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On the functional equation

$$f(x)+\sum_{i=1}^{n} g_i(y_i) = h\big(T(x, y_1, y_2, \dots, y_n)\big)$$

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Dedicated to Professor S. Gołąb
on his 70th Birthday

1. Introduction. In this paper we study the functional equation

$$(1.0)\qquad f(x)+\sum_{i=1}^{n} g_i(y_i) = h\big(T(x, y_1, y_2, \dots, y_n)\big) \quad \text{for all } x \in X,\ y_i \in Y_i$$

on topological spaces X and Y_i with real-valued f, g_i, h and T. Unless otherwise stated the followings are supposed:

PX: (A) X is a locally connected Hausdorff space such that any pair of points is contained in some open and connected set with compact closure,

or

(B) X is arcwise connected in the sense that with each x_1 and x_2 in X, $x_1 \neq x_2$, there can be associated a bicontinuous mapping (continuous, one to one, and with a continuous inverse) $\varphi\colon [0,1] \to X$ so that $\varphi(0) = x_1$ and $\varphi(1) = x_2$.

PY: $n \geqslant 1$ and for each $1 \leqslant i \leqslant n$, Y_i is a connected space.

Among other more general results the following uniqueness theorem will be proved.

THEOREM 1.0. *Suppose $T\colon X\times Y_1\times Y_2\times\ldots\times Y_n \to R$ is continuous in its first variable $x \in X$ and jointly continuous in its n variables $(y_1, y_2, \ldots, y_n) \in Y_1\times Y_2\times\ldots\times Y_n$. Suppose $(f^0, g_1^0, g_2^0, \ldots, g_n^0, h^0)$ is a particular system of solutions of* (1.0) *consisting of continuous f^0 such that f^0 and at least*

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one of the g_i^0, *s are non-constant. Then any system of solutions* $(f, g_1, g_2, \ldots, g_n, h)$ *of* (1.0) *with continuous* f *is of the form*

$$
\begin{aligned}
f &= af^0+\beta,\\
g_i &= ag_i^0+\beta_i, \quad i = 1, 2, \ldots, n, \qquad (1.1)\\
h &= ah^0+\beta+\beta_1+\beta_2+\ldots+\beta_n,
\end{aligned}
$$

with real constants a, β *and* β_i $(i = 1, 2, \ldots, n)$.

The important question of uniqueness of the system of solutions of the equation

$$f(x)+g(y) = h\big(T(x, y)\big) \tag{1.2}$$

has been widely examined, since J. V. Pexider 1903, under various structures of the spaces and regularities of the mappings in question. Following the works of F. Radó 1958, J. Aczél and S. Gołąb 1960 and R. D. Luce and J. W. Tukey 1964 (cf. Aczél [1], p. 147) Aczél gave in [1] the following analogous result.

THEOREM 1.1. *If the variables* x *and* y *in equation* (1.2) *range over a domain* $D \subseteq R$ *and also* $T(x, y) \in D$, *if further* (1.2) *has a system of solutions* (f^0, g^0, h^0) *consisting of functions mapping* D *onto* R *in a one to one way, one of which* (f^0) *is continuous, then any system of solutions* (f, g, h) *of* (1.2) *in which* f *can be majorized by a measurable function on a set of positive measure is of the form*

$$f = af^0+\beta, \quad g = ag^0+\beta_1, \quad h = ah^0+\beta+\beta_1,$$

with real constants a, β *and* β_1.

The simple proof Aczél gave in Theorem 1.1 is based on the transform of (1.2) into Pexider's equation $f(x)+g(y) = h(x+y)$ through the injections (f^0, g^0, h^0) of D onto R. This method, however, cannot be employed in proving Theorem 1.0 as the functions in question need not be injections. The present proofs are based on the work of Pfanzagl [5], where he gave the following result:

THEOREM 1.2A. *Let* X *be a locally connected Hausdorff space such that any pair of points is contained in some open and connected set with compact closure. Let* $T\colon X^2 \to R$ *be continuous in each variable. If the functional equation*

$$f(x)+f(y) = h\big(T(x, y)\big) \quad \text{for all } x, y \in X \tag{1.3}$$

has two solutions (f^i, h^i) *with continuous* f^i $(i = 0, 1)$ *and non-constant* f^0, *then*

$$f^1 = af^0+\beta, \tag{1.4}$$

for some real constants a *and* β.

