisfied in the sets $\bar{I}^u_{0i}$ and $\bar{I}^u_{bi}$, respectively.

LEMMA 3. *Let Assumptions* H_1–H_5 *hold. Then for every* $u \in B(a, P, Q, \omega)$ *the function* $Su\colon I_a \to \mathbf{R}^n$ *is continuous.*

The proof of this lemma is similar to the proof of Lemma 3 [14].

LEMMA 4. *Suppose that Assumptions* H_1–H_5 *are satisfied. Then for every* $u \in B(a, P, Q, \omega)$ *the function Su satisfies in* $\bar{I}^u_{\varphi i}$ *a Lipschitz condition in y with some constant* Q^S_φ.

Proof. Let $(x, y), (x, \bar{y}) \in \bar{I}^u_{\varphi i}$. Then, by Lemma 1, we have

$$|(Ru)_i(x, y) - (Ru)_i(x, \bar{y})| \leqslant \Phi L_1 |y - \bar{y}|.$$

Furthermore, by integration by parts, we get

$$\begin{aligned}
&|(Tu)_j(x, y) - (Tu)_j(x, \bar{y})| \\
&\quad = \Big| \sum_{k=1}^{n} \{[A^u_{jk}[x, y] - A^u_{jk}[x, \bar{y}]\,[u_k(x, y) - u_k[0, g_j]]\} - \\
&\qquad - \int_0^x [A^u_{jk}[t, g_j] - A^u_{jk}[t, \bar{g}_j]] \frac{d}{dt} u_k[t, g_j]\, dt \\
&\qquad + \int_0^x \frac{d}{dt} A^u_{jk}[t, \bar{g}_j]\,[u_k[t, g_j] - \\
&\qquad - u_k[t, \bar{g}_j] - \varphi_k(g_j(0; x, y)) + \varphi_k(g_j(0; x, \bar{y}))]\, dt \Big| \leqslant \alpha |y - \bar{y}|,
\end{aligned}$$

and

$$|(Zu)_j(x, y) - (Zu)_j(x, \bar{y})| \leqslant K_1 L_1 |y - \bar{y}|,$$

where

$$K_1 = K_1(a) = \int_0^a \{k_1(t) + k_2(t) Q + k_3(t)[c_3(t) Q + d_3(t)]\}\, dt,$$

$$\alpha = n[D(P + Q\Lambda)(1 + L_1) + (E + D\Lambda)(Q + \Phi) L_1]\, a, \qquad \bar{g}_j = g_j(t; x, \bar{y}).$$

Hence

$$\begin{aligned}
&|(Su)_i(x, y) - (Su)_i(x, \bar{y})| \\
&\quad \leqslant |(Ru)_i(x, y) - (Ru)_i(x, \bar{y})| + \Big| \sum_{j=1}^{n} [A^{-1}_{ij}(x, y, u(x, y), (V_1 u)(x, y)) - \\
&\qquad - A^{-1}_{ij}(x, \bar{y}, u(x, \bar{y}), (V_1 u)(x, \bar{y}))] \cdot [(Tu)_j(x, y) + (Zu)_j(x, y)] \Big|
\end{aligned}$$

$$+\Big|\sum_{j=1}^{n} A_{ij}^{-1}(x,\bar{y},u(x,\bar{y}),(V_1 u)(x,\bar{y}))[(Tu)_j(x,y)-(Tu)_j(x,\bar{y})]\Big|+$$

$$+\Big|\sum_{j=1}^{n} A_{ij}^{-1}(x,\bar{y},u(x,\bar{y}),(V_1 u)(x,\bar{y}))[(Zu)_j(x,y)-(Zu)_j(x,\bar{y})]$$

$$\leqslant [\Phi L_1+nD'(G+F)a+nM'(\alpha+K_1 L_1)]|y-\bar{y}|,$$

where $D' = m_1'+m_2' Q+m_3'(c_1 Q+d_1)$.

Thus, we can take $Q_\varphi^S = \Phi L_1+nD'(G+F)a+nM'(\alpha+K_1 L_1)$. This ends the proof.

LEMMA 5. *Let Assumptions* H_1–H_5 *hold. Then, for* $(x,y)\in \bar{I}_{\varphi i}^u$, $u\in B(a,P,Q,\omega)$, *the function Su is Lipschitzian in x with some constant* P_φ^S.

Proof. Since $2\Lambda a<b$, in virtue of the theorem on prolongation of a solution to the boundary for an ordinary differential equation, for any two points $(x,y),(\bar{x},y)\in \bar{I}_{\varphi i}^u$ $(x\leqslant \bar{x})$, we can finde the point $(x,\bar{y})\in \bar{I}_{\varphi i}^u$ such that $\bar{y}=g_i(x;\bar{x},y)$. Since the points $(x,\bar{y})$ and $(\bar{x},y)$ belong to the same characteristic $\xi=g_i(t;\bar{x},y)$, we have

$$|(Su)_i(\bar{x},y)-(Su)_i(x,y)|$$
$$\leqslant n(G+F)(E'a+M')|x-\bar{x}|+[nD'(G+F)a+Q_\varphi^S]|y-\bar{y}|,$$

where $E' = m_0'+m_2' P+m_3'(p_1 P+q_1)$. On the other hand, we have $|y-\bar{y}|\leqslant \Lambda|x-\bar{x}|$. Hence

$$|(Su)_i(x,y)-(Su)_i(\bar{x},y)|\leqslant (W+Q_\varphi^S\Lambda)|x-\bar{x}|,$$

where $W=n(G+F)(E'a+M'+D'\Lambda a)$. It means that we can take $P_\varphi^S = W+Q_\varphi^S \Lambda$, and the proof is complete.

LEMMA 6. *Assume that Assumptions* H_1–H_5 *are satisfied. Then, in the sets* $\bar{I}_{0i}^u$ *and* $\bar{I}_{bi}^u$, *the function Su satisfies a Lipschitz condition in y with some constant* Q_0^S.

Proof. Observe that the assumption $2\Lambda a<b$ gives $\bar{I}_{0i}^u\cap \bar{I}_{bi}^u=\emptyset$. Let us take $(x,y),(x,\bar{y})\in \bar{I}_{0i}^u$ and $y\leqslant \bar{y}$ (the proof for $\bar{I}_{bi}^u$ is similar). Then on account of Lemma 2, it follows that

$$|(Ru)_i(x,y)-(Ru)_i(x,\bar{y})|\leqslant \tilde{H}\Lambda_0^{-1} L_1|y-\bar{y}|,$$

where $\tilde{H}=H_1+H_2P+H_3(p_0P+q_0)$. Furthermore, because of $\tau_i(x,y,u)\geqslant \tau_i(x,\bar{y},u)$ for $y\leqslant \bar{y}$, and by Lemma 2, we find

$$|(Tu)_j(x,y)-(Tu)_j(x,\bar{y})|$$
$$\leqslant \Big|\sum_{k\in J_0}\{[A_{jk}^u[x,y]-A_{jk}^u[x,\bar{y}]][u_k(x,y)-h_{0k}^u[\tau_j]]\}\Big|+$$

$$+n(E+DA)(Q+\tilde{H}A_0^{-1})L_1 a|y-\bar{y}|+$$
$$+\Big|\sum_{k\notin J_0}\{[A^u_{jk}[x, y]-A^u_{jk}[x, \bar{y}]][u_k(x, y)-u_k[\tau_j, 0]]\}\Big|+$$
$$+n(E+DA)(Q+PA_0^{-1})L_1 a|y-\bar{y}|+GA_0^{-1}L_1|y-\bar{y}|\leqslant\beta|y-\bar{y}|,$$

where $\beta = 2nD(P+QA)a+nL_1(E+DA)[2Q+(\tilde{H}+P)A_0^{-1}]a+GA_0^{-1}L_1$.

Furthermore $|(Zu)_j(x, y)-(Zu)_j(x, \bar{y})|\leqslant\eta|y-\bar{y}|$, where $\eta=(K_1+FA_0^{-1})L_1$. Hence

$$|(Su)_i(x, y)-(Su)_i(x, \bar{y})|\leqslant[\tilde{H}A_0^{-1}L_1+nD'(G+F)a+nM'(\beta+\eta)]|y-\bar{y}|.$$

Thus, we can put $Q_0^S = \tilde{H}A_0^{-1}L_1+nD'(G+F)a+nM'(\beta+\eta)$. This ends the proof.

