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A NOTE ON WEAKLY $\mathfrak{m}$-DISTRIBUTIVE BOOLEAN ALGEBRAS

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We assume throughout that $\mathfrak{m}$ is an infinite cardinal number satisfying $2^{\mathfrak{m}} = \mathfrak{m}^+$. It is the purpose of this note to give a condition which is sufficient for a weakly $\mathfrak{m}$-distributive Boolean $\mathfrak{m}$-algebra to be $\mathfrak{m}$-distributive. The notation used is the same as in [2].

We shall use the following result, which is due to Banach and Kuratowski [1]. Let X be a set of power $2^{\mathfrak{m}} = \mathfrak{m}^+$, let $\boldsymbol{F}$ be the field of all subsets of X, and let Δ be any proper $\mathfrak{m}$-ideal of $\boldsymbol{F}$ which contains all the atoms of $\boldsymbol{F}$. Then $\boldsymbol{F}/\Delta$ is not weakly $\mathfrak{m}$-distributive. In particular, it is proved (see [2], p. 130) that there is an $\mathfrak{m}$-indexed set $\{B_{t,s}\}_{t\in T, s\in S}$ of subsets of X satisfying

(i) $$\bigcup_{s\in S} B_{t,s} = X \quad \text{for every } t\in T,$$

(ii) $$\text{for every } \Phi\in S^T \bigcap_{t\in T} B_{t,\Phi(t)} \text{ has power} \leqslant \mathfrak{m}.$$

Definition. Let $\mathfrak{k}$ and $\mathfrak{n}$ be infinite cardinal numbers with $\mathfrak{n} \leqslant \mathfrak{k}$. A Boolean $\mathfrak{n}$-algebra $\boldsymbol{A}$ is said to be $(\mathfrak{k}, \mathfrak{n})$-*representable* if $\boldsymbol{A}$ is isomorphic to a quotient algebra $\boldsymbol{F}/\Delta$, where $\boldsymbol{F}$ is a $\mathfrak{k}$-field of sets and Δ is an $\mathfrak{n}$-ideal of $\boldsymbol{F}$.

THEOREM. *Assume $\boldsymbol{A}$ is a Boolean $\mathfrak{m}$-algebra which is $(2^{\mathfrak{m}}, \mathfrak{m})$-representable. If $\boldsymbol{A}$ is weakly $\mathfrak{m}$-distributive, then $\boldsymbol{A}$ is $\mathfrak{m}$-distributive.*

Proof. Let $\boldsymbol{F}$ be a $2^{\mathfrak{m}}$-field of sets and let Δ be an $\mathfrak{m}$-ideal of $\boldsymbol{F}$ such that $\boldsymbol{F}/\Delta$ is isomorphic to $\boldsymbol{A}$.

Assume $\boldsymbol{F}/\Delta$ is not $\mathfrak{m}$-distributive. It suffices to prove the existence of an $\mathfrak{m}$-indexed set $\{a_{t,s}\}_{t\in T, s\in S}$ of elements of $\boldsymbol{F}$ together with an element d of $\boldsymbol{F}$, satisfying

(a) $$d \notin \Delta,$$

(b) $$\bigcup_{s\in S} a_{t,s} = d \quad \text{for every } t\in T,$$

(c) $$\text{for every } \Phi\in S^T, \bigcap_{t\in T} a_{t,\Phi(t)} \in \Delta.$$

Since F/Δ is not $\mathfrak{m}$-distributive Δ is not a $2^{\mathfrak{m}}$-ideal. There is a set $X \subseteq \Delta$ such that $\overline{\overline{X}} = 2^{\mathfrak{m}}$ and $\bigcup_{x \in X} x \notin \Delta$. In fact, we may assume that distinct elements of X are disjoint and that all elements of X are different from Λ.

Define $d = \bigcup_{x \in X} x$; then (a) holds. Choose an $\mathfrak{m}$-indexed set $\{B_{t,s}\}_{t \in T, s \in S}$ of subsets of X satisfying (i) and (ii). We may write each

$$B_{t,s} = \{x_{t,s,q}\colon q \in Q\}$$

with $\overline{\overline{Q}} = 2^{\mathfrak{m}}$. (Some of the $x_{t,s,q}$ may, of course, be equal to Λ.) For each $(t, s) \in T \times S$ define

$$a_{t,s} = \bigcup_{q \in Q} x_{t,s,q}.$$

It follows from (i) that (b) holds. In order to prove (c) consider an arbitrary but fixed $\Phi \in S^T$. For every $t \in T$ set $R_t = \Phi(t) \times Q$ and also set $R = \bigtimes_{t \in T} R_t$. We have

$$\begin{aligned}\bigcap_{t \in T} a_{t,\Phi(t)} &= \bigcap_{t \in T} \bigcup_{s \in \Phi(t)} a_{t,s} \\ &= \bigcap_{t \in T} \bigcup_{(s,q) \in R_t} x_{t,s,q} \\ &= \bigcup_{\theta \in R} \bigcap_{t \in T} x_{t,\theta(t)} \\ &= \bigcup_{\theta \in R} y_\theta,\end{aligned}$$

