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ON THE STATICAL COMPUTATION OF CONOIDAL SHELLS

Introduction. *Conoidal shells* are used in the modern industrial building for roofing (fig. 1).

The *conoid* is a surface described by a straight line moving parallel to a given constant plane and sliding along two constant curves. We

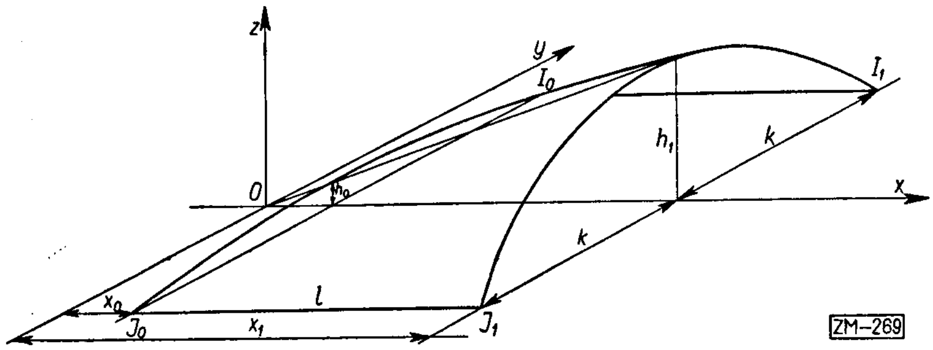

Fig. 1

shall consider only the case when the curves lie in two parallel planes which are perpendicular to the above-mentioned plane.

As far as I know, Fauconnier [1] first employed conoidal shells for the roofing of industrial buildings. But as far as the complete statical computation of conoidal shells is concerned no publications exist in which the boundary conditions are considered. There have been published only some particular solutions of the differential equations of the stress inside the shell ([2], [3], [8], [9]).

The present paper gives a mathematical method of statical computation of a conoidal shell if the exterior forces and the geometrical and elastic conditions on the boundary of the shell are known. This method is based on the following assumptions:

1. We assume that the stresses in the normal section of the shell are tangential to the surface and that no moments occur. The consideration of moments would lead to a more complicated task. Therefore we restrict ourselves to the momentless theory of shells. For thin shells

this theory seems to be the most adequate. Consequently, we assume the shell to be "thin".

2. The boundary of the shell is strengthened with bars in which besides stresses also moments may be considered. We suppose that the stresses are connected in a certain way (which we shall describe in § 7) with the interior forces of the bars.

In this paper we show how to compute the stress of the shell if the exterior forces acting on it are given.

I am indebted to Professor S. Drobot for calling my attention to this problem and for the aid which he has given me in solving it.

1. The geometrical description of a conoidal shell. Fig. 1 shows a conoidal surface in the frame of coordinates xyz. The notation of this figure will be used in the sequel. We consider a conoidal surface which arises when moving a straight line parallel to the plane xOz in such a way that at every instant it cuts the y-axis and the curve

$$x = x_1, \quad z = f(y). \tag{1.1}$$

Here $f(y)$ is an even and convex function. The equation of this conoidal surface is

$$z = \frac{xf(y)}{x_1}. \tag{1.2}$$

We consider that part of the surface (1.2) which lies above the rectangle $I_0I_1J_0J_1$. The numbers $x_0, \ldots, h_1$ (fig. 1) are considered as known. The first two of them can be expressed by the length l, breadth $2k$ and the heights h_0, h_1 as follows:

$$x_0 = \frac{h_0 l}{h_1 - h_0}, \quad x_1 = \frac{h_1 l}{h_1 - h_0}. \tag{1.3}$$

Instead of the variables x, y we shall use in our considerations the variables ξ, η defined by

$$\xi = \frac{x}{x_1}, \quad \eta = \frac{f'(y)}{f'(k)}, \quad \xi_0 = \frac{x_0}{x_1}. \tag{1.4}$$

The variable y can be expressed uniquely by η since it follows from our assumptions that $f'(y)$ is a monotone function, and therefore an invertible one. Thus, we shall regard y as a function of η.

The parametric equations in the coordinates ξ, η of the part of the conoidal surface lying above the rectangle $I_0 \ldots J_1$ are

$$x = x_1\xi, \quad y = y(\eta), \quad z = \xi f[y(\eta)]; \quad \xi_0 \leqslant \xi \leqslant 1, \ -1 \leqslant \eta \leqslant 1. \tag{1.5}$$

In the sequel we employ the coefficients and the determinant of the first fundamental form which we have computed to be

$$(1.6)\qquad \begin{aligned} &A(\xi,\eta)=\sqrt{x_1^2+f^2[y(\eta)]}, \qquad B(\xi,\eta)=y'(\eta)\sqrt{1+(\varkappa\xi\eta)^2},\\ &C(\xi,\eta)=y'(\eta)\sqrt{x_1^2+(x_1\varkappa\xi\eta)^2+f^2[y(\eta)]}; \qquad \varkappa=f'(k).\end{aligned}$$

2. The equations of equilibrium of the conoidal shell. We shall apply the general equations of the equilibrium of a thin shell in arbitrary curvilinear coordinates. These equations written in invariant form may be found in [6]. For our purpose it is not necessary to write them down. Let us denote the technical components of the stress tensor by S, T_1, T_2 (fig. 2) and the components of the exterior loads in the fixed coordinate system $Oxyz$ by X^*, Y^*, Z^*. In the sequel we shall not use the variables X^*, Y^*, Z^* but the following functions of them, regarded as functions of ξ, η:

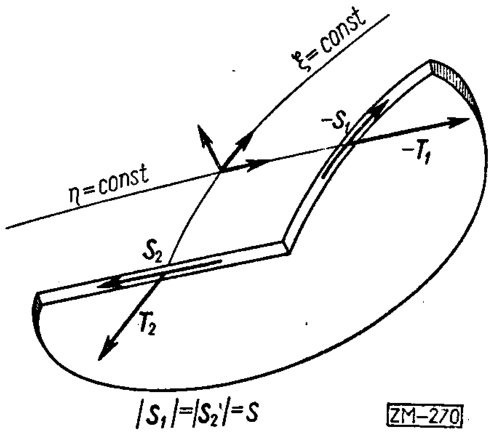

Fig. 2

$$(2.1)\qquad \begin{aligned} &X(\xi,\eta)=CX^*, \qquad Y(\xi,\eta)=CY^*,\\ &Z(\xi,\eta)=C\left[Z^*-\frac{1}{x_1}f(y)X^*-\varkappa\xi\eta Y^*\right].\end{aligned}$$

