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FACTORIZATION AND INVERSE EXPANSION THEOREMS FOR UNIFORMITIES

BY

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The aim of this paper is to prove two theorems on uniform spaces, both being generalizations of known theorems of Mardešić [6] concerning compact Hausdorff spaces and their further generalizations to more general topological spaces due to Pasynkov [8]. The first theorem is concerned, roughly speaking, with the possibility to get for each uniform map $X \to Y$ a factorization $X \to Z \to Y$ through a uniform space Z having uniform dimension the same as X and uniform weight the same as Y, and the second with the possibility to get an inverse expansion of a uniform space by means of metrizable uniform spaces having the same uniform dimension as the given space.

The main tool in our proofs is a quotient operation on pseudouniformities preserving dimension. This operation was defined by Bourbaki [2] without taking into consideration dimension properties. It was also considered by Arhangelskii [1] without specifying its uniform properties.

By using this operation, the proofs of uniform generalizations of Mardešić-Pasynkov theorems are natural and in a sense standard, which shows that the category of uniform spaces occurs to be a natural domain for the theorems of such a kind.

Uniformities are always regarded as families of coverings as in the book of Isbell [5].

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1. Preliminaries. The quotient of a pseudouniformity. Let X be a set and P a covering of X. If $x \in X$, then the *star of x with respect to P* is the set

$$\mathrm{st}(x, P) = \bigcup \{U\colon\ x \in U \in P\}.$$

If $U \in P$, then the *star of U with respect to P* is the set

$$\mathrm{st}(U, P) = \bigcup \{U'\colon\ U' \in P,\ U' \cap U \neq \emptyset\}.$$

If P and Q are coverings of X, then Q is said to be a *refinement* — respectively a *star refinement* — of P, in symbols $Q \succ P$ — respectively $Q \underset{*}{\succ} P$ — iff for each $V \in Q$ there exists a $U \in P$ such that $V \subset U$ — respectively $\mathrm{st}(V, Q) \subset U$.

A *pseudouniformity* on X is a family $\mathscr{U}$ of coverings of X such that

(1) $\mathscr{U}$ is a directed family with respect to the star refinement,

(2) if $Q \in \mathscr{U}$ and $Q \succ P$, then $P \in \mathscr{U}$.

A subfamily $\mathscr{B}$ of $\mathscr{U}$ such that each $Q \in \mathscr{U}$ has a refinement $P \in \mathscr{B}$ is said to be a *base* of $\mathscr{U}$. Of course, $\mathscr{B}$ satisfies (1).

If a family $\mathscr{B}$ of coverings of X satisfies (1), then it is a base for a pseudouniformity consisting of all coverings P of X such that $Q \succ P$ for some $Q \in \mathscr{B}$.

If a pseudouniformity $\mathscr{U}$ is such that

(3) for each two distinct points x' and x'' from X there exists P such that $x' \notin \mathrm{st}(x'', P)$,

then it is said to be a *uniformity*.

A family of coverings of X satisfying (1) and (3) is a base for a uniformity. For the brevity sake we shall denote the base and the uniformity induced by the base by the same symbols.

If $\mathscr{U}$ is a pseudouniformity on X, $\mathscr{V}$ is a pseudouniformity on Y and $f\colon X \to Y$ is a map, then f is said to be *uniform* with respect to $\mathscr{U}$ and $\mathscr{V}$ if for each $Q \in \mathscr{V}$ there is $f^{-1}(Q) \in \mathscr{U}$ ($f^{-1}(Q)$ means the covering $\{f^{-1}(V)\colon V \in Q\}$).

As usually, a pair $(X, \mathscr{U})$, where $\mathscr{U}$ is a pseudouniformity on X, is said to be *pseudouniform space*. Uniform maps will be denoted also by $(X, \mathscr{U}) \to (Y, \mathscr{V})$.

Each pseudouniformity leads to a uniformity by means of the following quotient operation.

