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ON THE DOMINATION OF FUNCTIONS BY GRAPHS

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In the study of various familiar properties in graphs, sufficient conditions are often given in terms of the degrees of the vertices. For example, for a graph G of order $p \geqslant 3$, the *minimum degree* $\delta(G) \geqslant \frac{1}{2}(p+m)$ suffices to insure that G contains a *hamiltonian path*, a *hamiltonian cycle*, is *hamiltonian-connected* [4], or is *panconnected* [3], [5] (i.e., for each pair u, v of distinct vertices of G, there exist $u-v$ paths of each length l, where $d(u, v) \leqslant l \leqslant p-1$ and $d(u, v)$ denotes the distance between u and v in G) for $m = -1, 0, 1$ or 2, respectively. In each case, every vertex of the graph G is required to possess sufficiently high degree, where this degree is a function of the order of G. The object of this paper is to introduce a concept in which each vertex of the graph has sufficiently large degree in an associated subgraph, where this degree is a function of the order of the subgraph. This enables us to present sufficient conditions for the above and other properties in terms of functions. In this process we obtain classes of graphs which have a nonempty intersection with the corresponding minimum degree classes, while neither of pairs of corresponding classes properly contains the other. Unless stated otherwise, the terminology of [1] will be used.

Definition. A graph G of order p *dominates* a function f if the vertices of G can be labeled $v_1, v_2, \ldots, v_p$ such that, for each i with $1 \leqslant i \leqslant p$,

$$\deg_{G_i} v_i \geqslant \min\{f(p_i), p_i - 1\},$$

where $G_i = \langle\{v_i, v_{i+1}, \ldots, v_p\}\rangle$ and $p_i = |V(G_i)|$.

It will be convenient to assume that if a graph G dominates f, then the vertices of G are already labeled $v_1, v_2, \ldots, v_p$ as in the definition. Obviously, if G dominates f, then each induced subgraph G_i also dominates f. The fact that $G - v_1 = G_2$ dominates f serves as an extremely useful tool for applying mathematical induction in our proofs. The following result is now obvious:

THEOREM 1. *A graph G dominates $f(x) = x-1$ if and only if G is complete.*

Graphs that dominate constant functions have a very familiar property, which we discuss next.

THEOREM 2. *Let $n \geqslant 1$. If a graph G of order $p \geqslant n+1$ dominates $f(x) = n$, then G is n-connected. Moreover, if G is connected, then G dominates $f(x) = 1$.*

Proof. If G is a graph of order $p = n+1$ and G dominates $f(x) = n$, then $G \simeq K_{n+1}$, and hence G is n-connected.

Assume that if H is a graph of order $p-1 \geqslant n+1$ which dominates $f(x) = n$, then H is n-connected, and let G be a graph of order p which dominates $f(x) = n$. Then the vertices of G can be labeled as $v_1, v_2, \ldots, v_p$ such that, for $1 \leqslant i \leqslant p-n$, $\deg_{G_i} v_i \geqslant n$ and $\langle\{v_{p-n+1}, \ldots, v_p\}\rangle \simeq K_n$. Since $G_2 = G - v_1$ dominates $f(x) = n$ and $p_2 = p-1$ is the order of G_2, by the induction hypothesis G_2 is n-connected.

Let S be an arbitrary set of vertices of G satisfying $|S| \leqslant n-1$. It suffices to show that $G-S$ is connected. If $v_1 \in S$, then $G-S = G_2 - (S - \{v_1\})$. Thus $G-S$ is connected, since $|S-\{v_1\}| \leqslant n-2$ and $\varkappa(G_2) \geqslant n$. So we may assume that $v_1 \notin S$. Then $S \subset V(G_2)$ and, since $|S| \leqslant n-1$ and $\varkappa(G_2) \geqslant n$, the graph $G_2 - S$ is connected. Since $\deg_{G_1} v_1 \geqslant n$, there is a vertex w of $G_2 - S$ such that $v_1 w \in E(G)$. This implies that $G-S$ is connected.