In our case the equations of the equilibrium of the conoidal shell are the following ones:

$$(2.2)\qquad \begin{aligned} &x_1\frac{\partial}{\partial\xi}\left(\frac{B}{A}T_1\right)+x_1\frac{\partial S}{\partial\eta}+X=0,\\ &\frac{\partial}{\partial\eta}\left(\frac{y'A}{B}T_2\right)+y'\frac{\partial S}{\partial\xi}+Y=0,\\ &\xi y'\frac{A}{B}T_2+2\eta y'S+\frac{1}{\varkappa}Z=0.\end{aligned}$$

Eliminating T_2 from the last two equations, we get

$$(2.3)\qquad \begin{aligned} &\xi\frac{\partial}{\partial\xi}(y'S)-2\eta\frac{\partial}{\partial\eta}(y'S)-2y'S-\frac{1}{\varkappa}Z'_\eta+\xi Y=0,\\ &\xi y'\frac{A}{B}T_2+2\eta y'S+\frac{1}{\varkappa}Z=0, \qquad x_1\frac{\partial}{\partial\xi}\left(\frac{B}{A}T_1\right)+x_1\frac{\partial S}{\partial\eta}+X=0.\end{aligned}$$

Here the variables S, T_1, T_2 are unknown functions of a point on the surface. We see that if we find S from the first equation, then the functions T_1, T_2 can be determined from the remaining two equations, namely

$$(2.4)\quad T_1 = -\frac{A}{x_1 B}\left[\int\left(x_1\frac{\partial S}{\partial\eta}+X\right)d\xi+\tau(\eta)\right],\qquad T_2 = -\frac{B}{\xi A}\left(2\eta S+\frac{1}{\varkappa y'}Z\right),$$

where $\tau(\eta)$ is an arbitrary function of the variable η. Later, in § 8.9, we shall determine $\tau(\eta)$ from the boundary conditions. Thus, we consider in the sequel the first equation (2.3) only.

3. Two Cauchy problems for the equations of equilibrium of the conoidal shell. In the first equation (2.3) let us write

$$(3.1)\qquad \sigma(\xi, \eta) = y'(\eta)S(\xi, \eta).$$

We get thus one equation in partial derivatives of the first order for one unknown function $\sigma(\xi, \eta)$:

$$(3.2)\quad \xi\frac{\partial\sigma}{\partial\xi}-2\eta\frac{\partial\sigma}{\partial\eta}-2\sigma+\xi Y-\frac{1}{\varkappa}Z'_\eta = 0.$$

Equation (3.2) determines the stress in the shell: if we know that in the rectangle $I_0 \ldots J_1$ (fig. 3),

$$(3.3)\qquad \xi_0 \leqslant \xi \leqslant 1,\qquad -1 \leqslant \eta \leqslant 1,$$

then, as has already been mentioned, we may determine S, T_1, and T_2 from equations (3.1) and (2.4).

Let us study equation (3.2) more closely. The characteristics of this equation are solutions of the system

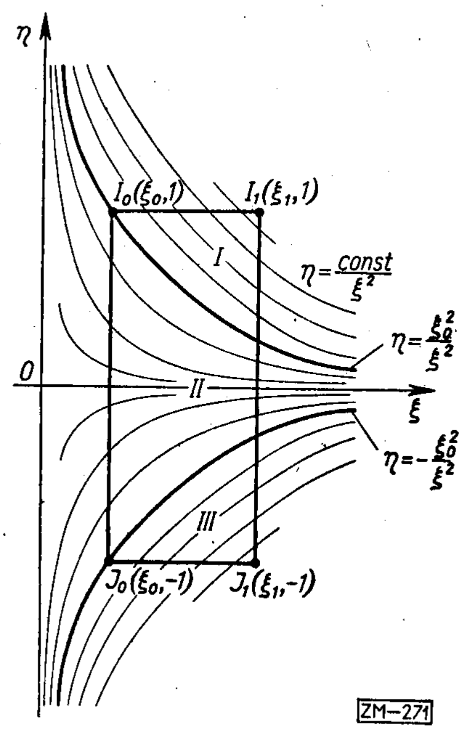

Fig. 3

$$(3.4)\qquad \frac{d\xi}{\xi} = \frac{d\eta}{-2\eta} = \frac{d\sigma}{2\sigma+\frac{1}{\varkappa}Z'_\eta-\xi Y}.$$

From the first of these equations we get

$$(3.5)\qquad \xi^2\eta = \text{const}.$$

This is a one-parameter family of curves which are projections on the plane $\xi O\eta$ of the characteristic lines (3.4) (fig. 3).

Let us now consider more exactly the behaviour of these characteristic lines, as it is of great importance for the sequel. Those characteristics which cut the sides I_0I_1 or J_0J_1 of the rectangle $I_0 \ldots J_1$ do not fill up the whole rectangle; consequently, if we know the boundary conditions for $\sigma(\xi, \eta)$ only on the sides I_0I_1 and J_0J_1, the determination of $\sigma(\xi, \eta)$ from (3.2) is impossible.

Therefore, in order to secure a unique solution of (3.2) in the whole rectangle (3.3), the boundary conditions for $\sigma(\xi, \eta)$ must be given

1° on the side I_1J_1, or

2° on the three sides I_0I_1, J_0J_1, I_0J_0,

since then the characteristic lines cutting these sides cover the whole rectangle.

In consequence there are two Cauchy problems for equation (3.2) and they depend upon which of the above-mentioned cases is admitted.

In the next two sections we shall consider these Cauchy problems.

4. The first boundary problem. The first boundary problem may be formulated as follows:

Find a function $\sigma(\xi, \eta)$ satisfying (3.2) in the rectangle (3.3), its values on the side I_1J_1 (fig. 3) being known.

More precisely: if a function $\sigma_1(\eta)$ is defined for $-1 \leqslant \eta \leqslant 1$ (on the side I_1J_1), find a function $\sigma(\xi, \eta)$ for which

$$\sigma(1, \eta) = \sigma_1(\eta) \quad \text{for} \quad -1 \leqslant \eta \leqslant 1. \tag{4.1}$$

This problem may be solved by known methods (see [7], p. 330), for example by the method of characteristics.