If $\mathscr{U}$ is a pseudouniformity on X, then sets

$$[x] = \bigcap \{\mathrm{st}(x, P)\colon P \in \mathscr{U}\}$$

form a partition of X. In fact, let $y \in [x] = \bigcap \{\mathrm{st}(x, P)\colon P \in \mathscr{U}\}$. For each $P \in U$ let us take $Q(P)$ such that $Q(P) \underset{*}{\succ} P$. There is $\mathrm{st}[y, Q(P)] \subset \mathrm{st}(x, P)$. Therefore

$$[y] = \bigcap \{\mathrm{st}(y, R)\colon R \in \mathscr{U}\} \subset \bigcap \{\mathrm{st}[y, Q(P)]\colon P \in \mathscr{U}\}$$
$$\subset \bigcap \{\mathrm{st}(x, P)\colon P \in \mathscr{U}\} = [x].$$

But if $y \in [x]$, then $x \in [y]$, because $x \notin [y]$ implies $x \notin \mathrm{st}(y, P)$ for some $P \in \mathscr{U}$, and hence $y \notin \mathrm{st}(x, P)$ for this P, contrary to $y \in [x]$. Thus it is proved that $[x]$ and $[y]$ are equal if they have a point in common.

Let $q\colon X \to X_{\mathscr{U}}$ be the quotient map onto the partition described above, $q(x) = [x]$ for $x \in X$. For each $P \in \mathscr{U}$ the family

$$_qP = \{X_{\mathscr{U}} - q(X-U)\colon\ U \in P\}$$

forms a covering of X (in fact, let $[x] \in X_{\mathscr{U}}$; then $[x] \subset \mathrm{st}(x, P)$; we take $Q \in \mathscr{U}$ such that $Q \underset{*}{\succ} P$; there is $[x] \subset \mathrm{st}(x, Q) \subset V$, where $V \in P$; if $y \subset X - V$, then $[x] \cap [y] = \emptyset$, thus $[x] \in X_{\mathscr{U}} - q(X-V)$ for $V \in P$ described above). The family $_q\mathscr{U} = \{_qP\colon P \in \mathscr{U}\}$ forms a base of a uniformity on $X_{\mathscr{U}}$. To see (1), let us take arbitrary coverings $_qP'$ and $_qP''$ from $_q\mathscr{U}$. Let $P \in \mathscr{U}$ be such that $P \underset{*}{\succ} P'$ and $P \underset{*}{\succ} P''$. Then, as is easy to see, $_qP \underset{*}{\succ} {_qP'}$ and $_qP \underset{*}{\succ} {_qP''}$. To check (3), let $[x] \neq [y]$; then there exists $P \in \mathscr{U}$ such that $y \notin \mathrm{st}(x, P)$. Thus $q(x) = [x] \in \mathrm{st}([x], {_qP})$ and $q(y) = [y] \notin \mathrm{st}([x], {_qP})$ are distinguished by a covering $_qP$ belonging to $_q\mathscr{U}$.

The map $q\colon X \to X_{\mathscr{U}}$ is uniform with respect to $\mathscr{U}$ and $_q\mathscr{U}$. In fact, $_qP \in {_q\mathscr{U}}$; we have

$$q^{-1}\{X_{\mathscr{U}} - q(X-U)\colon\ U \in P\} = \{X - q^{-1}[q(X-U)]\colon\ U \in P\} \prec Q,$$

where $Q \underset{*}{\succ} P$, $Q \in \mathscr{U}$. To check this, let $V \in Q$. There exists $U \in P$ such that $\mathrm{st}(V, Q) \subset U$. This implies that $V \subset X - q^{-1}[q(X-U)]$, since $x \in V \cap q^{-1}[q(X-U)]$ leads to $q(x) = q(y)$ for some $y \in X - U$. But $x \in V \subset \mathrm{st}(V, Q) \subset \mathscr{U}$ and $q(x) = q(y)$ implies $y \in \mathrm{st}(x, Q) \subset \mathrm{st}(V, Q) \subset \mathscr{U}$; a contradiction.

We shall show that $_q\mathscr{U}$ is the greatest uniformity $\mathscr{V}$ on $X_{\mathscr{U}}$ such that q is uniform with respect to $\mathscr{U}$ and $\mathscr{V}$. To see this, we shall show that if a covering Q of X is such that $q^{-1}(Q) \in \mathscr{U}$, then $Q \in {_q\mathscr{U}}$. In fact, if $q^{-1}(Q) \in \mathscr{U}$, then

$$q[q^{-1}(Q)] = \{X_U - q[X - q^{-1}(V)]\colon V \in Q\} \in {_q\mathscr{U}},$$

since $X_{\mathscr{U}} - q[X - q^{-1}(V)] = V$ for each $V \in Q$; we get $Q = q[q^{-1}(Q)]$.