Finally, let G be a connected graph of order p. If $p = 1$ or 2, then clearly G dominates $f(x) = 1$. If $p \geqslant 3$, let T be a spanning tree of G. By labeling an end vertex of T as v_1, labeling an end vertex of the tree $T - \{v_1, v_2, \ldots, v_{i-1}\}$ as v_i for $2 \leqslant i \leqslant p-1$, and labeling the remaining vertex of T as v_p, we observe that T, and hence G, dominates the function $f(x) = 1$.

We note that this result is the best possible. Clearly, every disconnected graph dominates $f(x) = 0$. For $n \geqslant 2$, consider disjoint graphs G_1 and G_2, where $G_1 \simeq K_{n-1}$ and $G_2 \simeq \bar{K}_m$ for $m \geqslant 2$. Let G be the graph obtained by adding all possible edges between G_1 and G_2. Then G is at least of order $n+1$ and G dominates $f(x) = n-1$, but G is not n-connected.

Next, we present a sequence of results related to the hamiltonian properties mentioned earlier.

THEOREM 3. *If G is a graph of order $p \geqslant 3$ and G dominates $f(x) = \frac{1}{2}(x-1)$, then G contains a hamiltonian path.*

Proof. The result is true for $p = 3$. Assume that if H is a graph of order $p-1 \geqslant 3$ and H dominates $f(x) = \frac{1}{2}(x-1)$, then H contains a hamiltonian path. Let G be a graph of order p which dominates $f(x) = \frac{1}{2}(x-1)$. Then $G_2 = G - v_1$ contains a hamiltonian path P by the induction hypothesis, since G_2 dominates $f(x) = \frac{1}{2}(x-1)$ and $|V(G_2)|$

$= p-1$. Let a and b denote the end vertices of P. If v_1a or v_1b is an edge of G, then P can be extended to a hamiltonian path of G. So we may assume that

$$\{v_1a, v_1b\} \cap E(G) = \emptyset.$$

Since G dominates $f(x) = \frac{1}{2}(x-1)$, we have $\deg_{G_1} v_1 \geqslant \frac{1}{2}(p-1)$, i.e., v_1 is adjacent to at least $\frac{1}{2}(p-1)$ of the $p-3$ vertices in $P-\{a, b\}$. Thus v_1 is adjacent to at least two adjacent vertices in $P-\{a, b\}$, which implies that P can be extended to a hamiltonian path of G.

A graph G of order $p \geqslant 3$ is said to be *n-hamiltonian*, $0 \leqslant n \leqslant p-3$, if the graph obtained from G by deleting any k vertices, $0 \leqslant k \leqslant n$, is hamiltonian. Thus, a 0-hamiltonian graph is simply a hamiltonian graph. It was shown in [2] that if G is a graph of order $p \geqslant 3$ and $\delta(G) \geqslant \frac{1}{2}(p+n)$, $0 \leqslant n \leqslant p-3$, then G is n-hamiltonian.

THEOREM 4. *Let n be a nonnegative integer. If a graph G of order $p \geqslant n+3$ dominates $f(x) = \frac{1}{2}(x+n)$, then G is n-hamiltonian.*

Proof. For $p = n+3$, $G \simeq K_{n+3}$, and the result holds. Assume that if H is a graph of order $p-1 \geqslant n+3$ and H dominates $f(x) = \frac{1}{2}(x+n)$, then H is n-hamiltonian. Let G be a graph of order p which dominates $f(x) = \frac{1}{2}(x+n)$. Then $G_2 = G-v_1$ is n-hamiltonian and $\deg_{G_1} v_1 \geqslant \frac{1}{2}(p+n)$.

Let $S \subset V(G)$, with $|S| = k$ and $0 \leqslant k \leqslant n$. We wish to show that $G-S$ is hamiltonian.

If $v_1 \in S$, then $G-S = G_2-(S-\{v_1\})$ is hamiltonian, since G_2 is n-hamiltonian. So suppose that $v_1 \notin S$. Now $0 \leqslant k \leqslant n$ and $S \subset V(G_2)$ imply that G_2-S contains a hamiltonian cycle C on $p-k-1$ vertices. Since

$$\deg_{G_1} v_1 \geqslant \tfrac{1}{2}(p+n) \geqslant \tfrac{1}{2}(p+k),$$

v_1 is adjacent to at least $\frac{1}{2}(p+k)-k = \frac{1}{2}(p-k)$ vertices of C, and hence is adjacent to two consecutive vertices of C. This implies the existence of a hamiltonian cycle in $G-S$.