From computations which we omit here it follows that

$$\sigma(\xi, \eta) = \xi^2 \left[\sigma_1(\xi^2\eta) + \int_1^\xi P\left(\lambda, \frac{\xi^2\eta}{\lambda^2}\right) d\lambda\right]; \tag{4.2}$$

$$P(\lambda, \mu) = \frac{1}{\lambda^3}\left[\frac{1}{\varkappa} \cdot \frac{\partial Z(\lambda, \mu)}{\partial \mu} - \lambda Y(\lambda, \mu)\right].$$

We infer from formulas (4.2) that the unknown function $\sigma(\xi, \eta)$, by use of which the stress in the shell is determined, depends on the function $\sigma_1(\eta)$ defined on the side I_1J_1. Consequently, we need to know the function $\sigma_1(\eta)$. We shall determine it later (in § 8) from assumptions formulated at a suitable time.

5. The second boundary problem. The second boundary problem is: Find $\sigma(\xi, \eta)$ in the rectangle (3.3), its values on the sides I_0I_1, J_0J_1, I_0J_0 being known.

Therefore

$$
(5.1)\qquad \begin{aligned}
\sigma(\xi, 1) &= \sigma_{I}(\xi) \quad \text{for } \xi_0 \leqslant \xi \leqslant 1,\\
\sigma(\xi_0, \eta) &= \sigma_{II}(\eta) \quad \text{for } -1 \leqslant \eta \leqslant 1,\\
\sigma(\xi, -1) &= \sigma_{III}(\xi) \quad \text{for } \xi_0 \leqslant \xi \leqslant 1.
\end{aligned}
$$

Since the function $\sigma(\xi, \eta)$ which we want to find ought to have continuous derivatives in the whole rectangle and on its sides, we must assume that the given functions $\sigma_I(\xi)$, $\sigma_{II}(\eta)$, $\sigma_{III}(\xi)$ also have continuous derivatives and satisfy at the vertices I_0, J_0 of the rectangle the following conditions of coincidence:

$$
(5.2)\qquad \begin{aligned}
&\sigma_{II}(1) = \sigma_I(\xi_0), \qquad \sigma_{II}(-1) = \sigma_{III}(\xi_0),\\
&\sigma'_{II}(1) = \frac{\xi_0}{2}\sigma'_I(\xi_0) - \sigma_I(\xi_0) - \frac{\xi_0^3}{2}P(\xi_0, 1),\\
&\sigma'_{II}(-1) = -\frac{\xi_0}{2}\sigma'_{III}(\xi_0) + \sigma_{III}(\xi_0) + \frac{\xi_0^3}{2}P(\xi_0, -1);
\end{aligned}
$$

$$
P(\lambda, \mu) = \frac{1}{\lambda^3}\left[\frac{1}{\varkappa}\cdot\frac{\partial Z(\lambda, \mu)}{\partial\mu} - \lambda Y(\lambda, \mu)\right].
$$

We shall now construct the unknown function $\sigma(\xi, \eta)$.

At first, let us observe that the projections of the characteristics $\xi^2\eta = \xi_0^2$, $\xi^2\eta = -\xi_0^2$ of equation (3.2) divide the rectangle $I_0 \ldots J_1$ into three regions as shown in fig. 3. The characteristics which originate in the segment I_0I_1 fill up region I, those which originate in I_0J_0 fill up region II, and those which have their initial points on J_0J_1 fill up region III. Thus, $\sigma_I(\xi)$ determines the unknown function $\sigma(\xi, \eta)$ in region I, $\sigma_{II}(\eta)$ in region II, and $\sigma_{III}(\xi)$ in region III, respectively.

The function $\sigma(\xi, \eta)$ can now be determined by known methods, for example by the method of characteristics. From computations which we omit here (solving (3.2)) for the initial conditions (5.1) we get

$$
(5.3)\qquad \sigma(\xi, \eta) = \begin{cases}
\dfrac{1}{\eta}\sigma_I(\xi\sqrt{\eta}) + \xi^2 \displaystyle\int_{\xi\sqrt{\eta}}^{\xi} P\left(\lambda, \frac{\xi^2\eta}{\lambda^2}\right) d\lambda & \text{in the region I},\\[2ex]
\left(\dfrac{\xi}{\xi_0}\right)^2 \sigma_{II}\left(\dfrac{\xi^2\eta}{\xi_0^2}\right) + \xi^2 \displaystyle\int_{\xi_0}^{\xi} P\left(\lambda, \frac{\xi^2\eta}{\lambda^2}\right) d\lambda & \text{in the region II},\\[2ex]
\dfrac{1}{\eta}\sigma_{III}(\xi\sqrt{-\eta}) + \xi^2 \displaystyle\int_{\xi\sqrt{-\eta}}^{\xi} P\left(\lambda, \frac{\xi^2\eta}{\lambda^2}\right) d\lambda & \text{in the region III}.
\end{cases}
$$

It follows from formula (5.1) that the unknown function depends on the functions σ_{I}, σ_{II}, σ_{III} defined on the sides I_0I_1, I_0J_0, J_0I_1, respectively. These functions will be determined later from the assumptions concerning the way in which the shell is strengthened with bars running along its boundary. In order to formulate these assumptions let us first consider the bars which strengthen the shell.

6. The equations of equilibrium for bars which strengthen the boundary of the shell. We assume that along the boundary of the shell there run strengthening bars. We distinguish two kinds of those bars. To the first kind belong curved bars which lie above the sides I_0J_0, I_1J_1 of the fundamental rectangle and the second kind consists of the straight bars along the sides I_0I_1, J_0J_1 of the fundamental rectangle. In this section we shall consider the equations of equilibrium of these bars.

6.1. The equations of equilibrium of curved bars. We parametrize the curved axes of those bars by η (fig. 4). The curved bar which lies

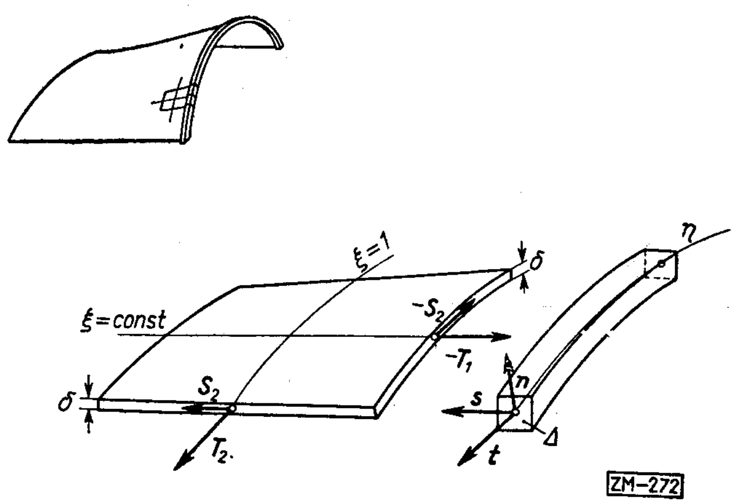

Fig. 4

above the side I_0J_0 will be called shortly the back bar. The equations of its axis are

$$x = x_1\xi_0, \quad y = y(\eta), \quad z = \xi_0 f[y(\eta)]. \tag{6.1}$$

The bar which lies above the side I_1J_1 will be called shortly the front bar. The equations of its axis are

$$x = x_1, \quad y = y(\eta), \quad z = f[y(\eta)]. \tag{6.2}$$

We assume that on these bars exterior loads and the forces originating in the shell are acting. Let the exterior load which acts on the unit length have the following components in the sytem xyz:

$$\text{in the back bar: } \bar{X}_0(\eta),\ \bar{Y}_0(\eta),\ \bar{Z}_0(\eta);$$
$$\text{in the front bar: } \bar{X}_1(\eta),\ \bar{Y}_1(\eta),\ \bar{Z}_1(\eta). \tag{6.3}$$