Equation (1.3) was first studied by Denny [3] on arcwise connected spaces X and G. Laube and Pfanzagl gave in [7] the following theorem which is a generalization of a result of Denny:

THEOREM 1.2B. *Let X be an arcwise connected Hausdorff space. Let n be any natural number greater than one. Assume that T: $X^n \to R$ is continuous in two variables. If the functional equation*

$$\sum_{i=1}^{n} f(x_i) = h\big(T(x_1, x_2, \ldots, x_n)\big) \quad \textit{for all } x_i \in X \tag{1.5}$$

has two solutions (f^i, h^i) with continuous f^i $(i = 0, 1)$ and non-constant f^0, then

$$f^1 = af^0 + \beta \tag{1.6}$$

with real constants a and β.

Theorem 1.0 is seen to be an extension of Theorems 1.2A and 1.2B. We shall adapt Pfanzagl's proof for Theorem 1.2A in a natural way with minor modifications.

2. Some uniqueness theorems. In what follows PX and PY of the previous section are always assumed.

THEOREM 2.0. *Let T: $X \times Y \to R$ be continuous in each variable. If the functional equation*

$$f(x) + g(y) = h\big(T(x, y)\big) \quad \textit{for all } x \in X, y \in Y \tag{2.0}$$

has two systems of solutions (f^i, g^i, h^i) with continuous f^i $(i = 0, 1)$ and non-constant f^0, then

$$g^1 = ag^0 + \beta_1$$

for some constants a, $\beta_1 \in R$.

Parallel to the formulation given in Pfanzagl [5] we prepare the proof by two lemmas. The first one is adopted directly from Pfanzagl [5]. We give the proof of the second one.

LEMMA 2.0. *Let X be a connected and locally connected Hausdorff space, θ: $X \to R$ a continuous function. Then for any $t_1, t_2 \in \theta(X)$, $t_1 < t_2$, there exists a connected set $B \subseteq X$ such that $\theta(B) =]t_1, t_2[$.*

LEMMA 2.1. *Suppose T: $X \times Y \to R$ is continuous in each variable and let (f, g, h) be a system of solutions of*

$$f(x) + g(y) = h\big(T(x, y)\big) \quad \textit{for all } x \in X, y \in Y, \tag{2.0}$$

with continuous f. If there exist $x_1, x_2 \in X$ such that $f(x_1) = f(x_2)$ and $T(x_1, y) \neq T(x_2, y)$ for all $y \in Y$, then g is constant on Y.

Proof. (A) Let us suppose PX(A).

There is an open and connected set $C \subset X$ with compact closure containing x_1 and x_2.

Let $y_0 \in Y$ be arbitrarily given. We shall prove that g is constant on some neighbourhood of y_0.

Let $t_i := T(x_i, y_0)$, $i = 1, 2$. We may assume $t_1 < t_2$. Since $f(x_1) = f(x_2)$ we have $h(t_1) = h(t_2)$.

According to Lemma 2.0 (applied for $X = C$, $\theta = T(\cdot, y_0)$) there exists a connected set $B \subseteq C$ such that $T(B, y_0) =]t_1, t_2[$. Let B^c be the closure of B taken in X. Then B^c is compact as B^c is closed in the compact set C^c, and so $T(B^c, y_0)$ is compact. On the other hand, $]t_1, t_2[= T(B, y_0) \subseteq T(B^c, y_0) \subseteq T(B, y_0)^c = [t_1, t_2]$ and so $T(B^c, y_0) = [t_1, t_2]$.

Hence there exist $x_i' \in B^c$ such that $T(x_i', y_0) = t_i$, $i = 1, 2$. We have now $f(x_1') + g(y_0) = h(t_1) = h(t_2) = f(x_2') + g(y_0)$ and so $f(x_1') = f(x_2')$.