The uniformity $_q\mathscr{U}$ on $X_{\mathscr{U}}$ will be said to be a *quotient* of pseudouniformity $\mathscr{U}$.

The quotient q of $\mathscr{U}$ is unique up to an isomorphism determined by the condition:

(5) for each map $f\colon X \to Y$ uniform with respect to a uniformity on Y there exists a unique uniform map filling up the diagram

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "q"'] & Y \\
X_{\mathscr{U}} \arrow[ur, dashed] &
\end{tikzcd}$$

The minimal cardinality of a base of a pseudouniformity $\mathscr{U}$ is called the *weight* of $\mathscr{U}$.

A covering P is said to be of *order* $\leqslant n$, $\operatorname{ord} P \leqslant n$, iff for each $x \in X$ there exists at most n elements U of P such that $x \in U$.

If a pseudouniformity $\mathscr{U}$ contains a base consisting of coverings of order $\leqslant n+1$, then it is said to be of *dimension* $\leqslant n$, $\dim \mathscr{U} \leqslant n$.

PROPOSITION 1. *If $\mathscr{U}$ is a pseudouniformity on X and ${}_q\mathscr{U}$ is the quotient of $\mathscr{U}$, then*

$$\text{weight}\ {}_q\mathscr{U} \leqslant \text{weight}\ \mathscr{U}, \tag{6}$$

$$\dim {}_q\mathscr{U} \leqslant \dim \mathscr{U}. \tag{7}$$

Proof. Property (6) follows immediately from the construction of the quotient. Property (7) easily follows if we note that if $P \in U$ and V_i is an element of P for $i = 1, 2, \ldots, k$, then

$$V_1 \cap V_2 \cap \ldots \cap V_k = \emptyset$$
$$\Rightarrow [X_{\mathscr{U}} - q(X - V_1)] \cap \ldots \cap [X_{\mathscr{U}} - q(X - V_k)] = \emptyset.$$

PROPOSITION 2. *Any map $f\colon X \to Y$ uniform with respect to a pseudouniformity $\mathscr{U}$ on X and a uniformity on Y admits a decomposition into*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "q"'] & Y \\ X_{\mathscr{U}} \arrow[ur] & \end{tikzcd}$$

maps uniform with respect to a uniformity ${}_q\mathscr{U}$ on $X_{\mathscr{U}}$ such that weight ${}_q\mathscr{U} \leqslant$ weight $\mathscr{U}$ and $\dim_q\mathscr{U} \leqslant \dim \mathscr{U}$.

Proof follows immediately from (5) by taking q as the quotient of $\mathscr{U}$. Factor map $X_{\mathscr{U}} \to Y$ is now uniquely determined by virtue of (5).

PROPOSITION 3. *For each subfamily $\mathscr{B}$ of a uniformity $\mathscr{U}$ on X there exists a pseudouniformity $\tilde{\mathscr{U}}$ such that*

$$\mathscr{B} \subset \tilde{\mathscr{U}} \subset \mathscr{U}, \quad \operatorname{card} \mathscr{B} = \text{weight}\ \tilde{\mathscr{U}}, \quad \dim \tilde{\mathscr{U}} \leqslant \dim \mathscr{U}. \tag{8}$$

Proof. For every two coverings $P', P'' \in \mathscr{B}$ take a covering $P \in \mathscr{U}$ such that $P \underset{*}{\succ} P'$ and $P \underset{*}{\succ} P''$ and $\operatorname{ord} P \leqslant \dim \mathscr{U} + 1$. Let $\mathscr{B}_1$ denote the family of all such coverings. Assume that families $\mathscr{B}_1, \ldots, \mathscr{B}_n$ are defined. Applying the above operation to $\bigcup \{\mathscr{B}_i\colon i = 1, \ldots, n\}$ we receive $\mathscr{B}_{n+1}$. Family $\tilde{\mathscr{U}} = \bigcup \{\mathscr{B}_i\colon i = 1, 2, \ldots\}$ forms a pseudouniformity satisfying (8).