The forces by which the shell acts on the bars are produced by the stresses S and T_1 (and not T_2) existing at the points at which the shell and the bars are joined. Here the shell acts

$$\text{on the back bar by the stresses } S(\xi_0, \eta),\ T_1(\xi_0, \eta);$$
$$\text{on the front bar by the stresses } S(1, \eta),\ \ T_1(1, \eta).$$

These stresses must be taken in opposite turns to those in the shell.

We shall now determine the interior forces produced in the cut of each bar by those loads. We shall decompose these forces into three components (fig. 5). The first component $\boldsymbol{t}(\eta)$ has a direction tangen-

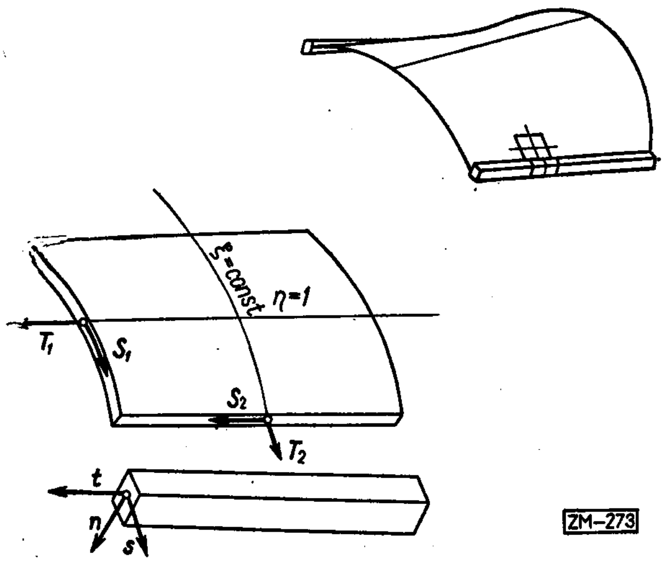

Fig. 5

tial to the axis of the bar, the second one, $\boldsymbol{s}(\eta)$, has the same direction as the stress $\boldsymbol{S}_2$, i. e. it is directed along the straight line directrix lying on the shell, while the third component $\boldsymbol{n}(\eta)$ lies in the plane passing through the vector $\boldsymbol{s}$ and through the chief-normals to the bar; the direction of this component is not yet defined and it depends on a certain function $\nu(\eta)$ which will be later so determined as to simplify the computations. The projections of the interior force $\boldsymbol{n}$ on the axes of the system xyz are the following: νn, $\varkappa\eta y' n$, $n(\nu f+y')$. Besides the interior forces $\boldsymbol{s}$, $\boldsymbol{t}$, $\boldsymbol{n}$, there may occur also moments in the cut of the bar but they have no influence on the computations which follow. If in practical computations the necessity of considering those moments arises, one can do it independently of what follows.

In order to obtain the equations of equilibrium in each of the curved bars, let us introduce the following notation. For the back bar:

(6.4) the interior forces $\boldsymbol{s}_0$, $\boldsymbol{t}_0$, $\boldsymbol{n}_0$;

$$A_0(\eta) = A(\xi_0, \eta), \qquad B_0(\eta) = B(\xi_0, \eta);$$

$\nu_0(\eta)$ is the function which defines the direction of the force $\boldsymbol{n}_0$.

For the front bar:

(6.5) the interior forces $\boldsymbol{s}_1$, $\boldsymbol{t}_1$, $\boldsymbol{n}_1$;

$$A_1(\eta) = A(1, \eta), \qquad B_1(\eta) = B(1, \eta);$$

$\nu_1(\eta)$ defines the direction of $\boldsymbol{n}_1$.

Considering the projections of all forces appearing in the bars on the axes of the system xyz we obtain the following equations of equilibrium:

for the back bar:

$$(6.6)\quad \begin{aligned} & x_1\frac{d}{d\eta}(s_0/A_0) - x_1\frac{d}{d\eta}(\nu_0 n_0) + B_0\overline{X}_0 + x_1\frac{B_0}{A_0}T_1(\xi_0, \eta) = 0, \\ & \frac{d}{d\eta}(y' t_0/B_0) + \varkappa\xi_0\frac{d}{d\eta}(\eta y' n_0) + B_0\overline{Y}_0 + y' S(\xi_0, \eta) = 0, \\ & \varkappa\xi_0\frac{d}{d\eta}(\eta y' t_0/B_0) + \frac{d}{d\eta}(f s_0/A_0) - \frac{d}{d\eta}[(\nu_0 f + y')n_0] + B_0\overline{Z}_0 + \\ & \qquad + \varkappa\xi_0\eta y' S(\xi_0, \eta) + \frac{fB_0}{A_0}T_1(\xi_0, \eta) = 0 \end{aligned}$$

for the front bar:

$$(6.7)\quad \begin{aligned} & x_1\frac{d}{d\eta}(s_1/A_1) - x_1\frac{d}{d\eta}(\nu_1 n_1) + B_1\overline{X}_1 + x_1\frac{B_1}{A_1}T_1(1, \eta) = 0, \\ & \frac{d}{d\eta}(y' t_1/B_1) + \varkappa\frac{d}{d\eta}(\eta y' n_1) + B_1\overline{Y}_1 - y' S(1, \eta) = 0, \\ & \varkappa\frac{d}{d\eta}(\eta y' t_1/B_1) + \frac{d}{d\eta}(f s_1/B_1) - \frac{d}{d\eta}[(\nu_1 f + y')n_1] + B_1\overline{Z}_1 - \\ & \qquad - \varkappa\eta y' S(1, \eta) - \frac{fB_1}{A_1}T_1(1, \eta) = 0. \end{aligned}$$

6.2. The equations of equilibrium of straight bars. On the axes of those bars we take the parameter ξ. The bar lying on the side I_0I_1 will be called shortly the right bar, and that which lies on the side J_0J_1 will be called the left bar.