Without loss of generality we may suppose that there exists $x_3 \in B$ such that $f(x_3) \geqslant f(x_1')$. We shall show that this implies the existence of $x_0 \in B^c$ such that $T(x_0, y_0) \in]t_1, t_2[$ and $f(x_0) \geqslant f(x)$ for all $x \in B^c$. As B^c is compact there exists $x_0 \in B^c$ such that $f(x_0) \geqslant f(x)$ for all $x \in B^c$. If $f(x_0) > f(x_1') = f(x_2')$, then $h(T(x_0, y_0)) > h(t_1) = h(t_2)$ and so $T(x_0, y_0) \neq t_i$, $i = 1, 2$; hence $T(x_0, y_0)$ is in $]t_1, t_2[$. If $f(x_0) = f(x_1')$ we may take $x_0 = x_3 \in B$ and so $T(x_0, y_0) \in T(B, y_0) =]t_1, t_2[$.

With $t_0 := T(x_0, y_0)$ we define

$$U := \{y \in Y \colon t_1 < T(x_0, y) < t_2\}, \qquad V := \{y \in Y \colon T(x_1', y) < t_0 < T(x_2', y)\}.$$

Since T is continuous in the second variable, U and V are both open in Y; furthermore as $y_0 \in U \cap V$, $U \cap V$ is a neighbourhood of y_0. We shall show that g is constant on $U \cap V$.

For any $y \in U$ there exists $x \in B$ such that $T(x, y_0) = T(x_0, y)$. Hence $f(x) + g(y_0) = h(T(x, y_0)) = h(T(x_0, y)) = f(x_0) + g(y)$. As $f(x) \leqslant f(x_0)$ we have $g(y) \leqslant g(y_0)$.

For any $y \in V$, $t_0 \in [T(x_1', y), T(x_2', y)] \subseteq T(B^c, y)$ (because $x_i' \in B^c$, B^c is connected and $T(\cdot, y)$ is continuous) and hence there exists $x \in B^c$ such that $t_0 = T(x, y)$. Now we have $f(x_0) + g(y_0) = h(t_0) = h(T(x, y)) = f(x) + g(y)$, and as $f(x_0) \geqslant f(x)$ we have $g(y) \geqslant g(y_0)$.

Hence for any $y \in U \cap V$ we have $g(y) = g(y_0)$.

As $y_0 \in Y$ is arbitrary, the function g is locally constant at each point of the connected space Y and is therefore constant on Y.

(B) Let us suppose PX(B).

Since X is arcwise connected there exists a bicontinuous mapping $\varphi \colon [0, 1] \to X$ so that $\varphi(0) = x_1$ and $\varphi(1) = x_2$. The simple arc $\varphi([0, 1])$ joining x_1 and x_2 is homeomorphic with $[0, 1]$ and is a space satisfying PX(A). If we consider the retriction $f|_{\varphi([0,1])}$ of f to $\varphi([0, 1])$ instead of f and $\varphi([0, 1])$ instead of X in part (A) we see that g is constant on Y.

It has been pointed out to me by M. A. McKiernan that continuity of φ is sufficient.

Proof of Theorem 2.0. Since f^0 is non-constant there exist $x_1, x_2 \in X$ such that $f^0(x_1) \neq f^0(x_2)$. This implies $T(x_1, y) \neq T(x_2, y)$ for all $y \in Y$. There exist real constants a, β such that $f^1(x_i) = af^0(x_i)+\beta$, $i = 1, 2$. The assumptions of Lemma 2.1 are fulfilled for $f\colon = f^1 - af^0$, $g : = g^1 - ag^0$ and $h\colon = h^1 - ah^0$ and hence $g^1 - ag^0$ equals a real constant, say β_1.