CONCLUSION. From Lemmas 4 and 6 it follows that the function Su satisfies in I_a the Lipschitz condition in y with the constant $Q^S=\max\{Q^S_\varphi, Q^S_0\}$. Obviously $Q^S\geqslant Q^S_\varphi$ and $Q^S\geqslant Q^S_0$. If the points (x, y) and $(x, \bar{y})$ belong to the different sets $\bar{I}^u_{\varphi i}$, $\bar{I}^u_{0i}$, $\bar{I}^u_{bi}$, then this case reduces, in view of Lemma 3, to the considered already one.

LEMMA 7. *Let Assumptions* H_1–H_5 *hold. Then the function Su satisfies in* I_a *a Lipschitz condition in* x *with the constant* $P^S=Q^S A+W$.

The proof of this lemma runs similarly as the proof of Lemma 7 [14].

Remark 5. In particular, without loss of generality, we may assume that $A\geqslant 1$. Then, by Lemmas 4,6 and 7, we conclude that the function Su satisfies in I_a the Lipschitz condition with respect to both variables with the constant P^S.

LEMMA 8. *Suppose that Assumptions* H_1–H_5 *are satisfied, and* $a\in(0, a_0]$ *is sufficiently small such that*

$$a\leqslant\omega[\tilde{H}+\Phi A+nM'(G+F)]^{-1}. \tag{15}$$

Then the operator S *maps* $B(a, P, Q, \omega)$ *into* $B(a, P^S, Q^S, \omega)$.

Proof. This will be proved by showing that

$$|(Su)(x, y)-\varphi(y)|_n\leqslant\omega, \tag{16}$$

and

$$(Su)(0, y)=\varphi(y), \qquad y\in[0, b], \tag{17}$$

for $u\in B(a, P, Q, \omega)$.

First, let $(x, y)\in\bar{I}^u_{\varphi i}$. Then

$$|(Su)_i(x, y)-\varphi_i(y)|\leqslant[\Phi A+nM'(G+F)]a. \tag{18}$$

Now, let $(x, y)\in\bar{I}^u_{0i}$ (the proof for $(x, y)\in\bar{I}^u_{bi}$ is analogous). Then, taking into consideration compatibility condition (iii) of H_4, and initial condition (2), we get

$$|(Ru)_i(x, y) - \varphi_i(y)| \leqslant (\tilde{H} + \Phi\Lambda)\, a,$$

since $y \leqslant \Lambda x \leqslant \Lambda a$ for $(x, y) \in \bar{I}^u_{0i}$.

Hence, by using estimates (14), we obtain in this case that

$$|(Su)_i(x, y) - \varphi_i(y)| \leqslant [\tilde{H} + \Phi\Lambda + nM'(G+F)]\, a. \tag{19}$$

Combining (18) and (19) we get (16) for a satisfying condition (15).

It is obvious that (17) is satisfied. Finally, let us observe that from Lemmas 4,6 and 7 it follows that Su satisfies in I_a a Lipschitz condition with respect to x and y with constants P^S and Q^S, respectively. Thus the proof is complete.

LEMMA 9. *If Assumptions* H_1–H_5 *are satisfied, then for all* $(x, y) \in \bar{I}^u_{\varphi i} \cap \bar{I}^v_{\varphi i}$, $u, v \in B(a, P, Q, \omega)$, *we have*

$$|(Su)(x, y) - (Sv)(x, y)|_n \leqslant \nu_1 \|u - v\|_a,$$

where ν_1 *is some constant such that* $\nu_1 \to 0^+$ *as* $a \to 0^+$.

Proof. By Lemma 1, we have $|(Ru)_i(x, y) - (Rv)_i(x, y)| \leqslant \Phi L_1 L_2 \|u - v\|_a$. Moreover, by manipulations and integration by parts, we get

$$|(Tu)_j(x, y) - (Tv)_j(x, y)| \leqslant \gamma \|u - v\|_a,$$

where

$$\gamma = n\{[2(m_2 + m_3 r_1) + DL_1 L_2](P + Q\Lambda) + (E + D\Lambda)(Q + \Phi) L_1 L_2 + 1]\}\, a,$$

and

$$|(Zu)_j(x, y) - (Zv)_j(x, y)| \leqslant (K_1 L_1 L_2 + K_2) \|u - v\|_a,$$

where $K_2 = K_2(a) = \int_0^a [k_2(t) + k_3(t) r_3(t)]\, dt$.

Combining the estimates above, we get

$$|(Su)_i(x, y) - (Sv)_i(x, y)| \leqslant \nu_1 \|u - v\|_a,$$

where $\nu_1 = \Phi L_1 L_2 + n(m_2' + m_3' r_1)(G + F)\, a + nM'(\gamma + K_1 L_1 L_2 + K_2)$.

It is obvious that $\nu_1 \to 0^+$ as $a \to 0^+$, since $L_2 \to 0^+$, $K_1 \to 0^+$, $K_2 \to 0^+$ as $a \to 0^+$. This completes the proof.

LEMMA 10. *Let Assumptions* H_1–H_5 *hold. Then for every* $(x, y) \in \bar{I}^u_{0i} \cap \bar{I}^v_{0i}$ *or* $(x, y) \in \bar{I}^u_{bi} \cap \bar{I}^v_{bi}$ *and* $u, v \in B(a, P, Q, \omega)$ *we have*

$$|(Su)(x, y) - (Sv)(x, y)|_n \leqslant \nu_2 \|u - v\|_a,$$

where ν_2 *is a constant.*

Proof. Let us take $(x, y)\in\bar{I}^{u}_{0i}\cap\bar{I}^{v}_{0i}$; *then by Lemma* 2, *we get*

$$|(Ru)_i(x, y)-(Rv)_i(x, y)| \leqslant (\tilde{H}L_1 L_2 \Lambda_0^{-1}+H_2+H_3 r_0)\|u-v\|_a.$$

Furthermore, let us assume that $\tau_j(x, y, u) \leqslant \tau_j(x, y, v)$; then by using Lemmas 1 and 2, we see that

$$|(Tu)_j(x, y)-(Tv)_j(x, y)| \leqslant [\zeta+na(E+D\Lambda)(H_2+H_3 r_0)]\|u-v\|_a,$$

where

$$\zeta = n\{(P+Q\Lambda)[3(m_2+m_3 r_1)+DL_1 L_2(\Lambda\Lambda_0^{-1}+1)]a+G\Lambda_0^{-1}L_1 L_2+ \\ +a(E+D\Lambda)[L_1 L_2(2Q+P\Lambda_0^{-1}+\tilde{H}\Lambda_0^{-1})+3]\}.$$

Moreover,

$$|(Zu)_j(x, y)-(Zv)_j(x, y)| \leqslant \sigma\|u-v\|_a,$$

where $\sigma = (K_1+F\Lambda_0^{-1})L_1 L_2+K_2$.

Hence

$$|(Su)_i(x, y)-(Sv)_i(x, y)| \leqslant \nu_2\|u-v\|_a,$$

where

$$\nu_2 = \tilde{H}L_1 L_2\Lambda_0^{-1}+H_2+H_3 r_0+n(m_2'+m_3' r_1)(F+G)a+nM'(\zeta+\sigma).$$

This proves the lemma.

LEMMA 11. *Suppose that Assumptions* H_1–H_5 *are satisfied. Then for every* $(x, y)\in\bar{I}^{u}_{\varphi i}\cap\bar{I}^{v}_{0i}$ (*or* $(x, y)\in\bar{I}^{u}_{\varphi i}\cap\bar{I}^{v}_{bi}$, *or* $(x, y)\in\bar{I}^{u}_{0i}\cap\bar{I}^{v}_{\varphi i}$, *or* $(x, y)\in\bar{I}^{u}_{bi}\cap I^{v}_{\varphi i}$) *and* $u, v\in B(a, P, Q, \omega)$ *we have*

$$|(Su)(x, y)-(Sv)(x, y)|_n \leqslant \nu_3\|u-v\|_a,$$

where ν_3 *is a constant.*

Proof. We consider only the case when $(x, y)\in\bar{I}^{u}_{\varphi i}\cap\bar{I}^{v}_{0i}$ (the remaining cases may be handled in the same way). Since $\tau_i(x, y, u) = 0$ for $(x, y) = I^{u}_{\varphi i}$, and $y \leqslant \Lambda a$ for $(x, y)\in\bar{I}^{v}_{0i}$, by Lemma 2, we have $|\tau_i(x, y, v)| \leqslant \Lambda_0^{-1}L_1 L_2\|u-v\|_a$.