We assume that on these bars known exterior loads and also forces produced by the shell are acting. Let us denote the components of the exterior load in the coordinate system xyz as follows:

$$\begin{aligned} &\bar{X}_2(\xi),\ \bar{Y}_2(\xi),\ \bar{Z}_2(\xi) \quad \text{in the right bar;} \\ &\bar{X}_3(\xi),\ \bar{Y}_3(\xi),\ \bar{Z}_3(\xi) \quad \text{in the left bar.} \end{aligned} \tag{6.8}$$

The forces produced by the shell and acting on the straight bars originate in the stresses T_2 and S (fig. 5) which act at those points at which the shell is joined with the bars. Hence the shell acts

on the right bar by the stresses $S(\xi, 1)$, $T_2(\xi, 1)$;

on the left bar by the stresses $S(\xi, -1)$, $T_2(\xi, -1)$.

Those stresses and the exterior loads produce interior forces in the cross-section of the bar. We decompose those forces into three components (fig. 5). The first component $\boldsymbol{t}(\xi)$ acts in the direction of the bar axis, the second component $\boldsymbol{s}(\xi)$ has the same direction as the stress $\boldsymbol{S}_1$, while the third component $\boldsymbol{n}$ acts in the direction of the z-axis but in the opposite turn.

The moments which may also occur in the cross-section are not regarded for the same reasons as for the curved bars.

Let us denote the components of the interior forces which act in the cross-section of the right bar by $\boldsymbol{s}_2(\xi)$, $\boldsymbol{t}_2(\xi)$, $\boldsymbol{n}_2(\xi)$ and of these which act in the left bar by $\boldsymbol{s}_3(\xi)$, $\boldsymbol{t}_3(\xi)$, $\boldsymbol{n}_3(\xi)$.

Projecting all forces appearing in those bars we get the following equations of equilibrium:

for a right bar:

$$\begin{aligned} &\frac{d}{d\xi}(t_2)+x_1[\bar{X}_2-S(\xi,1)]=0, \\ &y'(1)\frac{d}{d\xi}[s_2/B(\xi,1)]+x_1\left[\bar{Y}_2-\frac{y'(1)}{B(\xi,1)}T_2(\xi,1)\right]=0, \\ &\varkappa y'(1)\frac{d}{d\xi}[s_2/B(\xi,1)]+\frac{d}{d\xi}(n_2)+x_1\left[\bar{Z}_2-\frac{\varkappa y'(1)}{B(\xi,1)}T_2(\xi,1)\right]=0 \end{aligned} \tag{6.9}$$

for a left bar:

$$\frac{d}{d\xi}(t_3)+x_1[\bar{X}_3+S(\xi,-1)]=0,$$

(6.10) $$y'(-1)\frac{d}{d\xi}[s_3/B(\xi,-1)]+x_1\left[\bar{Y}_3+\frac{y'(-1)}{B(\xi,-1)}T_2(\xi,-1)\right]=0,$$

$$\varkappa y'(-1)\frac{d}{d\xi}[s_3/B(\xi,-1)]-\frac{dn_3}{d\xi}+x_1\left[\frac{\varkappa y'(-1)}{B(\xi,-1)}T_2(\xi,-1)-\bar{Z}_3\right]=0.$$

7. The conditions of coincidence on the boundary of the shell and in the strengthening bars. We now make certain assumptions about the nature of the connexion between the stresses in the shell and the interior forces in the bars, as mentioned in the introduction. Namely, we assume that the interior force $\boldsymbol{t}$ in the bar is proportional to the stress $\boldsymbol{T}_2$ and the interior force $\mathbf{s}$ in the bar is proportional to the stress $\boldsymbol{S}_2$, stresses $\boldsymbol{T}_2$ and $\boldsymbol{S}_2$ acting at the point at which the bar and the shell are joined. More exactly, in the back bar we assume

(7.1) $$t_0=b_0(\eta)T_2(\xi_0,\eta),\qquad s_0=a_0(\eta)S(\xi_0,\eta),$$

and in the front bar

(7.2) $$t_1=b_1(\eta)T_2(1,\eta),\qquad s_1=a_1(\eta)S(1,\eta),$$

where $a_0(\eta)$, $b_0(\eta)$, $a_1(\eta)$, $b_1(\eta)$ are certain functions defined along the bar. They have the following statical interpretation.

Let us first consider the connections between interior forces $\boldsymbol{t}$ in the bar and the stresses $\boldsymbol{T}_2$ in the shell (see fig. 4). Let δ denote the thickness of the shell and the area Δ of that cross-section of the bar in which the interior force $\boldsymbol{t}$ acts. For the cross-section of the bar given in § 6 the area Δ depends on the parameter η along the bar. Now, since δ is small we assume that in every section of the shell which has the length 1 and the thickness δ there acts a stress T_2/δ (expressed in kG/m²).

Similarly, since the area Δ is small we suppose that in the section Δ there exists a stress equal to t/Δ (in kG/m²). Our assumption means that the stresses T_2/δ and t/Δ are equal, i. e. $t=\Delta T_2/\delta$. Hence the coefficient b_0 is equal to Δ_0/δ, and the coefficient b_1 to Δ_1/δ, where Δ_0 and Δ_1 denote the areas of the sections of the back and of the front bar, respectively.

Let us now consider the connections between the interior forces s in the bar and the stresses S in the shell (see fig. 4). We suppose that the stress S/δ (expressed in kG/m²) in the shell is equal to the average shearing stress s/Δ in the section of the bar. (By the way, we are not concerned with the decomposition of the shearing stresses in the section of the bar, but only with the average stress. In the statical computation of the bar one can also consider that decomposition but for our aim it is unnecessary.) Hence we obtain an analogous statical interpretation of the functions $a_0(\eta)$, $a_1(\eta)$.

Since we are considering the momentless theory of shells in which no normal stresses exist, we make no assumptions about the interior normal forces n in the bar. Consequently, the normal forces n in the bar must always be such as follows from the equations of equilibrium.