THEOREM 2.1. *Suppose $T\colon X \times Y \to R$ is continuous in each variable, and suppose (f^0, g^0, h^0) is a particular system of solutions of*

$$f(x)+g(y) = h\big(T(x, y)\big) \quad \textit{for all } x \in X, y \in Y \tag{2.0}$$

with non-constant continuous f^0 and non-constant g^0. Then the general system of solutions (f, g, h) of (2.0) with continuous f is given by

$$f = af^0+\beta, \quad g = ag^0+\beta_1, \quad h = ah^0+\beta+\beta_1. \tag{2.1}$$

Proof. Let (f^1, g^1, h^1) be a solution of (2.0) with continuous f^1. By Theorem 2.0 there exist real constants a and β_1 such that

$$g^1 = ag^0+\beta_1. \tag{2.2}$$

Now $(f^1-af^0, \beta_1, h^1-ah^0)$ is a system satisfying (2.0), and if possible that f^1-af^0 is not a constant then by Theorem 2.0 again there exist constants $\gamma, \delta \in R$ such that $g^0 = \gamma\cdot\beta_1+\delta$ and contradicts the assumption that g^0 is non-constant. Thus f^1-af^0 is a constant, say $\beta \in R$, and we have

$$f^1 = af^0+\beta. \tag{2.3}$$

Now $h^1\big(T(x, y)\big) = f^1(x)+g^1(y) = af^0(x)+\beta+ag^0(y)+\beta_1 = ah^0\big(T(x, y)\big)+ +\beta+\beta_1$ for all $x \in X, y \in Y$ and hence

$$h^1 = ah^0+\beta+\beta_1 \quad (\text{on } T(X, Y) \text{ of course}). \tag{2.4}$$

Equations (2.2), (2.3) and (2.4) imply that (f^1, g^1, h^1) is of the form (2.1). The fact that any system (f, g, h) defined by (2.1) satisfies (2.0) with continuous f is obvious.

THEOREM 2.2 (1.0). *Suppose $T\colon X \times Y_1 \times Y_2 \times \ldots \times Y_n \to R$ is continuous in its first variable $x \in X$ and jointly continuous in its n variables $(y_1, y_2, \ldots, y_n) \in Y_1 \times Y_2 \times \ldots \times Y_n$. Suppose $(f^0, g_1^0, g_2^0, \ldots, g_n^0, h^0)$ is a particular system of solutions of the functional equation*

$$f(x) + \sum_{i=1}^{n} g_i(y_i) = h\big(T(x, y_1, y_2, \ldots, y_n)\big) \quad \textit{for all } x \in X, y_i \in Y_i \tag{1.0}$$

consisting of continuous f^0 such that f^0 and at least one of the g_i^0's are non-

constant. Then any system of solutions $(f, g_1, g_2, \ldots, g_n, h)$ *of* (1.0) *with continuous* f *is of the form*

$$
\begin{aligned}
f &= af^0+\beta,\\
g_i &= ag_i^0+\beta_i \quad (i = 1, 2, \ldots, n), \qquad (1.1)\\
h &= ah^0+\beta+\beta_1+\beta_2+\ldots+\beta_n
\end{aligned}
$$

with real constants a, β *and* β_i $(i = 1, 2, \ldots, n)$ *and vice versa.*

Proof. The space $Y: = Y_1\times Y_2\times\ldots\times Y_n$ is connected. By applying Theorem 2.1 (for $X = X$, $Y: = Y_1\times Y_2\times\ldots\times Y_n$, $T = T$, $f^0 = f^0$, $g^0(y_1, y_2, \ldots, y_n): = \sum_{i=1}^{n} g_i^0(y_i)$, $h^0 = h^0$) we can write the general system of solutions $(f, g_1, g_2, \ldots, g_n, h)$ as

$$
\begin{aligned}
f &= af^0+\beta,\\
\sum_{i=1}^{n} g_i(y_i) &= a\Big(\sum_{i=1}^{n} g_i^0(y_i)\Big)+\gamma_1 \quad \text{for all } y_i\in Y_i \ (i = 1, 2, \ldots, n), \qquad (2.5)\\
h &= ah^0+\beta+\gamma_1,
\end{aligned}
$$

where a, β, γ_1 are real constants. Equation (2.5) is equivalent to (1.1) with appropriate constants β_i.

THEOREM 2.3. *Suppose* $T: X\times Y_1\times Y_2\times\ldots\times Y_n\to R$ *is continuous in each of its* $n+1$ *variables, and suppose* $(f^0, g_1^0, g_2^0, \ldots, g_n^0, h^0)$ *is a particular system of solutions of the functional equation* (1.0) *consisting of non-constant continuous* f^0 *and non-constant* g_i^0 $(i = 1, 2, \ldots, n)$. *Then the general solution of* (1.0) *with continuous* f *is given by* (1.1).