Since $i\in J_0$ for $(x, y)\in\bar{I}^{v}_{0i}$, in virtue of assumptions (iv) of H_2, we have $\lambda_i(x, y, u, v) \geqslant \Lambda_0$. because of $i\in J_0$ and $y \leqslant \varepsilon_0$. Therefore, the function $g_i[u](t; x, y)$ is increasing in t for $t\in[0, x]$. Whence

$$g_i[u](0; x, y) \leqslant g_i[u](\tau_i(x, y, v); x, y).$$

In view of Lemma 1, we get

$$|g_i[u](\tau_i(x, y, v); x, y)-g_i[v](\tau_i(x, y, v); x, y)| \leqslant L_1 L_2\|u-v\|_a.$$

Hence, we have $g_i[u](0; x, y) \leqslant L_1 L_2\|u-v\|_a$.

Now, on account of compatibility conditions (iii) of H_4 and initial condition (2), we find

$$|(Ru)_i(x, y)-(Rv)_i(x, y)| \leqslant L_1 L_2(\Phi+\tilde{H}\Lambda_0^{-1})\|u-v\|_a.$$

Moreover, by manipulations and integration by parts, we get

$$|(Tu)_j(x, y)-(Tv)_j(x, y)| \leqslant [\zeta+a(E+D\Lambda)L_1 L_2(\Phi+Q)]\|u-v\|_a,$$

and

$$|(Zu)_j(x, y)-(Zv)_j(x, y)| \leqslant \sigma\|u-v\|_a.$$

Combining the estimates above we conclude that

$$|(Su)_i(x, y)-(Sv)_i(x, y)| \leqslant v_3\|u-v\|_a,$$

where

$$v_3 = L_1 L_2(\Phi+\tilde{H}\Lambda_0^{-1})+n(m_2'+m_3' r_1)\times$$
$$\times(F+G)a+nM'[\zeta+a(E+D\Lambda)L_1 L_2(\Phi+Q)+\sigma].$$

Thus, the proof is finished.

Conclusion. From the assumption $2\Lambda a<b$ it follows that $\bar{I}_{0i}^u\cap\bar{I}_{bi}^v=\emptyset$, $\bar{I}_{bi}^u\cap\bar{I}_{0i}^v=\emptyset$. Thus, the cases considered in Lemmas 9, 10 and 11 cover all rectangle I_a. These lemmas show that in I_a, we have

$$\|Su-Sv\|_a \leqslant v_4\|u-v\|_a, \tag{20}$$

where $v_4 = v_3+(H_2+H_3 r_0)[1+a(E+D\Lambda)]$ (since $v_1\leqslant v_3$).

Note that, generally it has not to be $H_2+H_3 r_0<1$. Thus, in general case the operator S is not a contraction.

4. Properties of the operator S^2. Now we are interested in properties of the operator $SS = S^2$.

Lemma 12. *Let Assumptions H_1–H_5 hold. Then for $a\in(0, a_0]$ sufficiently small and P, Q sufficiently large, the operator S^2 maps $B(a, P, Q, \omega)$ into itself.*

Proof. Applying Lemma 8 to the function $Su\in B(a, P^S, Q^S, \omega)$, we get

$$|(S^2u)(x, y)-\varphi(y)|_n\leqslant\omega,\quad |(S^2u)(x, y)|_n\leqslant\Omega,\quad (S^2u)(0, y) = \varphi(y),$$

provided $a\leqslant\omega[\tilde{H}^S+\Phi\Lambda+nM'(G^S+F)]^{-1}$, where $\tilde{H}^S = H_1+H_2P^S+H_3\times(p_0P^S+q_0)$, $G^S = n(E^S+D^S\Lambda)(P^S+Q^S\Lambda)a$, $E^S = m_0+m_2P^S+m_3(p_1P^S+q_1)$, $D^S = m_1+m_2Q^S+m_3(c_1Q^S+d_1)$.

From Lemmas 4, 5, 6 and 7 it follows that the function S^2u satisfies in I_a a Lipschitz condition with respect to both variables with the constants P^{SS} and Q^{SS}, respectively. Since now the arguments of the operator S are not arbitrary functions of $B(a, P^S, Q^S, \omega)$, but the functions of the form Su. Therefore the Lipschitz constants of the function $S^2 u$ can be made more precise.

Indeed, for any two points $(x, y), (x, \bar{y}) \in \bar{I}^u_{\varphi i}$, by Lemma 4, we have

$$|(S^2 u)_i(x, y) - (S^2 u)_i(x, \bar{y})| \leqslant [\Phi L_1^S + nD'^S(G^S + F)a + nM'(\alpha^S + K_1^S L_1^S)]\,|y - \bar{y}|,$$

where

$$L_1^S = \exp\Big(\int_0^a [l_1(t) + l_2(t) Q^S + l_3(t)(c_2(t) Q^S + d_2(t))]\,dt\Big),$$

$$D'^S = m_1' + m_2' Q^S + m_3'(c_1 Q^S + d_1),$$

$$K_1^S = \int_0^a \{k_1(t) + k_2(t) Q^S + k_3(t)[c_3(t) Q^S + d_3(t)]\}\,dt.$$

Let now $(x, y), (x, \bar{y}) \in \bar{I}^u_{0i}$ (the same proof works in the case when $(x, y), (x, \bar{y}) \in \bar{I}^u_{bi}$). Then we have that $i \in J_0$, therefore for $j \notin J_0$ the point $(x, 0)$ belongs to the set $\bar{I}^u_{\varphi i}$. Hence, by Lemma 5, we obtain

$$|(Su)_j(x, 0) - (Su)_j(\bar{x}, 0)| \leqslant P^S_\varphi |x - \bar{x}|,$$

where $j \notin J_0$. According to assumption (i) of H_4 the function $h_{0i}(x, Su, V_0 Su)$ does not depend on $(Su)_j$ and $(V_0 Su)_j$, for $j \in J_0$, thus for $(x, y), (x, \bar{y}) \in \bar{I}^{Su}_{0i}$, by Lemmas 2 and 5, we have

$$\begin{aligned} |(RSu)_i(x, y) - (RSu)_i(x, \bar{y})| &\leqslant H_1 |\tau_i(x, y, Su) - \tau_i(x, \bar{y}, Su)| + \\ &+ H_2 \max_{j \notin J_0} \big|(Su)_j(\tau_i(x, y, Su), 0) - (Su)_j(\tau_i(x, \bar{y}, Su), 0)\big| + \\ &+ H_3 \max_{j \notin J_0} \big|(V_0 Su)_j(\tau_i(x, y, Su), 0) - (V_0 Su)_j(\tau_i(x, \bar{y}, Su), 0)\big| \\ &\leqslant \tilde{H}^S \Lambda_0^{-1} L_1^S |y - \bar{y}|. \end{aligned}$$

Furthermore, by Lemma 6, we get

$$|(TSu)_j(x, y) - (TSu)_j(x, \bar{y})| \leqslant \beta^S |y - \bar{y}|,$$

and

$$|(ZSu)_j(x, y) - (ZSu)_j(x, \bar{y})| \leqslant \eta^S |y - \bar{y}|,$$

where β^S and η^S are defined by β and η with P and Q replaced by P^S and Q^S, respectively.

Hence

$$\begin{aligned} &|(S^2 u)_i(x, y) - (S^2 u)_i(x, \bar{y})| \\ &\qquad \leqslant [\tilde{H}^S \Lambda_0^{-1} L_1^S + nD'^S(G^S + F)a + nM'(\beta^S + \eta^S)]\,|y - \bar{y}|. \end{aligned}$$

Therefore, for the function $S^2 u$ as a Lipschitz constant in y, we can take

$$Q^{SS} = \Phi L_1^S + \tilde{H}^S \Lambda_0^{-1} L_1^S + nD'^S(G^S + F)a + nM'(\alpha^S + \beta^S + \eta^S).$$

Consequently, in virtue of Lemma 7, we conclude that the function $S^2 u$ satisfies in I_a a Lipschitz condition in x with the constant

$$P^{SS} = Q^{SS} \Lambda + W^S, \quad \text{where } W^S = n(G^S + F)(E_a'^S + M' + D'^S \Lambda a).$$

Obviously, without loss of generality, we may assume that $\Lambda \geqslant 1$. Hence, in particular, we can take P^{SS} as a Lipschitz constant of the function $S^2 u$ with respect to both variables.