The assumptions about the connections between the stresses in the shell and the interior forces in the straight bars are similar to those suppositions which were made for curved bars. Namely, we now suppose that the interior force t in the bar is proportional to the stress T_1 and the interior force s in the bar is proportional to the stress S_1, stresses T_1 and S_1 acting at the point at which the bar and the shell are joined (see fig. 5). This means that along the left bar we assume

$$t_2 = b_2(\xi)T_1(\xi, 1), \quad s_2 = a_2(\xi)S(\xi, 1), \tag{7.3}$$

and in the right bar

$$t_3 = b_3(\xi)T_1(\xi, -1), \quad s_3 = a_3(\xi)S(\xi, -1). \tag{7.4}$$

Here $a_2(\xi)$, $a_3(\xi)$, $b_2(\xi)$, $b_3(\xi)$ are certain functions of the parameter ξ along the bar.

8. Determination of stresses in the shell by solution of the first boundary problem. The solving of the system of equations of equilibrium was reduced above (in §3) to the solution of (3.2), where the unknown function is $\sigma(\xi, \eta)$. For that equation two boundary problems have been considered and solved. In order to find the stress in the shell we shall apply here solution (4.2) of the first boundary problem for (3.2). The stress in the shell for this problem, as follows from formulas (4.2), (3.1), (2.4), depends on two functions of one variable: on the function $\sigma_1(\eta)$ defined on the side I_1J_1 of the fundamental rectangle, and on an arbitrary function $\tau(\eta)$ which appears in the first of formulas (2.4).

In order to simplify the computations let us take in (2.4) the integral between the boundaries 1 and ξ. Then on the side I_1J_1 of the fundamental rectangle the stress T_1 does not depend on S and X but only on an arbitrary function of the parameter η, which we now denote by $\tau_1(\eta)$:

$$T_1(\xi, \eta) = -\frac{A}{B}\left[\int_1^{\xi}\left(\frac{\partial S}{\partial \eta} + \frac{1}{x_1}X\right)d\xi + \tau_1(\eta)\right]. \tag{8.1}$$

Hence the stress-system depends on two functions, $\sigma_1(\eta)$ and $\tau_1(\eta)$, given on the side I_1J_1 or, in other words, on the stresses S, T_1 and T_2 which appear on the front boundary of the shell, since the stresses on this boundary can be expressed by those functions, as follows:

$$(8.2)\qquad S(1,\eta)=\frac{\sigma_1(\eta)}{y'},\qquad T_1(1,\eta)=-\frac{A_1}{B_1}\tau_1(\eta),$$
$$T_2(1,\eta)=-\frac{B_1}{y'A_1}\left[2\eta\sigma_1(\eta)+\frac{1}{\varkappa}Z(1,\eta)\right].$$

In order to solve our problem completely, we must know the functions $\sigma_1(\eta)$ and $\tau_1(\eta)$ (as remarked in § 4). We determine those functions from the assumptions about the way in which the shell is joined with the front bar, namely, from the connections obtained in § 6 and § 7. Substituting in (6.7) the stresses $S(1,\eta)$, $T_1(1,\eta)$, $T_2(1,\eta)$ from (8.2) and using the conditions of coincidence (7.2), we obtain the following system of differential equations (where $\sigma_1(\eta)$, $\tau_1(\eta)$ and $n_1(\eta)$ are the unknown functions):

$$x_1\frac{d}{d\eta}\left(\frac{a_1\sigma_1}{y'A_1}\right)-x_1\frac{d}{d\eta}(\nu_1n_1)+x_1\tau_1+B_1\overline{X}_1=0,$$

$$(8.3)\qquad -\frac{d}{d\eta}\left[\frac{b_1}{A_1}\left(2\eta\sigma_1+\frac{1}{\varkappa}Z\right)\right]+\varkappa\frac{d}{d\eta}[\eta y'n_1]-\sigma_1+B_1\overline{Y}_1=0,$$

$$-\varkappa\frac{d}{d\eta}\left[\frac{\eta b_1}{A_1}\left(2\eta\sigma_1+\frac{1}{\varkappa}Z\right)\right]+\frac{d}{d\eta}\left(\frac{a_1f\sigma_1}{y'A_1}\right)-\frac{d}{d\eta}[(\nu_1f+y')n_1]+$$
$$+f\tau_1-\varkappa\eta\sigma_1+B_1\overline{Z}_1=0.$$

We introduce the following notation:

$$(8.4)\qquad \alpha_1=\frac{a_1}{A_1},\qquad \beta_1=\frac{b_1}{A_1};$$
$$Q=\frac{1}{\varkappa}\left(\frac{\alpha_1\beta_1Z}{2\beta_1-\alpha_1}\right)'+\frac{\beta_1Z}{\varkappa\eta(2\beta_1-\alpha_1)}-\frac{2}{\varkappa}\left(\frac{\beta_1B_1\overline{Z}_1}{2\beta_1-\alpha_1}\right)'-\frac{B_1\overline{Z}_1}{\varkappa\eta(2\beta_1-\alpha_1)}+$$
$$+2\left(\frac{\eta\beta_1B_1\overline{Y}_1}{2\beta_1-\alpha_1}\right)'+\frac{B_1\overline{Y}_1}{2\beta_1-\alpha_1}+\frac{2}{x_1\varkappa}\left[\frac{\beta_1fB_1\overline{X}_1}{\eta(2\beta_1-\alpha_1)}\right]'+\frac{fB_1\overline{X}_1}{x_1\varkappa\eta(2\beta_1-\alpha_1)}.$$

Eliminating from equations (8.3) the functions σ_1 and τ_1 we obtain an equation of the second order where the unknown function is n_1:

$$(8.5)\quad \frac{2\beta_1\eta(1+\varkappa^2\eta^2)}{\varkappa(2\beta_1-\alpha_1)}\cdot\frac{d^2}{d\eta^2}(y'n_1)+$$
$$+\left\{\frac{2\eta}{\varkappa}\left[\frac{\beta_1(1+\varkappa^2\eta^2)}{2\beta_1-\alpha_1}\right]'+\frac{1+\varkappa^2(4\beta_1-\alpha_1)\eta^2}{\varkappa(2\beta_1-\alpha_1)}+\frac{2\beta_1\nu_1\eta^2}{2\beta_1-\alpha_1}\right\}\frac{d}{d\eta}(y'n_1)+$$
$$+\left[2\dot{\varkappa}\eta\left(\frac{\beta_1\eta}{2\beta_1-\alpha_1}\right)'+2\eta\left(\frac{\eta\beta_1\nu_1}{2\beta_1-\alpha_1}\right)'+\frac{\varkappa(2\beta_1-\alpha_1)+\nu_1}{2\beta_1-\alpha_1}\eta\right]y'n_1+\eta\dot{Q}=0.$$