where, for each $\theta \in R$, y_θ denotes $\bigcap_{t \in T} x_{t,\theta(t)}$. Since distinct elements of X are disjoint either $y_\theta = \Lambda$ or $y_\theta \in X$. The case $y_\theta \in X$ occurs precisely when $x_{t,\theta(t)} = x_{t',\theta(t')}$ for all $t, t' \in T$. In order to show that $\bigcap_{t \in T} a_{t,\Phi(t)} \in \Delta$ it is sufficient to show that $\{y_\theta\colon \theta \in R \text{ and } y_\theta \neq \Lambda\}$ has power $\leqslant \mathfrak{m}$. Let $y_\theta \neq \Lambda$. If t' is any element of T, then

$$y_\theta = x_{t',\theta(t')} = x_{t',s',q'},$$

where $\theta(t') = (s', t')$. However,

$$x_{t',s',q'} \in B_{t',s'} \subseteq B_{t',\Phi(t')}.$$

The last inclusion follows from the definition of $R_{t'}$. Thus

$$\{y_\theta\colon \theta \in R \text{ and } y_\theta \neq \Lambda\} \subseteq \bigcap_{t \in T} B_{t'\Phi(t)}.$$

This proves condition (c).

Of course not all $\mathfrak{m}$-representable Boolean $\mathfrak{m}$-algebras are $(2^{\mathfrak{m}}, \mathfrak{m})$-representable. Even an $\mathfrak{m}$-field may not be $(2^{\mathfrak{m}}, \mathfrak{m})$-representable.

Example. Let $\boldsymbol{A}$ be any $\mathfrak{m}$-field which has power $2^{\mathfrak{m}}$ and also has a subset of power $2^{\mathfrak{m}}$ consisting of disjoint elements. One example is obtained from a set X of power $2^{\mathfrak{m}}$ by letting $\boldsymbol{A}$ consist of those subsets a of X such that either a or $X \sim a$ has power $\leqslant \mathfrak{m}$. Another is the free $\mathfrak{m}$-representable Boolean $\mathfrak{m}$-algebra with exactly $\mathfrak{m}$ free $\mathfrak{m}$-generators (see [2], p. 135). Choose a disjoint set $\{a_\alpha: \alpha < 2^{\mathfrak{m}}\}$ of non-zero elements of $\boldsymbol{A}$ in such a way that $a_\alpha \neq a_\beta$ whenever $\alpha \neq \beta$. (Here $2^{\mathfrak{m}}$ is considered to be the set of all ordinal numbers of power less than $2^{\mathfrak{m}}$.)

Assume $\boldsymbol{F}$ is a $2^{\mathfrak{m}}$-field and h is an $\mathfrak{m}$-homomorphism from $\boldsymbol{F}$ onto $\boldsymbol{A}$. There is a set $\{B_\alpha: \alpha < 2^{\mathfrak{m}}\}$ of elements of $\boldsymbol{F}$ satisfying

$$h(B_\alpha) = a_\alpha \quad \text{for every } \alpha < 2^{\mathfrak{m}}, \tag{1}$$

$$B_\alpha \cap B_\beta = \Lambda \quad \text{whenever } \alpha \neq \beta. \tag{2}$$

To prove this choose elements $\{C_\alpha: \alpha < 2^{\mathfrak{m}}\}$ of $\boldsymbol{F}$ such that for every $\alpha < 2^{\mathfrak{m}} h(C_\alpha) = a_\alpha$. Define $B_0 = C_0$ and $B_\alpha = C_\alpha \cap [-\bigcup_{\beta<\alpha} C_\beta]$ for $0 < \alpha < 2^{\mathfrak{m}}$. An easy calculation, using $\mathfrak{m}^+ = 2^{\mathfrak{m}}$, shows that condition (1) holds. Clearly (2) holds.

For every subset $S \subseteq 2^{\mathfrak{m}}$, we define

$$S^* = \bigcup_{\alpha \in S} B_\alpha.$$

Clearly, $S^* \in \boldsymbol{F}$. If S and T are distinct subsets of $2^{\mathfrak{m}}$, then $h(S^*) \neq h(T^*)$. For if $\alpha \in S$, $\alpha \notin T$, then $B_\alpha \subseteq S^*$ and by (2) $B_\alpha \cap T^* = \Lambda$. Thus $a_\alpha \leqslant h(S^*)$ and $a_\alpha \cdot h(T^*) = \Lambda$, and since $a_\alpha \neq \Lambda$, we have $h(S^*) \neq h(T^*)$. This implies that the cardinality of $\boldsymbol{A}$ is greater than $2^{\mathfrak{m}}$, a contradiction.

REFERENCES

[1] S. Banach et C. Kuratowski, *Sur une généralisation du problème de la mesure* Fundamenta Mathematicae 14 (1929), p. 127-131.

[2] R. Sikorski, *Boolean algebras*, Berlin — Göttingen — Heidelberg 1964 (2nd ed.).

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