Proof. Let $i \leqslant n$ be arbitrarily fixed and let us leave the y_j's, say $y_j = b_j$, fixed in (1.0) for each $j \neq i$. We then have

$$f(x)+\Big(g_i+\sum_{j\neq i} g_j(b_j)\Big)(y_i) = h\big(T(x, b_1, b_2, \ldots, b_{i-1}, y_i, b_{i+1}, \ldots, b_n)\big) \quad \text{for all } x\in X, y_i\in Y_i.$$

By Theorem 2.1 there exist constants a, β and γ_i such that

$$f = af^0+\beta, \quad g_i+\sum_{j\neq i} g_j(b_j) = a\Big(g_i^0+\sum_{j\neq i} g_j^0(b_i)\Big)+\gamma_i. \qquad (2.6)$$

As f^0 is non-constant, a and β are uniquely determined and are independent of i. Putting $\beta_i: = \gamma_i+\sum_{j\neq i}[ag_j^0(b_j)-g_j(b_j)]$ in (2.6) gives (1.1) for f and g_i, and of course $h = ah^0+\beta+\beta_1+\beta_2+\ldots+\beta_n$ then follows.

3. Some applications. In 1970 J. Aczél, D. Z. Djoković and J. Pfanzagl gave in [2] the following result concerning the uniqueness of scales derived from canonical representations $P(a, b) = F(n(b) - m(a))$.

THEOREM 3.0. *Let m and n be strictly increasing functions defined on ordered sets A and B, respectively, having the same non-degenerated real interval I as range. Let m^* and n^* be another such pair of functions. If these functions satisfy the equation*

$$F(n(b) - m(a)) = F^*(n^*(b) - m^*(a)), \quad F(0) = F^*(0) = \tfrac{1}{2} \tag{3.0}$$

with F and F^* strictly increasing, then

$$m^*(a) = am(a) + \beta, \quad n^*(b) = an(b) + \beta, \quad F^*(t) = F\left(\frac{t}{a}\right), \tag{3.1}$$

where a, β are appropriate real constants.

We shall now solve the functional equation (3.0) without the standardization assumption $F(0) = F^*(0) = \frac{1}{2}$, and without assuming that m and n are ranging the same interval.

THEOREM 3.1. *Let m and n be strictly increasing functions defined on ordered sets A and B having ranges I and J non-degenerated intervals of R respectively. Let m^* and n^* be another such pair of functions. If these functions satisfy the equation*

$$F(n(b) - m(a)) = F^*(n^*(b) - m^*(a)) \tag{3.2}$$

with F and F^* injective, then

$$\begin{cases} m^*(a) = am(a) - \beta & \text{for all } a \in A, \\ n^*(b) = an(b) + \beta_1 & \text{for all } b \in B, \\ F^*(t) = F\left(\dfrac{t - \beta - \beta_1}{a}\right) & \text{for all } t \in J^* - I^*, \end{cases} \tag{3.3}$$

where $a \neq 0, \beta, \beta_1$ are real constants.

Conversely if m^*, n^* and F^* are defined by (3.3), *then they satisfy* (3.2).

Proof. Let us topologize A and B with the order topologies (cf. [4], p. 57). A and B are homeomorphic with I and J under m and n respectively; thus A and B are topologically equivalent to an interval of R. With these topologies m^* and n^* are continuous injections.

We can now apply Theorem 2.1 for $X = A$, $Y = B$, $T(a, b) = F(n(b) - m(a))$, $f^0 = -m$, $g^0 = n$, $h^0 = F^{-1}$, $f^1 = -m^*$, $g_1 = n^*$ and $h^1 = F^{*-1}$. Thus there exist real constants $a \neq 0, \beta$ and β_1 such that

$$-m^* = a(-m) + \beta, \quad n^* = an + \beta_1, \quad F^{*-1} = aF^{-1} + \beta + \beta_1$$

and this is exactly (3.3).

Other applications can be found in Aczél [1] and Pfanzagl [5].

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