We shall reduce this equation to an equation of the first order. For this aim we shall take a suitable definition of the function $\nu_1(\eta)$ which was introduced in § 6 (formula 6.5) and which determines the direction of the component n_1 of the interior force in the front bar. Namely we choose ν_1 as follows:

$$(8.6)\quad \nu_1(\eta)=\frac{4\beta_1-\alpha_1-1}{1-2\beta_1}\varkappa+\frac{2\beta_1-\alpha_1}{\varkappa\eta}\left(\frac{1+\varkappa^2\eta^2}{2\beta_1-\alpha_1}\right)'-\frac{2\beta_1'(1+\varkappa^2\eta^2)}{\varkappa\eta(1-2\beta_1)}.$$

After substituting in (8.5) the above value of $\nu_1(\eta)$ we are able to reduce this equation by integrations to an equation of the first order. Solving this equation of the first order we get

$$(8.7)\quad y'n_1=\frac{2\beta_1-\alpha_1}{1+\varkappa^2\eta^2}\exp(-\omega)\left[-\frac{\varkappa}{2}\int\frac{\int\eta Q d\eta+c_1}{\beta_1\eta}\exp\omega\, d\eta+c_2\right],$$

where

$$\omega=\int\left[\frac{\varkappa^2(2\beta_1-\alpha_1)\eta}{2\beta_1(1-2\beta_1)(1+\varkappa^2\eta^2)}+\frac{1-2\beta_1}{2\beta_1\eta}-\frac{2\beta_1'}{1-2\beta_1}\right]d\eta,$$

and c_1, c_2 are constants of integration. These constants will be determined from the boundary conditions for the front bar.

The unknown functions $\sigma_1(\eta)$ and $\tau_1(\eta)$ can be expressed by the function ν_1, which is defined by (8.7). Namely

$$(8.8)\quad \begin{aligned}\sigma_1(\eta)&=\left[\varkappa+\frac{(1-2\beta_1)(1+\varkappa^2\eta^2)}{2\varkappa\beta_1(2\beta_1-\alpha_1)\eta^2}\right]y'n_1+\frac{1}{2\beta_1\eta^2}\left(\int\eta Q d\eta+c_1\right)+\\&\quad+\frac{B_1\bar{Z}_1-\beta_1 Z}{\varkappa\eta(2\beta_1-\alpha_1)}+\frac{B_1\bar{Y}_1}{\alpha_1-2\beta_1}+\frac{fB_1\bar{X}_1}{x_1\varkappa\eta(\alpha_1-2\beta_1)},\\ \tau_1(\eta)&=\frac{d}{d\eta}(\nu_1 n_1-\alpha_1\sigma_1/y')-B_1\bar{X}_1/x_1.\end{aligned}$$

Substituting the obtained σ_1 and τ_1 into formulas (8.1), (4.2), (3.1), (2.4) we obtain the functions $S(\xi, \eta)$, $T_1(\xi, \eta)$, $T_2(\xi, \eta)$ describing the stress in the shell for a given exterior load acting on it and for arbitrary elastic boundary conditions along the front bar.

If we assume that the shell is free along its front boundary, then we assume

$$\sigma_1 = 0, \quad \tau_1 = 0. \tag{8.9}$$

This also follows from equations (8.3) if we substitute in them

$$\alpha_1 = \beta_1 = \nu_1 = \overline{X}_1 = \overline{Y}_1 = \overline{Z}_1 = 0.$$

In the engineering practice one often deals with bars which give no resistance in a certain direction. Let us now consider the case of a bar which gives way in the direction of $\boldsymbol{T}_1$. It is also possible to describe that case by formulas (8.8), (8.7), but it involves great difficulties in computations since the function α_1 which occurs in them is expressed by unknown functions n_1; therefore we choose a different way to this aim.

From the assumption that no forces are passed through the bar produced by the shell in the direction of $\boldsymbol{T}_1$ it follows that $T_1(1, \eta) = -A_1\tau_1/B_1 = 0$, i. e. $\tau_1 = 0$. We also suppose that $\overline{X}_1 = 0$. Then the first equation of the system (8.2) implies $\alpha_1 = \nu_1 y' n_1/\sigma_1$. Eliminating from the other two equations α_1 and n_1 and substituting

$$\sigma_1(\eta) = u''(\eta), \tag{8.10}$$

we obtain a differential equation of the second order for the unknown function $u(\eta)$:

$$2\beta_1\eta(1+\varkappa^2\eta^2)u'' + (1+\varkappa^2\eta^2)u' - \varkappa^2\eta^2 u = q, \tag{8.11}$$

where

$$q = -\frac{\beta_1}{\varkappa}(1+\varkappa^2\eta^2)Z(1, \eta) + \int B_1\overline{Y}_1 d\eta + \varkappa\eta\int B_1\overline{Z}_1 d\eta + c_1.$$

If we put

$$\beta_1 = \frac{\lambda+\varkappa^2\eta^2}{4(1+\varkappa^2\eta^2)}, \quad \text{where} \quad \lambda = \text{const}, \tag{8.12}$$

then equation (8.11) can be solved by integrations, namely

$$u(\eta) = \eta^{1-2/\lambda}(\lambda+\varkappa^2\eta^2)^{1/\lambda}\Big[2\int\int q d\eta\cdot(\lambda+\varkappa^2\eta^2)^{-1-1/\lambda}\eta^{2/\lambda-2}d\eta + c_2\Big]. \tag{8.13}$$

The constants of integration can be determined from the boundary conditions of the front bar.

From formulas (8.10) and (8.13) obtained here it follows that the stress in the shell, when determined by applying the first boundary problem, depends on the way in which the front boundary is fastened. The way of supporting the shell along its remaining sides has no influence on the stress in the shell but it must be such as to ensure that the supports will stand the stress from the shell. In the next section we shall consider a problem in which the back, right and left sides have the decisive influence on the stress.

9. The determination of stresses in the shell by solution of the second boundary problem. In order to find the stress in the shell we shall apply here solution (5.3) of the second boundary problem for equation (3.2). Now the stress in the shell depends on four functions of one variable, namely on $\sigma_{\mathrm{I}}(\xi)$, $\sigma_{\mathrm{II}}(\eta)$, $\sigma_{\mathrm{III}}(\xi)$, which are defined on the sides I_0I_1, I_0J_0, J_0J_1 of the fundamental rectangle respectively (see figs 1 and 3) and on the function $\tau(\eta)$, which appears in formula (2.4).