2. Factorization theorem for uniformities.

THEOREM 1. *Let $f\colon X \to Y$ be a map uniform with respect to a uniformity $\mathscr{U}$ on X and a uniformity $\mathscr{V}$ on Y. Then there exists a factorization of f (in general, not unique)*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "q"', "\text{onto}"] & Y \\ Z \arrow[ur, "h"'] & \end{tikzcd}$$

uniform with respect to a uniformity $\mathscr{W}$ on Z, and such that

(9) $$\dim \mathscr{W} \leqslant \dim \mathscr{U},$$

(10) $$\text{weight}\, \mathscr{W} \leqslant \text{weight}\, \mathscr{V}.$$

Proof. Let $\mathscr{B}$ be a base of the uniformity $\mathscr{V}$ on Y such that card $\mathscr{B}$ $\leqslant$ weight $\mathscr{V}$. Consider the family $f^{-1}(\mathscr{B})$ of coverings on X. Since f is uniform, we have $f^{-1}(\mathscr{B}) \subset \mathscr{U}$. According to Proposition 3, the family $f^{-1}(\mathscr{B})$ may be extended to a base of a pseudouniformity $\tilde{\mathscr{U}} \subset \mathscr{U}$ such that $\dim \tilde{\mathscr{U}} \leqslant \dim \mathscr{U}$ and weight $\tilde{\mathscr{U}} \leqslant \operatorname{card} f^{-1}(\mathscr{B})$. Now $f: X \to Y$ is uniform with respect to $\tilde{\mathscr{U}}$ and $\mathscr{V}$. Applying Proposition 2 we get the required factorization of f.

The factorization $f = h \circ g$ satisfying (9) and (10) is not unique: it depends on the choosing of a pseudouniformity $\tilde{\mathscr{U}}$ containing $f^{-1}(\mathscr{B})$ (it is easy to see that it does not depend on the chosing of $\mathscr{B}$, the base for $\mathscr{V}$).

We shall call factorization $f = h \circ g$ *finer* than a factorization $f = h' \circ g'$ if there exists a uniform map $k : Z \to Z'$ such that $k \circ g = g'$ (hence $h \circ k = h'$) and uniformity on Z satisfies (9) and (10).

PROPOSITION 4. *There does not exist the finest factorization for f satisfying* (9) *and* (10), *unless weight $\mathscr{U} \leqslant$ weight $\mathscr{V}$.*

Proof. It suffices to show that for every covering P from U there exists a factorization

(11) $$(X, \mathscr{U}) \xrightarrow{q_P} (Z_P, \mathscr{W}_P) \to (Y, \mathscr{V})$$

having properties (9) and (10) and such that $q_P^{-1}({}_qP) \succ P$, where ${}_qP \in \mathscr{W}_P$. Since the finest factorization $(X, \mathscr{U}) \xrightarrow{g} (Z, \mathscr{W}) \to (Y, \mathscr{V})$ of f is finer than each of (11), we infer that $g\colon (X, \mathscr{U}) \to (Z, \mathscr{W})$ is a uniform isomorphism. Thus we get, according to (10), weight $\mathscr{U} \leqslant$ weight $\mathscr{V}$.

To get the decomposition (11) for each $P \in \mathscr{U}$, let us consider the family $\mathscr{B}'$ of coverings from $\mathscr{U}$ such that $P \in \mathscr{B}'$ and $f^{-1}(Q) \in \mathscr{B}'$ for each $Q \in \mathscr{V}$ and let us extend this $\mathscr{B}'$ to a pseudouniformity $\mathscr{W}_P \subset \mathscr{U}$ in the way described in Proposition 3. The quotient map $(X, \mathscr{U}) \xrightarrow{q_P} (Z_P, \mathscr{W}_P)$ is the first map in (11) and the second map of (11) is uniquely determined by ${}_q\mathscr{P}$ as was shown in Proposition 2.

PROPOSITION 5. *If the set S of factorizations of f is such that*

(12) *each factorization of f satisfying* (9) *and* (10) *is isomorphic to a factorization from S,*

then the cardinality of S is greater than the weight of $\mathscr{V}$.