THEOREM 5. *If a graph G of order $p \geqslant 3$ dominates $f(x) = \frac{1}{2}(x+1)$, then G is hamiltonian-connected.*

Proof. The result holds when $p = 3$, for then $G \simeq K_3$. Let $p \geqslant 4$ and assume that the result holds for all graphs H of order less than p. Let G be a graph of order p which dominates $f(x) = \frac{1}{2}(x+1)$. Then $G_2 = G-v_1$ is hamiltonian-connected and $\deg_{G_1} v_1 \geqslant \frac{1}{2}(p+1)$. Let $a, b \in V(G)$. If $v_1 \notin \{a, b\}$, let P be a hamiltonian $a-b$ path in G_2. Then $\deg_{G_1} v_1 \geqslant \frac{1}{2}(p+1)$ implies that v_1 is adjacent to two consecutive vertices on P. Thus P can be extended to a hamiltonian $a-b$ path in G. Otherwise, without loss of generality, assume $v_1 = a$. Since $\deg_{G_1} v_1 \geqslant \frac{1}{2}(p+1)$, there exists a vertex $w \neq b$ in G_2 which is adjacent to v_1. Then the edge v_1w together with a hamiltonian $w-b$ path in G_2 yields a hamiltonian $a-b$ path in G.

It has been recently established that if $\delta(G) \geqslant \frac{1}{2}(p+2)$ for a graph G of order $p \geqslant 4$, then between every pair of distinct vertices of G there exist paths of each length l, where $2 \leqslant l \leqslant p-1$. Thus, the minimum degree requirement implies that G is *panconnected* [3], [5]. The following example shows that it is not possible to obtain a similar result by requiring that G dominate the function $f(x) = \frac{1}{2}(x+2)$.

Let G be the graph of even order $p \geqslant 12$, where

$$V(G) = \{v_1, v_2, \ldots, v_p\},$$

with

$$H = \langle\{v_2, \ldots, v_{p-1}\}\rangle \simeq K_{p-2}$$

and

$$E(G) = E(H) \cup \{v_1 v_i \mid 2 \leqslant i \leqslant \tfrac{1}{2}(p+2)\} \cup \{v_j v_p \mid \tfrac{1}{2}(p+4) \leqslant j \leqslant p-1\} \cup \{v_1 v_p\}.$$

It can be verified that with this labeling of the vertices, G dominates $(fx) = \frac{1}{2}(x+2)$. However, there is no $v_1 - v_p$ path of length two in G.

The following examples (a)-(c) illustrate that the results obtained in Theorems 3, 4, and 5 are "best possible". Example (d) indicates that, for each $m \geqslant -1$, the class of graphs which are not complete and which dominate $f(x) = \frac{1}{2}(x+m)$ is not contained in the class consisting of graphs G such that $\delta(G) \geqslant \frac{1}{2}(|V(G)|+m)$.

Let m and p be integers satisfying $-2 \leqslant m \leqslant p-3$ with $p \geqslant 3$, and let G be the graph of order p such that

$$V(G) = \{v_1, v_2, \ldots, v_p\}, \qquad H = G - v_p \simeq K_{p-1},$$

and

$$E(G) = E(H) \cup \{v_p v_i \mid p-m-2 \leqslant i \leqslant p-1\}.$$

(For $m = -2$, $E(G) = E(H)$.)

Then G dominates $f(x) = \frac{1}{2}(x+m)$ and $\deg_G v_p = m+2$. We now observe the following:

(a) For $m = -2$, G dominates $f(x) = \frac{1}{2}(x-2)$, but G does not contain a hamiltonian path.

(b) For $m = n-1 \geqslant -1$, G dominates $f(x) = \frac{1}{2}(x+n-1)$, but G is not n-hamiltonian.

(c) For $m = 0$ and $p \geqslant 4$, G dominates $f(x) = \frac{1}{2}x$, but G is not hamiltonian-connected.