Thus, in order to show that the operator S maps $B(a, P, Q, \omega)$ into $B(a, P^S, Q^S, \omega)$, and the operator S^2 maps the ball $B(a, P, Q, \omega)$ into itself, one needs the following restrictions on the constants $\omega \in (0, \Omega - \Phi_1]$, $P \geqslant 0$, $Q \geqslant \Phi$, $a \in (0, a_0]$:

$$(21) \qquad \begin{aligned} &[\tilde{H} + \Phi\Lambda + mM'(G+F)]\, a \leqslant \omega, \qquad \Lambda a \leqslant \varepsilon_0, \ 2\Lambda a < b, \\ &[\tilde{H}^S + \Phi\Lambda + nM'(G^S + F)]\, a \leqslant \omega, \qquad P^{SS} \leqslant P, \ Q^{SS} \leqslant Q. \end{aligned}$$

Observe that, if ω, P, Q are fixed, and $a \to 0^+$, then

$$P^S \to \Lambda \max\{\Phi, \Lambda_0^{-1}(\tilde{H} + nM'F)\} + nM'F,$$

$$Q^S \to \max\{\Phi, \Lambda_0^{-1}(\tilde{H} + nM'F)\},$$

$$P^{SS} \to U\Lambda + nM'F, \qquad Q^{SS} \to U,$$

where $U = \{H_1 + H_2(nM'F + \Phi\Lambda) + H_3[p_0(nM'F + \Phi\Lambda) + q_0] + nM'F\}\Lambda_0^{-1}$.

Therefore, for arbitrary $\omega \in (0, \Omega - \Phi_1]$, if $P > U\Lambda + nM'F$, $Q > U$, then, for sufficiently small $a \in (0, a_0]$, all inequalities of (21) are satisfied. Thus, we have

$$(22) \qquad S^2\colon B(a, P, Q, \omega) \to B(a, P, Q, \omega),$$

which ends the proof.

In the sequel we shall assume that the constants P, Q and a are chosen in such a way that (22) is satisfied.

We shall now prove that the operator S^2 is a contraction.

LEMMA 13. *Suppose that Assumptions* H_1–H_5 *are satisfied. Then for all* $u, v \in B(a, P, Q, \omega)$ *we have*

$$|(S^2 u)(x, y) - (S^2 v)(x, y)|_n \leqslant v^S \|u - v\|_a,$$

where the coefficient $v^S \to 0^+$ *as* $a \to 0^+$.

Proof. Let $(x, y) \in \bar{I}_{\varphi i}^{Su} \cap \bar{I}_{\varphi i}^{Sv}$. By using Lemma 9, we get

$$|(S^2 u)(x, y) - (S^2 v)(x, y)|_n \leqslant v_1^S \|Su - Sv\|_a,$$

with $v_1^S = \varphi L_1^S L_2 + n(m_2' + m_3' r_1)(G^s + F)a + nM(\gamma^S + K_1^S L_1^S L_2 + K_2)$.

Let $(x, y) \in \bar{I}_{\varphi i}^{Su} \cap \bar{I}_{0i}^{Sv}$ (analogously we can consider the cases when $(x, y) \in \bar{I}_{\varphi i}^{Su} \cap \bar{I}_{bi}^{Sv}$, $(x, y) \in \bar{I}_{0i}^{Su} \cap \bar{I}_{\varphi i}^{Sv}$, $(x, y) \in \bar{I}_{\varphi i}^{Sv} \cap \bar{I}_{bi}^{Su}$). Then the assumptions of

Lemma 11 are satisfied, and we have

$$|(S^2 u)(x, y)-(S^2 v)(x, y)|_n \leqslant \nu_3^S \|Su-Sv\|_a,$$

where

$$\nu_3^S = L_1^S L_2(\Phi+\tilde{H}^S \Lambda_0^{-1})$$
$$+n(m_2'+m_3' r_1)(G^S+F)a+nM'[\zeta^S+a(E^S+D^S \Lambda) L_1^S L_2(\Phi+Q^S)+\sigma^S].$$

Let now $(x, y)\in \bar{I}_{0i}^{Su} \cap \bar{I}_{0i}^{Sv}$ (the case $(x, y)\in \bar{I}_{bi}^{Su} \cap \bar{I}_{bi}^{Sv}$ is similar). This implies that $i\in J_0$. Thus, the point $(x, 0)$ belongs to $\bar{I}_{\varphi j}^{u} \cap \bar{I}_{\varphi j}^{v}$ for $j\notin J_0$. Consequently, by Lemma 9, we obtain

$$|(Su)_j(x, 0)-(Sv)_j(x, 0)| \leqslant \nu_1 \|u-v\|_a.$$

In virtue of assumption (i) of H_4, the function $h_{0i}(x, Su, V_0, Su)$, $i\in J_0$, does not depend on $(Su)_j$ and $(V_0 Su)_j$ for $j\in J_0$. Therefore, we obtain

$$|(RSu)_i(x, y)-(RSv)_j(x, y)| \leqslant \tilde{H}^S \Lambda_0^{-1} L_1^S L_2 \|Su-Sv\|_a+(H_2+H_3 r_0)\nu_1 \|u-v\|_a.$$

We may assume that $\tau_j(x, y, Su) \leqslant \tau_j(x, y, Sv)$ then, by Lemma 10, we have

$$|(TSu)_j(x, y)-(TSv)_j(x, y)| \leqslant [\zeta^S+a(E^S+D^S \Lambda)(H_2+H_3 r_0)] \|Su-Sv\|_a,$$

and

$$|(ZSu)_j(x, y)-(ZSv)_j(x, y)| \leqslant \sigma^S \|Su-Sv\|_a.$$

Hence, in $\bar{I}_{0i}^{Su} \cap \bar{I}_{0i}^{Sv}$ (also in $\bar{I}_{bi}^{Su} \cap \bar{I}_{bi}^{Sv}$), we get

$$|(S^2 u)(x, y)-(S^2 v)(x, y)|_n \leqslant \nu_2^S \|Su-Sv\|_a+(H_2+H_3 r_0)\nu_1 \|u-v\|_a,$$

where

$$\nu_2^S = \tilde{H}^S \Lambda_0^{-1} L_1^S L_2+n(m_2'+m_3' r_1) \times$$
$$\times(G^S+F)a+nM'[\zeta^S+a(E^S+D^S \Lambda)(H_2+H_3 r_0)+\sigma^S].$$

Thus, combining the estimates above, we find (we remind that $\bar{I}_{0i}^{Su} \cap \bar{I}_{bi}^{Sv} = \bar{I}_{bi}^{Su} \cap \bar{I}_{bi}^{Sv} = \emptyset$, since $2\Lambda a < b$) in I_a

$$|(S^2 u)(x, y)-(S^2 v)(x, y)|_n$$
$$\leqslant [\nu_2^S+nM'a(E^S+D^S \Lambda)(\Phi+Q^S) L_1^S L_2] \|Su-Sv\|_a+(H_2+H_3 r_0)\nu_1 \|u-v\|_a.$$

Finally, by (20), we get

$$|(S^2 u)(x, y)-(S^2 v)(x, y)|_n$$
$$\leqslant \{[\nu_2^S+nM'a(E^S+D^S\Lambda)(\Phi+Q^S) L_1^S L_2]\nu_4+(H_2+H_3 r_0)\nu_1\} \|u-v\|_a.$$

It should be remarked that $\nu_2^S \to 0$, $L_1^S \to 1$, $L_2 \to 0$, $\nu_1 \to 0$, $\nu_4 \to H_2+H_3 r_0$, $E^S \to \text{const}$, $D^S \to \text{const}$, as $a \to 0^+$.

Taking $v^S = [v_2^S + nM'a(E^S + D^S A)(\Phi + Q^S) L_1^S L_2] v_4 + (H_2 + H_3 r_0) v_1$, where certainly $v^S \to 0^+$ as $a \to 0^+$, we get the assertion of Lemma 13.

5. The main theorem.

THEOREM 1. *Let Assumptions* H_1-H_5 *holds. Then, for any* $\omega \in (0, \Omega - \Phi_1]$, *and any sufficiently large constants* P, Q, *there are number* a, $a \in (0, a_0]$, *and a function* $u\colon I_a \to \mathbf{R}^n$, $u \in B(a, P, Q, \omega)$, *satisfying* (1) *a.e. in* I_a *and* (2), (3) *everywhere in* $[0, b]$, $[0, a]$, *respectively. Furthermore,* u *is unique in* $B(a, P, Q, \omega)$.

Proof. Let us choose P, Q and a in such a way that inequalities (21) are satisfied. Then, by Lemma 12, we get $S\colon B(a, P, Q, \omega) \to B(a, P^S, Q^S, \omega)$, $S^2\colon B(a, P, Q, \omega) \to B(a, P, Q, \omega)$.