Proof. Suppose that there exists a set S of factorizations of f having property (12) and such that the cardinality of S is not greater than the weight of $\mathscr{V}$. Thus the product $(Z_*, \mathscr{W}_*)$ of the "middle" spaces in these factorizations $(X, \mathscr{U}) \xrightarrow{g} (Z, \mathscr{W}) \xrightarrow{h} (Y, \mathscr{V})$, where $\mathscr{W} \in S$, is of weight not greater than the weight of $\mathscr{V}$. The maps $g, \mathscr{W} \in S$ induce a map $g_*\colon (X, \mathscr{U}) \to (Z_*, \mathscr{W}_*)$. There exists a factorization $(X, \mathscr{U}) \xrightarrow{g_*} (Z_*, \mathscr{W}_*) \to (Y, \mathscr{V})$ of f. By Proposition 1, the map g_* may be factorized through a map $g'\colon (X, \mathscr{U}) \to (Z, \mathscr{W}')$ such that $\dim \mathscr{W}' \leqslant \dim \mathscr{U}$ and weight $\mathscr{W}' \leqslant$ weight $\mathscr{W}_* \leqslant$ weight $\mathscr{V}$. The factorization of g_* through g' induces a factorization of f satisfying (9) and (10) which is the finest among having these properties; a contradiction with Proposition 4.

In order to get topological corollaries of Theorem 1, let us recall some facts connecting uniform and topological notions.

If X is a set, then any uniformity $\mathscr{U}$ on X determines a *topology* $T(\mathscr{U})$ on X: a subset A of X is *open* in $T(\mathscr{U})$ if for each $x \in A$ there exists $P \in \mathscr{U}$ such that $\mathrm{st}(x, P) \subset A$. $T(\mathscr{U})$ is a completely regular topology on X. Conversely, each completely regular topology on X is induced by a uniformity, not necesserily unique. The uniqueness holds, e.g., for compact spaces.

We say that the weight of a topology T is not greater than τ, weight $T \leqslant \tau$, if there exists a base for T with the cardinality $\leqslant \tau$. If $\mathscr{B}$ is a base for uniformity $\mathscr{U}$, then it is easy to verify that interiors of elements of coverings belonging to $\mathscr{B}$ form a covering belonging to $\mathscr{B}$ and that interiors of elements of all coverings from $\mathscr{B}$ form a base for the topology induced by the uniformity $\mathscr{U}$. Thus, if $\mathscr{B}$ consists of finite coverings of X, then the weight of the topology T on X is not greater than card $\mathscr{B}$. In the case of compact spaces the weight of T coincides with the weight of the unique uniformity consisting of finite coverings.

If T is a completely regular topology on X, then it seems reasonable to assume that the covering dimension of T is defined by means of functionally open coverings as follows: $\dim T \leqslant n$ iff for each finite covering P of X consisting of functionally open subsets there exists a finite covering Q consisting of functionally open subsets such that $Q \succ P$ and $\operatorname{ord} Q \leqslant n+1$ (see the book of Engelking [4]).

As was shown by Pasynkov [8], $\dim T \leqslant n$ iff $\dim \mathscr{U}_{\max} \leqslant n$, where $\mathscr{U}_{\max}$ is the greatest uniformity on X inducing the topology T.

If the topology is fixed, it is convenient to write $\dim X$ and weight X instead of $\dim T$ and weight T.

COROLLARY (Mardešić [6]). *Let X and Y be compact Hausdorff spaces and let $f\colon X \to Y$ be a continuous map. There exist a compact space Z such that* $\dim Z \leqslant \dim X$ *and weight $Z \leqslant$ weight Y and a factorization $X \underset{\text{onto}}{\overset{g}{\to}} Z \overset{h}{\to} Y$ of f into continuous maps.*

Proof. The map $f\colon X \to Y$, being uniform with respect to the unique uniformities on X and Y inducing the given topologies, admits, by virtue of Theorem 1, a factorization $X \underset{\text{onto}}{\overset{g}{\to}} Z \overset{h}{\to} Y$, where g and h are uniform with respect to a uniformity $\mathscr{W}$ on Z such that $\dim \mathscr{W} \leqslant \dim X$ and weight $\mathscr{W} \leqslant$ weight Y. Since g is continuous, Z is compact. Hence $\dim \mathscr{W} = \dim Z$ and weight $\mathscr{W} =$ weight Z, which proves the theorem.