(d) For $m \geqslant -1$ and $p \geqslant m+5$, G dominates $f(x) = \frac{1}{2}(x+m)$, but $\delta(G) \geqslant \frac{1}{2}(p+m)$ fails to hold.

Finally, we wish to show that the classes of graphs obtained by the domination of $f(x) = \frac{1}{2}(x+m)$ for each $m \geqslant -1$ do not contain the corresponding minimum degree classes. Towards this end we derive the following necessary condition for G to dominate $f(x) = \frac{1}{2}(x+m)$. As usual, $\{y\}$ will denote the least integer not less than y.

THEOREM 6. *Let p and m be integers of the same parity satisfying $-1 \leqslant m \leqslant p-4$. If a graph G of order p dominates $f(x) = \frac{1}{2}(x+m)$, then $|E(G)| > \frac{1}{4}p(p+m)$.*

Proof. Let $q = |E(G)|$. Then clearly

$$q = \sum_{i=1}^{p} \deg_{G_i} v_i,$$

where, by definition, $\deg_{G_i} v_i \geqslant \frac{1}{2}(p_i+m)$ if $\frac{1}{2}(p_i+m) < p_i - 1$ and $\deg_{G_i} v_i = p_i - 1$ if $\frac{1}{2}(p_i+m) \geqslant p_i - 1$.

Since $p_i = |V(G_i)| = p-i+1$, we have $\frac{1}{2}(p_i+m) \geqslant p_i - 1$ if and only if $i \geqslant p-m-1$. Hence $i \geqslant p-m-1$ implies $\deg_{G_i} v_i = p_i - 1$, i.e., $G_{p-m-1} \simeq K_{m+2}$. Furthermore, for $i \leqslant p-m-2$,

$$\deg_{G_i} v_i \geqslant \left\{\frac{1}{2}(p_i+m)\right\} = \left\{\frac{1}{2}(p+m-i+1)\right\}.$$

Now, considering the parities of i, $p+m$, and $p-m-2$, we see that

$$\sum_{i=1}^{p-m-2} \left\{\frac{1}{2}(p+m-i+1)\right\} = \sum_{i=1}^{p-m-2} \frac{1}{2}(p+m-i) + \frac{3}{4}(p-m-2).$$

Thus,

$$\begin{aligned} q &= \sum_{i=1}^{p} \deg_{G_i} v_i \geqslant \sum_{i=1}^{p-m-2} \left\{\frac{1}{2}(p+m-i+1)\right\} + |E(K_{m+2})| \\ &= \sum_{i=1}^{p-m-2} \frac{1}{2}(p+m-i) + \frac{3}{4}(p-m-2) + |E(K_{m+2})| \\ &= \frac{1}{4}(p-m-2)(p+3m+1) + \frac{3}{4}(p-m-2) + \frac{1}{2}(m+2)(m+1) \\ &= \frac{1}{4}p(p+m) + \frac{1}{4}(p-m-2)(m+2) \\ &> \frac{1}{4}p(p+m). \end{aligned}$$

Now, let p and m be integers satisfying $-1 \leqslant m \leqslant p-4$ such that $p+m$ is divisible by 4. Under these conditions, there exist $\frac{1}{2}(p+m)$-regular graphs of order p. For such a graph G, $\delta(G) \geqslant \frac{1}{2}(p+m)$, but by the preceding theorem G does not dominate $f(x) = \frac{1}{2}(x+m)$.

The work presented here is a preliminary report on the concept introduced, and further related investigations will form the basis for future presentations.

REFERENCES

[1] M. Behzad and G. Chartrand, *Introduction to the theory of graphs*, Allyn and Bacon, Boston 1971.

[2] G. Chartrand, S. F. Kapoor and D. R. Lick, *n-Hamiltonian graphs*, Journal of Combinatorial Theory 9 (1970), p. 308-312.

[3] R. J. Faudree, C. C. Rousseau and R. H. Schelp, *Theory of path length distributions I*, Discrete Mathematics 6 (1973), p. 35-52.

[4] O. Ore, *Hamilton connected graphs*, Journal de Mathématiques Pures et Appliquées 42 (1963), p. 21-27.

[5] J. E. Williamson, *Panconnected graphs*, Notices of the American Mathematical Society 21 (1974), p. A-37.

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