Now, let us take $a \in (0, a_0]$, so that $v^S < 1$. Then, by Lemma 13, we conclude that the operator S^2 is a contraction. Hence, on account of completeness of $B(a, P, Q, \omega)$, it follows that there exists a function $u \in B(a, P, Q, \omega)$ such that $S^2 u = u$. It is known that the fixed point of any power of an operator is the fixed point of this operator. Thus, we have $Su = u$. For this fixed element we derive from (7) the integral equations

$$u_i(x, y) = \sum_{j=1}^{n} A_{ij}^{-1}(x, y, u(x, y), (V_1 u)(x, y))[(\tilde{R}u)_j(x, y) + (\tilde{T}u)_j(x, y) + (Zu)_j(x, y)], \quad i = 1, \dots, n, \tag{23}$$

where $\tilde{R}$, $\tilde{T}$ and Z are defined by (8), (9) and (10), respectively.

Now, we only need to show that the fixed point u of the operator S satisfies system (1) a.e. in I_a and conditions (2), (3) everywhere in $[0, b]$ and $[0, a]$, respectively. We consider only the case when $(x, y) \in I_{\varphi i}^{u}$. If we write j instead of i in the relations (23), then by multiplication by $A_{ij}(x, y, u(x, y), (V_1 u)(x, y))$, summation with respect to j, and usual simplifications, we have

$$\sum_{j=1}^{n} A_{ij}(x, y, u(x, y), (V_1 u)(x, y)) u_j(x, y) = (\tilde{R}u)_i(x, y) + (\tilde{T}u)_i(x, y) + (Zu)_i(x, y). \tag{24}$$

By integration by parts and further simplifications, we obtain

$$\int_0^x \Big[-\sum_{k=1}^{n} A_{ik}\big(t, g_i(t; x, y), u(t; g_i(t; x, y)), (V_1 u)(t, g_i(t; x, y))\big) \times \frac{d}{dt} u_k\big(t, g_i(t; x, y)\big) + f_i\big(t, g_i(t; x, y), u(t, g_i(t; x, y)), (V_3 u)(t, g_i(t; x, y))\big)\Big] dt = 0,$$

and this relation holds for all $(x, y) \in I_a$, $i = 1, \ldots, n$. By taking $y = g_i(x; 0, \eta)$ and making use of (6), we get

$$(25)\qquad \int_0^x \Big[-\sum_{k=1}^n A_{ik}\big(t, g_i(t; 0, \eta), u(t, g_i(t; 0, \eta)), (V_1 u)(t, g_i(t; 0, \eta))\big) \times \\ \times \frac{d}{dt} u_k\big(t, g_i(t; x, y)\big)\big|_{y = g_i(x; 0, \eta)} + \\ + f_i\big(t, g_i(t; 0, \eta), u(t, g_i(t; 0, \eta)), (V_1 u)(t, g_i(t; 0, \eta))\big)\Big] dt = 0.$$

By force of (5) and Chain Rule Differentiation Lemma (4.ii) of [6], the derivative in (25) becomes

$$\frac{d}{dt} u_k\big(t, g_i(t; x, y)\big)\big|_{y = g_i(x; 0, \eta)} = D_t u_k\big(t, g_i(t; 0, \eta)\big) + \\ + \lambda_i\big(t, g_i(t; 0, \eta), u(t, g_i(t; 0, \eta)), (V_1 u)(t, g_i(t; 0, \eta))\big) D_y u_k\big(t, g_i(t; 0, \eta)\big)$$

and this relation holds a.e. in the region I_a of the (t, η) space.

By differentiating (25) with respect to x we obtain

$$\sum_{k=1}^n A_{ik}\big(t, g_i(t; 0, \eta), u(t, g_i(t; 0, \eta)), (V_1 u)(t, g_i(t; 0, \eta))\big) \big[D_t u_k(t, g_i(t; 0, \eta)) \\ + \lambda_i\big(t, g_i(t; 0, \eta), u(t, g_i(t; 0, \eta)), (V_1 u)(t, g_i(t; 0, \eta))\big) D_y u_k\big(t, g_i(t; 0, \eta)\big)\big] \\ = f_i\big(t, g_i(t; 0, \eta), u(t, g_i(t; 0, \eta)), (V_3 u)(t, g_i(t; 0, \eta))\big), \qquad i = 1, \ldots, n,$$

and this relation holds a.e. in I_a. Finally, by taking $y = g_i(x; 0, \eta)$, that is, returning to the variables xy, we get equality (1). Since the transformation $y = g_i(x; 0, \eta)$ preserves sets of measure zero (being Lipschitzian), we conclude that (1) holds a.e. in I_a as stated.

Similar arguments apply to the cases when $(x, y) \in I^u_{0i}$ or $(x, y) \in I^u_{bi}$ (cf. proof of Theorem [14]), and the proof is finished.

6. Continuous dependence on output data. We consider together with problem (1)–(3) the following systems

$$(26)\qquad \sum_{j=1}^n A^*_{ij}(x, y, u^*(x, y), (V^*_1 u^*)(x, y)) \big[D_x u^*_j(x, y) \\ + \lambda^*_i(x, y, u^*(x, y), (V^*_2 u^*)(x, y)) D_y u^*_j(x, y)\big] \\ = f^*_i(x, y, u^*(x, y), (V^*_3 u^*)(x, y)),$$

$(x, y) \in I_{a_0}$, $i = 1, \ldots, n$, with the initial condition

$$(27)\qquad u^*(0, y) = \varphi^*(y), \qquad y \in [0, b],$$

and the boundary conditions

$$(28)\quad \begin{aligned} &u_i^*(x, 0) = h_{0i}^*(x, u^*(x, 0), (V_0^* u^*)(x, 0)),\\ &\quad i \in J_0^* = \{i\colon \operatorname{sgn} \lambda_i^*(0, 0, 0, 0) = 1\},\\ &u_i^*(x, b) = h_{bi}^*(x, u^*(x, b), (V_0^* u^*)(x, b)),\\ &\quad i \in J_b^* = \{i\colon \operatorname{sgn} \lambda_i^*(0, b, 0, 0) = -1\}, \qquad x \in [0, a_0]. \end{aligned}$$

ASSUMPTION H_6. Suppose that

(i) the functions A_{ij}^*, λ_i^*, f_i^*, φ_i^*, h_{0i}^*, h_{bi}^*, and the operators V_s^*, $s = 0, 1, 2, 3$, satisfy Assumptions H_1–H_5 with the same constants M, m_s, $s = 0, 1, 2, 3$, Λ, Λ_0, F, H_k, $k = 1, 2, 3$, Φ, Φ_1, c_1, d_1, p_s, q_s, r_s, $s = 0, 1$, and the functions l_s, k_s, $s = 1, 2, 3$, c_l, d_l, r_l, $l = 2, 3$;

(ii) $\max_{I_{a_0}} \{|u(x, y)|_n, |u^*(x, y)|_n\} \leqslant \Omega$;

(iii) $J_0^* = J_0$, $J_b^* = J_b$.

Now, let the functions $u \in B(a_0, P, Q) \subset D(a_0)$ and $u^* \in B^*(a_0, P^*, Q^*) \subset D(a_0)$ be generalized solutions of problems (1)–(3) and (26)–(28), respectively. Where, by $B^*(a_0, P^*, Q^*)$, we denote the set of all functions $u \in D(a_0)$ satisfying the following conditions

$$u^*(0, y) = \varphi^*(y), \qquad y \in [0, b],$$

$$|u^*(x, y)|_n \leqslant \Omega, \qquad (x, y) \in I_{a_0},$$

$$|u^*(x, y) - u^*(\bar{x}, \bar{y})|_n \leqslant P^* |x - \bar{x}| + Q^* |y - \bar{y}|$$

for all (x, y), $(\bar{x}, \bar{y}) \in I_{a_0}$ (existence of the such solutions, for $a \in (0, a_0]$, follows from Theorem 1).

Remark 6. Obviously, without loss of generality we may assume that $P = P^*$ and $Q = Q^*$.

Analogously as before, for $u^* \in D(a_0)$, we denote by $g_i^* = g_i^*[u^*](t; x, y)$ the solution of the problem

$$(29)\quad \begin{aligned} &D_t g^*(t; x, y)\\ &\quad = \lambda_i^*\big(t, g^*(t; x, y), u^*(t, g^*(t; x, y)), (V_2^* u^*)(t, g^*(t; x, y))\big),\\ &\qquad g^*(x; x, y) = y, \end{aligned}$$

and by $\tau_i^*(x, y, u^*)$ the smallest value of the argument x for which the solution $g_i^* = g_i^*[u^*](t; x, y)$ of problem (29) is defined.