Pasynkov [8] proved that if X is a completely regular space, Y is a metric space and $f\colon X \to Y$ is continuous, then there exist a metric space Z such that $\dim Z \leqslant \dim X$ and weight $Z \leqslant$ weight X and a factorization $X \underset{\text{onto}}{\overset{g}{\to}} Z \overset{h}{\to} Y$ of f into continuous maps.

Main assertions of the Pasynkov result follow from Theorem 1 immediately. Namely, consider $f\colon X \to Y$ as a uniform map $f\colon (X, \mathscr{U}_{\max}) \to (Y, \mathscr{V})$, where $\mathscr{V}$ is a metrizable uniformity (i.e., weight $\mathscr{V} \leqslant \aleph_0$), Theorem 1 gives a factorization $(X, \mathscr{U}_{\max}) \to (Z, \mathscr{W}) \to (Y, \mathscr{V})$ such that $\dim \mathscr{W} \leqslant \dim \mathscr{U}_{\max}$ and weight $\mathscr{W} \leqslant$ weight $\mathscr{V}$. The latter inequality means that Z is metrizable, the former according to a theorem of Nagata ([7], p. 126, Th. V. 1.) means that $\dim Z \leqslant \dim \mathscr{U}_{\max}$. Since $\dim \mathscr{U}_{\max} = \dim X$, we get $\dim Z \leqslant \dim X$.

To get the assertion concerning topological weight of Z an additional procedure is needed in which the cardinalities of coverings are taken into consideration.

3. Inverse expansions associated with a given uniformity. Let be given a uniform space $(X, \mathscr{U})$ and a family F of subsets of X. We say that F *contains arbitrarily small sets* iff for every covering $P \in \mathscr{U}$ there exists $A \in F$ such that $A \subset V$ for a $V \in P$.

We say that uniformity $\mathscr{U}$ in a set X is a *complete uniformily* iff for each family F of closed subsets of X (in the topology induced by $\mathscr{U}$) such that F has the finite intersection property and contains arbitrarily small sets, the intersection $\bigcap\{A: A \in F\}$ is non-empty.

THEOREM 2. *Let X be a completely regular space and let $\mathcal{U}$ be a uniformity on X. Then there exists a uniform dense embedding of the uniform space $(X, \mathcal{U})$ into a uniform space $(\tilde{X}, \tilde{\mathcal{U}})$, where $(\tilde{X}, \tilde{\mathcal{U}})$ is the inverse limit of a system over a directed set M such that*

1. *spaces of the system are metrizable and of dimension not greater than* $\dim \mathcal{U}$,
2. *maps $\pi_\beta^\alpha: X_\alpha \to X_\beta$ of the system are uniform and onto,*
3. $\operatorname{card} M \leqslant \operatorname{weight} \mathcal{U}$.

If, in addition, $\mathcal{U}$ is a complete uniformity, then $(X, \mathcal{U})$ is uniformly isomorphic with $(\tilde{X}, \tilde{\mathcal{U}})$.

Note. If a space X is an inverse limit of a system of metrizable spaces, then X being a closed subspace of the product of metrizable spaces, i.e. a closed subspace of complete space, is a complete space. Thus, Theorem 2 gives the possibility to expand a space into a system of metrizable spaces under the best possible conditions.

Before proving Theorem 2, let us quote the following immediate topological corollaries:

COROLLARY 1 (Mardešić [6]). *If a topological space is Hausdorff compact, then it is homeomorphic with the inverse limit of a system of compact metrizable spaces; moreover, the cardinality of the system is not greater than the weight of the space and the dimension of each space in the system is not greater than the dimension of the given space.*

A completely regular space is said to be *complete* (Dieudonné [3]) iff there exists a complete uniformity on X inducing the topology.

COROLLARY 2 (Pasynkov [8]). *If X is a completely regular space, then there exists an extension of X to a complete space which is an inverse limit of a system of metrizable spaces; moreover, the cardinality of the system is not greater than the weight of X and the dimension of each space in the system is not greater than that of X.*

Dieudonné [3] has shown that each complete space is a closed subspace of a product of metrizable spaces. Thus the existence of an inverse expansion of a complete space by means of metrizable spaces is obvious if we require no dimensional conditions on spaces in the system.

COROLLARY 3 (cf. Isbell [5]). *If $(X, \mathcal{U})$ is a uniform space, then there exists a compactification X^* of X (in the topology induced by $\mathcal{U}$) such that* $\dim X^* \leqslant \dim \mathcal{U}$.

Proof. It follows from Theorem 2 that $(X, \mathcal{U})$ admits a dense embedding in the inverse limit of a system consisting of uniform maps $\pi_m^n: X_n$

$\to X_m$, where all spaces X_n in the system are metrizable and $\dim X_n \leqslant \dim \mathscr{U}$. Consider the system of maps $\beta\pi_m^n\colon \beta X_n \to \beta X_m$, where β is the symbol of Čech-Stone compactification. There is $\dim \beta X_m = \dim X_m \leqslant \dim \mathscr{U}$ each $m \in M$, and the inverse limit space X^* of the system of maps $\beta\pi_m^n$ is such that $\dim X^* \leqslant \dim X_m \leqslant \dim \mathscr{U}$. It is easy to see that X admits a dense (topological) embedding into X^*, $X \to X^*$, being the composition of the uniform embedding $(X, \mathscr{U}) \to (\tilde{X}, \tilde{\mathscr{U}})$ and the map $\tilde{X} \to X^*$ induced by Čech-Stone embeddings $X_m \to \beta X_m$ for $m \in M$.

4. Proof of Theorem 2. Let $\mathscr{B}$ be a base for $\mathscr{U}$ such that card $\mathscr{B}$ = weight $\mathscr{U}$. We shall construct a family M of pseudouniformities a on X such that $a \subset \mathscr{U}$ for every $a \in M$. The family M will be directed with respect to inclusions. We construct M in the following way. For each covering $P \in \mathscr{B}$ we construct by induction a sequence $a(P) = \{P_i\colon i = 1, 2, \ldots\}$ of coverings from $\mathscr{U}$ such that

$$P = P_1 \underset{*}{\prec} P_2 \underset{*}{\prec} P_3 \underset{*}{\prec} \ldots \tag{1}$$

and

$$\operatorname{ord} P_i \leqslant 1 + \dim \mathscr{U} \quad \text{for } i = 1, 2, \ldots \tag{2}$$

Each family $a(P) = \{P_i\colon i = 1, 2, \ldots\}$ forms a base for a pseudouniformity. Of course, $\dim a(P) \leqslant \dim \mathscr{U}$ and weight $a(P)$ is countable. Let $\mathscr{B}_1 = \{a(P)\colon P \in \mathscr{B}\}$ be the family of pseudouniformities formed in this way. Assume that the families $\mathscr{B}_1, \ldots, \mathscr{B}_{n-1}$ of pseudouniformities having dimension not greater than $\dim \mathscr{U}$ and of countable weight are already defined. Now we define $\mathscr{B}_n$. For each two pseudouniformities a' and a'' from $\mathscr{B}_1 \cup \ldots \cup \mathscr{B}_{n-1}$ we form, according to Proposition 3, a pseudouniformity a such that $a' \cup a'' \subset a$, $\dim a \leqslant \dim \mathscr{U}$, and that the weight a is countable. Let $M = \{a\colon a \in \mathscr{B}_i, i = 1, 2, \ldots\}$. By the above construction M is directed with respect to inclusions of pseudouniformities and card $M \leqslant$ weight $\mathscr{U}$.