By Theorem 1 it follows that the functions u and u^* satisfy the following equalities $u = Su$, $u^* = S^* u^*$, where the operator S^* is defined on $D(a_0)$ by the formula

$$(S^* u^*)_i (x, y) = (R^* u^*)_i (x, y) + \sum_{j=1}^{n} A_{ij}^{*-1}(x, y, u^*(x, y), (V_1^* u^*)(x, y))[(T^* u^*)_j (x, y) + (Z^* u^*)_j (x, y)], \quad i = 1, \ldots, n,$$

and R^*, T^* and Z^* are defined by (12), (13), (10) with φ_i, h_{0i}, h_{bi}, g_i, A_{ij}, τ_i, V_s, $s = 0, 1, 2, 3$, replaced by φ_i^*, h_{0i}^*, h_{bi}^*, g_i^*, A_{ij}^*, τ_i^*, V_s^*, $s = 0, 1, 2, 3$, respectively.

Put $\mu(x) = \max_{(t,y)\in I_x} |u(t, y) - u^*(t, y)|_n$.

We shall denote by C_i $(i = 1, 2, \ldots)$ the constants dependent only on the given constants.

Let $\delta = \max\{\delta_k,\ k = 0, 1, \ldots, 7\}$, where

$$\delta_0 = \max_{1\le i\le n} \max_{[0,b]} |\varphi_i(y) - \varphi_i^*(y)|, \qquad \delta_1 = \max_{i\in J_0} \max_{E_{a_0}^1} |h_{0i}(x, u, v) - h_{0i}^*(x, u, v)|,$$

$$\delta_2 = \max_{i\in J_b} \max_{E_{a_0}} |h_{bi}(x, u, v) - h_{bi}^*(x, u, v)|,$$

$$\delta_3 = \max_{1\le i,j\le n} \max_{E_{a_0}^1} |A_{ij}(x, y, u, v) - A_{ij}^*(x, y, u, v)|,$$

$$\delta_4 = \max_{1\le i,j\le n} \max_{E_{a_0}} |A_{ij}^{-1}(x, y, u, v) - A_{ij}^{*-1}(x, y, u, v)|,$$

$$\delta_5 = \max_{1\le i\le n} \max_{E_{a_0}} |f_i(x, y, u, v) - f_i^*(x, y, u, v)|,$$

$$\delta_6 = \max_{1\le i\le n} \max_{E_{a_0}} |\lambda_i(x, y, u, v) - \lambda_i^*(x, y, u, v)|,$$

$$\delta_7 = \max_{0\le s\le 3} \max_{u\in B(a_0,P,Q)} \max_{I_{a_0}} |(V_s u)(x, y) - (V_s^* u)(x, y)|.$$

In the remainder of this section we assume: $a \in (0, a_0]$, $2\Lambda a < b$, $\Lambda a \le \varepsilon_0$, $(H_2 + H_3 r_0 + 2)\varrho a < 1$, where $\varrho = n[(G+F)(m_2' + m_3' r_1) + nM'(m_2 + m_3 r_1)(P + Q\Lambda)]$.

LEMMA 14. *Suppose that Assumptions* H_1–H_6 *are satisfied. Then, for* $u \in B(a, P, Q)$, $u^* \in B^*(a, P, Q)$, *and* $t \in [\max\{\tau_i(x, y, u), \tau_i^*(x, y, u^*)\}, x]$, *we have*

$$(30) \qquad |g_i[u](t; x, y) - g_i^*[u^*](t; x, y)| \le L_1 \Big(\delta L_3 + \int_0^a [l_2(s) + l_3(s) r_2(s)]\, \mu(s)\, ds\Big),$$

where $L_3 = L_3(a_0) = \int_0^{a_0} [l_3(s) + 1]\, ds$.

Proof. Writing the integral equations for g_i and g_i^*, and subtracting them, we get

$$|g_i[u](t;x,y)-g_i^*[u^*](t;x,y)|$$
$$\leqslant \int_t^x [l_1(s)|g_i[u](s;x,y)-g_i^*[u^*](s;x,y)|+$$
$$+l_2(s)|u(s,g_i[u](s;x,y))-u^*(s,g_i^*[u^*](s;x,y))|_n+$$
$$+l_3(s)|(V_2u)(s,g_i[u](s;x,y))-(V_2^*u^*)(s,g_i^*[u^*](s;x,y))|_n+\delta]\,ds$$
$$\leqslant (\delta L_3+\int_0^a [l_2(s)+l_3(s)r_2(s)]\,\mu(s)\,ds)+$$
$$+\int_t^x [l_1(s)+l_2(s)Q+l_3(s)(c_2(s)Q+d_2(s))]\,|g_i[u](s;x,y)$$
$$-g_i^*[u^*](s;x,y)|\,ds,$$

where $\max\{\tau_i(x,y,u),\tau_i^*(x,y,u^*)\}\leqslant t\leqslant x$. Hence, and by Gronwall's inequality we obtain (30). This ends the proof.

LEMMA 15. *Let Assumptions* H_1–H_6 *hold. Then, for* $u\in B(a,P,Q)$, $u^*\in B^*(a,P,Q)$, *and* $i\in J_0$ *or* $i\in J_b$, *we have*

$$|\tau_i(x,y,u)-\tau_i^*(x,y,u^*)|\leqslant \Lambda_0^{-1}L_1(\delta L_3+\int_0^a [l_2(s)+l_3(s)r_2(s)]\,\mu(s)\,ds).$$

Proof. Let suppose that $\tau_i(x,y,u)\leqslant\tau_i^*(x,y,u^*)$. Since $g_i[u](\tau_i(x,y,u);x,y)=0$, by using the mean value theorem, we have

$$(31)\quad g_i[u](\tau_i^*(x,y,u^*);x,y)=g_i[u](\tau_i^*(x,y,u^*);x,y)-$$
$$-g_i[u](\tau_i(x,y,u);x,y)=D_tg_i[u](\xi;x,y)[\tau_i^*(x,y,u^*)-\tau_i(x,y,u)]$$

where $\tau_i(x,y,u)\leqslant\xi\leqslant\tau_i^*(x,y,u^*)$.

Since $2\Lambda a<b$ and $\Lambda a\leqslant\varepsilon_0$, so we have $|\lambda_i(x,y,u,v)|\geqslant\Lambda_0$, $|\lambda_i^*(x,y,u^*,v^*)|\geqslant\Lambda_0$, for the sets $\bar{I}_{0i}^u$ and $\bar{I}_{0i}^{u*}$ (similar arguments apply to the sets $\bar{I}_{bi}^u$ and $\bar{I}_{bi}^{u*}$, and $\bar{I}_{0i}^u\cap\bar{I}_{bi}^{u*}=\bar{I}_{bi}^u\cap\bar{I}_{0i}^{u*}=\emptyset$).

Hence, for $(x,y)\in\bar{I}_{0i}^u\cap\bar{I}_{0i}^{u*}$, by (31) and Lemma 14, we obtain

$$\tau_i^*(x,y,u^*)-\tau_i(x,y,u)$$
$$\leqslant \Lambda_0^{-1}g_i[u](\tau_i^*(x,y,u^*);x,y)$$
$$=\Lambda_0^{-1}[g_i[u](\tau_i^*(x,y,u^*);x,y)-g_i^*[u^*](\tau_i^*(x,y,u^*);x,y)]$$
$$\leqslant \Lambda_0^{-1}L_1(\delta L_3+\int_0^a [l_2(s)+l_3(s)r_2(s)]\,\mu(s)\,ds).$$

Analogously we can consider the case $\tau_i^*(x, y, u^*) \leqslant \tau_i(x, y, u)$, and the case when $(x, y) \in \bar{I}_{bi}^{u} \cap \bar{I}_{bi}^{u^*}$, and the proof is complete.

LEMMA 16. *Suppose that Assumptions* H_1–H_6 *are satisfied. Then, for* $(x, y) \in \bar{I}_{\varphi i}^{u} \cap \bar{I}_{\varphi i}^{u^*}$, $u \in B(a, P, Q)$, $u^* \in B^*(a, P, Q)$, *we have*

$$|(Su)_i(x, y) - (S^* u^*)_i(x, y)| \leqslant \varrho a \mu(a) + C_3 \Big(\delta + \int_0^a L(s) \mu(s) ds\Big),$$

where $L(s) = l_2(s) + l_3(s) r_2(s) + k_2(s) + k_3(s) r_3(s) + 1$.