For each $a \in M$ we form the quotient uniform map $q_a\colon X \to X_a$ of pseudouniformity a. According to Proposition 1, the uniformities on X have countable bases consisting of coverings of order not greater than $n+1$ ($\dim \mathscr{U} \leqslant n$), and so, by the above quoted theorem of Nagata [7], each topological space X_a with topology induced by the uniformity a is metrizable and $\dim X_a \leqslant \dim \mathscr{U}$, $a \in M$. Fix the following symbols (cf. Section 1):

$$\begin{cases} [x]_a = \bigcap \{\operatorname{st}(x, P)\colon P \in a\}, \qquad q_a(x) = [x]_a, \\ {}_aP = \{X_a - q_a(X - V)\colon V \in P\}, \qquad \text{where } P \in a, \\ \mathscr{U}_a = \{{}_aP\colon P \in a\}, \end{cases} \tag{3}$$

i.e. $\mathscr{U}_a$ is the quotient of pseudouniformity a with respect to $\mathscr{U}$.

Define maps $\pi^{\alpha}_{\beta}: X_{\alpha} \to X_{\beta}$ for $\alpha \supset \beta$ by the formula

$$\pi^{\alpha}_{\beta}([x]_{\alpha}) = [x]_{\beta}. \tag{4}$$

Thus

$$\pi^{\beta}_{\alpha} \circ \pi^{\gamma}_{\beta} = \pi^{\alpha}_{\gamma} \quad \text{for } \gamma \supset \beta \supset \alpha. \tag{5}$$

The maps π^{α}_{β} are well defined, because if $\alpha \supset \beta$, then

$$[x]_{\beta} = \bigcap \{\mathrm{st}(x, P): P \in \beta\} \subset \bigcap \{\mathrm{st}(x, P): P \in \alpha\} = [x]_{\alpha}.$$

Maps π^{α}_{β} are uniform with respect to the uniformities $\mathscr{U}_{\alpha}$ and U_{β}. To prove this let us note that

$$\pi^{\alpha}_{\beta} q_{\alpha} = q_{\beta} \quad \text{for } \alpha \supset \beta \tag{6}$$

and that for every $P \in \alpha$ we have $q_{\alpha}(P) = \{q_{\alpha}(V): V \in P\} \in \mathscr{U}_{\alpha}$, whence $(\pi^{\alpha}_{\beta})^{-1}({}_{\beta}P) = q_{\alpha}[q_{\beta}^{-1}({}_{\beta}P)] \in \mathscr{U}_{\alpha}$.

Let $(\tilde{X}, \tilde{\mathscr{U}})$ be the inverse limit of the above directed system and let $\pi_{\alpha}: \tilde{X} \to X_{\alpha}$, $\alpha \in M$, be the projections. According to (6), the maps $q_{\alpha}: X \to X_{\alpha}$, $\alpha \in M$, induce a uniform map $q: X \to \tilde{X}$. The map is a uniform embedding. In fact, it suffices to show that for every $P \in \mathscr{U}$ there exists a covering $P' \in \tilde{\mathscr{U}}$ such that $q^{-1}(P') \succ P$. Since the coverings $\pi_{\alpha}^{-1}({}_{\alpha}P)$ forms a base for the uniformity $\tilde{\mathscr{U}}$ and since $q_{\alpha}^{-1}({}_{\alpha}P) \succ P$ (see Section 1), it implies that q is a uniform embedding.

The image $q(X)$ is dense in the space $\tilde{X}$ with the topology induced by the uniformity $\tilde{\mathscr{U}}$. In fact, let $y \in \tilde{X}$ and let $W \subset \tilde{X}$ be a non-empty open subset of $\tilde{X}$. We shall show that $W \cap q(X) \neq \emptyset$. It is sufficient to consider only $W = \pi_{\alpha}^{-1}(W_{\alpha})$, where W_{α} is open in X_{α}. Let $y = \{y_{\alpha}\} \in W$, $\pi_{\alpha}(y) = y_{\alpha} \in W_{\alpha}$ and $\pi_{\alpha}^{-1}(y_{\alpha}) \subset W$. Since the maps $q_{\alpha}: X \to X_{\alpha}$ are onto, hence there exists $x \in X$ that $q_{\alpha}(x) = y_{\alpha}$. But $y_{\alpha} \in W_{\alpha}$ implies $q(x) \in \pi_{\alpha}^{-1}(y_{\alpha}) \subset W$. This means that $\pi_{\alpha}^{-1}(W_{\alpha}) \cap q(X) \neq \emptyset$. If $\mathscr{U}$ is a complete uniformity, then $q(X) = \tilde{X}$.

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