Proof. Applying Lemma 14, we get

$$\begin{aligned}|(Ru)_i(x, y) - (R^* u^*)_i(x, y)| &\leqslant \big|\varphi_i(g_i[u](0; x, y)) - \varphi_i(g_i^*[u^*](0; x, y))\big| + \\ &\quad + \big|\varphi_i(g_i^*[u^*](0; x, y)) - \varphi_i^*(g_i^*[u^*](0; x, y))\big| \\ &\leqslant \Phi L_1 \Big(\delta L_3 + \int_0^a [l_2(s) + l_3(s) r_2(s)] \mu(s) ds\Big) + \delta.\end{aligned}$$

Now, by integration by parts, and further manipulations, we obtain

$$\begin{aligned}|(Tu)_j(x, y) - (T^* u^*)_j(x, y)| &= \Big|\sum_{k=1}^{n} \int_0^x \Big(\frac{d}{dt} A_{jk} - \frac{d}{dt} A_{jk}^*\Big)(u_k - \varphi_k) dt + \\ &+ \sum_{k=1}^{n} \int_0^x \frac{d}{dt} A_{jk}^* (u_k - u_k^* - \varphi_k + \varphi_k^*) dt\Big| \leqslant n(m_2 + m_3 r_1)(P + Q\Lambda) a \mu(a) \\ &+ C_1 \Big(\delta + \int_0^a [l_2(s) + l_3(s) r_2(s) + 1] \mu(s) ds\Big),\end{aligned}$$

where

$$\begin{aligned}C_1 = n \{(P + Q\Lambda)[2a_0(m_3 + 1) + m_2 + m_3 r_1] + (E + D\Lambda)(a_0 + 1) \\ + a_0 [(E + D\Lambda)(\Phi + Q) + D]\}(L_3 + 1).\end{aligned}$$

Furthermore

$$\begin{aligned}|(Zu)_j(x, y) - (Z^* u^*)_j(x, y)| &\leqslant K_1 L_1 \Big(\delta L_3 + \int_0^a [l_2 + l_3(s) r_2(s)] \mu(s) ds\Big) + \\ &+ \delta L_3 + \int_0^a [k_2(s) + k_3(s) r_3(s)] \mu(s) ds \leqslant C_2 \Big(\delta + \int_0^a [L(s) - 1] \mu(s) ds\Big),\end{aligned}$$

where $C_2 = (K_1 L_1 + 1)(L_3 + 1)$.

Hence, we have

$$|(Su)_i(x, y) - (S^* u^*)_i(x, y)|$$

$$\leqslant |(Ru)_i(x, y)-(R^* u^*)_i(x, y)|+\Big|\sum_{j=1}^{n} \big[A_{ij}^{-1}\big(x, y, u(x, y), (V_1 u)(x, y)\big)$$

$$-A_{ij}^{*-1}\big(x, y, u^*(x, y), (V_1^* u^*)(x, y)\big)\big]\cdot[(Tu)_j(x, y)+(Zu)_j(x, y)]\Big|$$

$$+\Big|\sum_{j=1}^{n} A_{ij}^{*-1}\big(x, y, u^*(x, y), (V_1^* u^*)(x, y)\big)[(Tu)_j(x, y)-(T^* u^*)_j(x, y)+$$

$$+(Zu)_j(x, y)-(Z^* u^*)_j(x, y)]\Big|$$

$$\leqslant \varrho a\mu(a)+C_3\big(\delta+\int_0^a L(s)\,\mu(s)\,ds\big),$$

where $C_3(\Phi L_1+1)(L_3+1)+n(G+F)(m_3'+1)a_0+nM'(C_1+C_2)$, which completes the proof.

LEMMA 17. *If Assumptions* H_1–H_6 *are satisfied, then for* $(x, y)\in (\bar{I}_{0i}^{u}\cap\bar{I}_{0i}^{u^*})\cup(\bar{I}_{bi}^{u}\cap\bar{I}_{bi}^{u^*})$ *we have*

$$|(Su)(x, y)-(S^* u^*)(x, y)|_n \leqslant \varrho(H_2+H_3 r_0+2)\,a\mu(a)+C_7\big(\delta+\int_0^a L(s)\,\mu(s)\,ds\big).$$

Proof. Suppose that $(x, y)\in\bar{I}_{0i}^{u}\cap\bar{I}_{0i}^{u^*}$. Then we have

$$|(Ru)_i(x, y)-(R^* u^*)_i(x, y)| \leqslant H_1\,|\tau_i(x, y, u)-\tau_i^*(x, y, u^*)|+$$

$$+H_2\max_{j\notin J_0}\big|u_j(\tau_i(x, y, u), 0)-u_j^*(\tau_i^*(x, y, u^*), 0)\big|+$$

$$+H_3\max_{j\notin J_0}\big|(V_0 u)_j(\tau_i(x, y, u), 0)-(V_0^* u^*)_j(\tau_i^*(x, y, u^*), 0)\big|+\delta$$

$$\leqslant \tilde{H}\,|\tau_i(x, y, u)-\tau_i^*(x, y, u^*)|+(H_2+H_3 r_0)\max_{j\notin J_0}\big|u_j(\tau_i(x, y, u), 0)-$$

$$-u_j^*(\tau_i^*(x, y, u), 0)\big|+(1+H_3)\,\delta.$$

However, the functions u and u^* are solutions of the adequate mixed problems. Thus $u = Su$ and $u^* = S^* u^*$. Furthermore, for $j\notin J_0$, the point $(x, 0)$ belongs to the set $\bar{I}_{\varphi j}^{u}\cap\bar{I}_{\varphi j}^{u}$. Hence, by Lemma 16, we get

$$\max_{j\notin J_0}\big|u_j(\tau_i(x, y, u), 0)-u_j^*(\tau_i^*(x, y, u^*), 0)\big|$$

$$= \max_{j\notin J_0}\big|(Su)_j(\tau_i(x, y, u), 0)-(S^* u^*)_j(\tau_i^*(x, y, u^*), 0)\big|$$

$$\leqslant \varrho a\mu(a)+C_3\big(\delta+\int_0^a L(s)\,\mu(s)\,ds\big).$$

Combining the estimates above, we obtain

$$|(Ru)_i(x, y)-(R^* u^*)_i(x, y)| \leqslant (H_2+H_3 r_0)\,\varrho a\mu(a)+C_4\big(\delta+\int_0^a L(s)\,\mu(s)\,ds\big),$$

where $C_4 = \tilde{H}\Lambda_0^{-1} L_1(L_3+1)+(H_2+H_3 r_0)\,C_3+H_3+1$.

Furthermore, assume that $\tau_i(x, y, u) \leqslant \tau_i^*(x, y, u^*)$. Then by manipulations and integration by parts, we see that

$$|(Tu)_j(x, y)-(T^* u^*)_j(x, y)| \leqslant \Bigg|\sum_{k\in J_0}\Bigg\{\int_{\tau_j^*(x,y,u)}^{x}\left(\frac{d}{dt}A_{jk}-\frac{d}{dt}A_{jk}^*\right)(u_k-h_{0k})\,dt+$$

$$+\int_{\tau_j^*(x,y,u^*)}^{x}\frac{d}{dt}A_{jk}^*(u_k-u_k^*-h_{0k}+h_{0k}^*)\,dt+\int_{\tau_j(x,y,u)}^{\tau_j^*(x,y,u^*)}\frac{d}{dt}A_{jk}(u_k-h_{0k})\,dt\Bigg\}\Bigg|+$$

$$+\Bigg|\sum_{k\notin J_0}\Bigg\{\int_{\tau_j^*(x,y,u^*)}^{x}\left(\frac{d}{dt}A_{jk}-\frac{d}{dt}A_{jk}^*\right)(u_k-u_k^*(\tau_j^*))\,dt+$$

$$+\int_{\tau_j^*(x,y,u^*)}^{x}\frac{d}{dt}A_{jk}^*(u_k-u_k^*-u_k(\tau_j)+u_k^*(\tau_j^*))\,dt\Bigg|+\Bigg|\int_{\tau_j(x,y,u)}^{\tau_j^*(x,y,u^*)}\frac{d}{dt}A_{jk}(u_k-u_k(\tau_j))\,dt\Bigg|$$

$$\leqslant 2n(m_2+m_3r_1)(P+QA)\,a\mu(a)+C_5\Big(\delta+\int_0^a[l_2(s)+l_3(s)r_2(s)+1]\,\mu(s)\,ds\Big),$$

where

$$C_5 = n\{2a_0[D(P+QA)+Q(E+DA)]L_1+(a_0\tilde{H}+a_0P+G)A_0^{-1}L_1$$
$$+(m_2+m_3r_1)(P+QA)+(E+DA)(H_2+H_3r_0+3)$$
$$+3(m_3+1)(P+QA)+H_3+1\}(L_3+1).$$

Finally, by Lemmas 14 and 15, we obtain

$$|(Zu)_j(x, y)-(Z^* u^*)_j(x, y)| \leqslant C_6\Big(\delta+\int_0^a[L(s)-1]\,\mu(s)\,ds\Big),$$

where $C_6 = [(K_1+FA_0^{-1})L_1+1](L_3+1)$.

Summarizing, we get

$$|(Su)_i(x, y)-(S^*u^*)_i(x, y)| \leqslant \varrho(H_2+H_3r_0+2)\,a\mu(a)+C_7\Big(\delta+\int_0^a L(s)\,\mu(s)\,ds\Big),$$

where $C_7 = C_4+nM'(C_5+C_6)+n(G+F)(m_3'+1)a_0$. this proves the lemma.

Lemma 18. *Let Assumptions* H_1–H_6 *hold. Then, for* $(x, y)\in\bar{I}^u_{\varphi i}\cap\bar{I}^{u^*}_{0i}$, *or* $(x, y)\in\bar{I}^u_{\varphi i}\cap\bar{I}^{u^*}_{hi}$, *or* $(x, y)\in\bar{I}^u_{0i}\cap\bar{I}^{u^*}_{\varphi i}$, *or* $(x, y)\in\bar{I}^u_{hi}\cap\bar{I}^{u^*}_{\varphi i}$, *we have*

$$\|(Su)(x, y)-(S^*u^*)(x, y)\|_n \leqslant 2\varrho\mu(a)+C_{10}\Big(\delta+\int_0^a L(s)\,ds\Big).$$

Proof. Assume that $(x, y)\in\bar{I}^u_{\varphi i}\cap\bar{I}^{u^*}_{0i}$. Since $(x, y)\in\bar{I}^u_{\varphi i}$, $\tau_i(x, y, u) = 0$, and $(x, y)\in\bar{I}^{u^*}_{0i}$ yields $y \leqslant Aa$. Since we assume that $Aa \leqslant \varepsilon_0$, it follows that the assumptions of Lemma 15 are satisfied with $\tau_i(x, y, u) = 0$.

Therefore

$$\tau_i^*(x, y, u^*) \leqslant \Lambda_0^{-1} L_1 \big(\delta L_3 + \int\limits_0^a [l_2(s) + l_3(s) r_2(s)] \mu(s) ds\big).$$

Furthermore, since $(x, y) \in \bar{\Gamma}_{0i}^u$, we see that $i \in J_0$. But for $i \in J_0$ and $y \leqslant \varepsilon_0$, in virtue of assumption (iv) of H_1 we have $\lambda_i(x, y, u, v) \geqslant \Lambda_0$. Then the function $g_i[u](t; x, y)$ is increasing in t, for $t \in [0, x]$. Whence $g_i[u](0; x, y) \leqslant g_i[u](\tau_i^*(x, y, u^*); x, y)$.

By Lemma 14, we get

$$|g_i[u](\tau_i^*(x, y, u^*); x, y) - g_i^*[u^*](\tau_i^*(x, y, u^*); x, y)|$$
$$\leqslant L_1 \big(\delta L_3 + \int\limits_0^a [l_2(s) + l_3(s) r_2(s)] \mu(s) ds\big),$$

since $g_i^*[u^*](\tau_i^*(x, y, u^*); x, y) = 0$.

In view of compatibility conditions (iii) of H_4, we find

$$|(Ru)_i(x, y) - (R^* u^*)_i(x, y)| \leqslant |\varphi_i(g_i[u](0; x, y) - \varphi_i(0))| +$$
$$+ |\varphi_i(0) - h_{0i}^*(0, u^*(0, 0), (V_0^* u^*)(0, 0))| + |h_{0i}^*(0, u^*(0, 0), (V_0^* u^*)(0, 0)) -$$
$$- h_{0i}^*(\tau_i^*(x, y, u^*), u^*(\tau_i^*(x, y, u^*), 0), (V_0^* u^*)(\tau_i^*(x, y, u^*), 0))|$$
$$\leqslant C_8 \big(\delta + \int\limits_0^a [l_2(s) + l_3(s) r_2(s)] \mu(s) ds\big),$$

where $C_8 = [L_1(\Phi + \tilde{H}\Lambda_0^{-1}) + 1](L_3 + 1)$.

By manipulations and integration by parts, we have

$$|(Tu)_j(x, y) - (T^* u^*)_j(x, y)|$$
$$\leqslant 2n(m_2 + m_3 r_1)(P + Q\Lambda) a\mu(a) + C_9 \big(\delta + \int\limits_0^a [l_2(s) + l_3(s) r_2(s) + 1] \mu(s) ds\big),$$

where

$$C_9 = n\{2a_0 L_1 [D(P + Q\Lambda) + (E + D\Lambda)(2Q + P\Lambda_0^{-1} + 4) + m_2 + m_3 r_1$$
$$+ a_0(m_3 + 1)(P + Q\Lambda + 1)\}(L_3 + 1).$$

Moreover,

$$|(Zu)_j(x, y) - (Z^* u^*)_j(x, y)|$$
$$\leqslant K_1 |g_j[u](t; x, y) - g_j^*[u^*](t; x, y)| + \delta L_3 + \int\limits_0^a [k_2(s) + k_3(s) r_3(s)] \mu(s) ds$$
$$+ \int\limits_0^{\tau_j^*(x,y,u^*)} F ds \leqslant C_6 \big(\delta + \int\limits_0^a [L(s) - 1] \mu(s) ds\big).$$

Summing up, we have

$$|(Su)_i(x, y)-(S^* u^*)_i(x, y)| \leqslant 2\varrho\mu(a)+C_{10}(\delta+\int_0^a L(s)\mu(s)\,ds),$$

where $C_{10} = C_8+nM'(C_9+C_6)+n(G+F)(m_3'+1)a_0$. This proves the lemma.

Conclusion. Since $2\Lambda a < b$, it follows that $\bar{I}^u_{0i} \cap \bar{I}^{u^*}_{bi} = \bar{I}^u_{bi} \cap \bar{I}^{u^*}_{0i} = \emptyset$. Then, the cases considered in Lemmas 16, 17 and 19 cover all rectangle I_a. Hence, in I_a, we have

$$(32)\qquad |(Su)(x, y)-(S^* u^*)(x, y)|_n \leqslant (H_2+H_3 r_0+2)\varrho a\mu(a) + C_{11}(\delta+\int_0^a L(s)\mu(s)\,ds),$$

where $C_{11} = \max\{C_3, C_7, C_{10}\}$.

Theorem 2. *Let Assumptions* H_1–H_6 *hold. Suppose that* $u \in B(a, P, Q)$ *and* $u^* \in B^*(a, P, Q)$ *are generalized solutions of problems* (1)–(3) *and* (26)–(28), *respectively. Then, for every* $\varepsilon > 0$ *there is* $\delta_0(\varepsilon) > 0$, *such that* $\delta \leqslant \delta_0(\varepsilon)$ *implies*

$$\max_{I_{a_0}} |u(x, y)-u^*(x, y)|_n < \varepsilon.$$

Proof. Let $\varepsilon > 0$ be fixed. Next, we divide, by means of parallel lines to y axis, the rectangle I_{a_0} into rectangles which width does not exceed the number $\min\{(2\Lambda)^{-1}b, [(H_2+H_3r_0+2)\varrho]^{-1}\}$. Then, for the every such rectangle all obtained estimates hold, with Φ replaced by Q. Now, we consider the first rectangle (near the y axis). Thus, for every point (t, y), $0 \leqslant t \leqslant x < \min\{(2\Lambda)^{-1}b, [(H_2+H_3r_0+2)\varrho]^{-1}\}$, applying (32), we get

$$|u(t, y)-u^*(t, y)|_n \leqslant (H_2+H_3 r_0+2)\varrho a\mu(x)+C_{11}(\delta+\int_0^x L(s)\mu(s)\,ds).$$

Now, from the definition of μ, we obtain

$$\mu(x) \leqslant (H_2+H_3 r_0+2)\varrho a\mu(x)+\tilde{C}_{11}(\delta+\int_0^x L(s)\mu(s)\,ds),$$

where $\tilde{C}_{11}$ is formed from C_{11} by replacing Φ by Q. Hence, and by Gronwall's inequality, we see that

$$\mu(x) \leqslant [1-(H_2+H_3 r_0+2)\varrho a]^{-1}\tilde{C}_{11}\delta\exp\{[1-(H_2+H_3 r_0+2)\varrho a]^{-1} \times \tilde{C}_{11}\int_0^{a_0} L(\tilde{C})\,ds\}.$$

Putting

$$\delta_0(\varepsilon) = \varepsilon \left\{[1-(H_2+H_3 r_0+2)\varrho a]^{-1} \tilde{C}_{11} \exp \left\{[1-(H_2+H_3 r_0+2)\varrho a]^{-1} \times \times \tilde{C}_{11} \int_0^{a_0} L(s)\,ds\right\}\right\}^{-1},$$

we obtain the assertion of the theorem.

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