As in § 8 (see formula 8.1) we substitute in (2.4) for the undefined integral an integral between certain limits ξ_0 and ξ. Then on the side I_0J_0 of the fundamental rectangle the stress T_1 does not depend on S and X but only on an arbitrary function of the parameter η which will be denoted by $\tau_0(\eta)$:

$$T_1(\xi,\eta) = -\frac{A}{B}\left[\int_{\xi_0}^{\xi}\left(\frac{\partial S}{\partial\eta}+\frac{1}{x_1}X\right)d\xi+\tau_0(\eta)\right]. \tag{9.1}$$

Thus, the stress in the shell depends on two functions, $\sigma_{\mathrm{II}}(\eta)$ and $\tau_0(\eta)$ defined on the side I_0J_0, on the function $\sigma_{\mathrm{I}}(\xi)$ defined on the side I_0I_1 and on the function $\sigma_{\mathrm{III}}(\xi)$ defined on the side J_0J_1. In other words, the stress in the shell depends on the stress given on the back, left and right boundaries; since the stress on those boundaries can be expressed by the above-mentioned functions, namely

1° on the back boundary

$$\begin{gathered} S(\xi_0,\eta) = \frac{\sigma_{\mathrm{II}}(\eta)}{y'}, \quad T_1(\xi_0,\eta) = -\frac{A_0}{B_0}\tau_0(\eta), \\ T_2(\xi_0,\eta) = -\frac{B_0}{\xi_0 y' A_0}\left[2\eta\sigma_{\mathrm{II}}(\eta)+\frac{1}{\varkappa}Z\right]; \end{gathered} \tag{9.2}$$

2° on the right boundary

$$\begin{gathered} S(\xi,1) = \frac{\sigma_{\mathrm{I}}(\xi)}{y'(1)}, \quad T_2(\xi,1) = -\frac{B(\xi,1)}{x_1y'(1)\xi}\left[2\sigma_{\mathrm{I}}(\xi)+\frac{1}{\varkappa}Z\right], \\ T_1(\xi,1) = -\frac{x_1}{2y'(1)B(\xi,1)}\left[\xi\sigma_{\mathrm{I}}-\frac{3y'(1)+2y''(1)}{y'(1)}\int_{\xi_0}^{\xi}\sigma_{\mathrm{I}}d\xi-\right. \\ \left.-\int_{\xi_0}^{\xi}\xi^3P(\xi,1)d\xi+\frac{2y'(1)}{x_1}\int_{\xi_0}^{\xi}Xd\xi+2y'(1)\tau_0(1)-\xi_0\sigma_{\mathrm{I}}(\xi_0)\right]; \end{gathered} \tag{9.3}$$

3° on the left boundary

(9.4)
$$S(\xi,-1)=-\frac{\sigma_{III}(\xi)}{y'(-1)},\qquad T_2(\xi,-1)=-\frac{B(\xi,-1)}{x_1y'(-1)\xi}\left[2\sigma_{III}+\frac{1}{\varkappa}Z\right],$$
$$T_1(\xi,-1)=-\frac{x_1}{2y'(-1)B(\xi,-1)}\times$$
$$\times\left[\xi\sigma_{III}-\frac{3y'(-1)-2y''(-1)}{y'(-1)}\int\limits_{\xi_0}^{\xi}\sigma_{III}d\xi+\int\limits_{\xi_0}^{\xi}\xi^3P(\xi,-1)d\xi+\right.$$
$$\left.+\frac{2y'(-1)}{x_1}\int\limits_{\xi_0}^{\xi}Xd\xi+2y'(-1)\tau_0(-1)-\xi_0\sigma_I(\xi)\right].$$

Hence we shall determine the functions σ_{II} and τ_0, as in § 8, from the way in which the shell is fastened along its back boundary and the functions $\sigma_I(\xi)$ and $\sigma_{III}(\xi)$ from the way in which the shell is fastened along its left and right boundaries.

Let us first consider the back boundary. We suppose that this boundary is strengthened by a bar. Then, substituting in (6.6) the stresses $S(\xi_0,\eta)$, $T_1(\xi_0,\eta)$, $T_2(\xi_0,\eta)$, from formulas (9.2) and using the conditions of coincidence (7.1), we obtain the following system of differential equations with the unknown functions $\sigma_{II}(\eta)$, $\tau_0(\eta)$, $n_0(\eta)$:

(9.5)
$$x_1\frac{d}{d\eta}\left(\frac{a_0\sigma_{II}}{y'A_0}\right)-x_1\frac{d}{d\eta}(\nu_0n_0)+x_1\tau_0+B_0\overline{X}_0=0,$$
$$-\frac{1}{\xi_0}\cdot\frac{d}{d\eta}\left[\frac{b_0}{A_0}\left(2\eta\sigma_{II}+\frac{1}{\varkappa}Z\right)\right]+\varkappa\xi_0\frac{d}{d\eta}(\eta y'n_0)+\sigma_{II}+B_0Y_0=0,$$
$$-\frac{\varkappa}{\xi_0}\cdot\frac{d}{d\eta}\left[\frac{\eta b_0}{A_0}\left(2\eta\sigma_{II}+\frac{1}{\varkappa}Z\right)\right]+\frac{d}{d\eta}\left(\frac{a_0f\sigma_{II}}{y'A_0}\right)-\frac{d}{d\eta}[(\nu_0f+y')n_0]+$$
$$+\varkappa\xi_0\eta\sigma_{II}+f\tau_0+B_0\overline{Z}_0=0.$$

Let us now introduce the following notation:

(9.6)
$$\alpha_0=\frac{a_0}{A_0},\qquad \beta_0=\frac{b_0}{A_0};$$
$$Q_0=\frac{1}{x\xi_0}\left(\frac{\alpha_0\beta_0Z}{2\beta_0-\alpha_0}\right)'-\frac{\beta_0Z}{\varkappa\eta(2\beta_0-\alpha_0)}-\frac{2}{\varkappa\xi}\left(\frac{\beta_0B_0\overline{Z}_0}{2\beta_0-\alpha_0}\right)'+\frac{B_0\overline{Z}_0}{\varkappa\eta(2\beta_0-\alpha_0)}+$$
$$+2\left(\frac{\eta B_0\overline{Y}_0}{2\beta_0-\alpha_0}\right)'-\frac{\xi_0B_0\overline{Y}_0}{2\beta_0-\alpha_0}+B_0Y_0+\frac{2}{\varkappa\xi_0x_1}\left(\frac{fB_0\overline{X}_0}{2\beta_0-\alpha_0}\right)'-\frac{fB_0\overline{X}_0}{\varkappa x_1\eta(2\beta_0-\alpha_0)}.$$

Eliminating from equation (9.5) $\sigma_{II}(\eta)$ and $\tau_0(\eta)$ we obtain an equation of the second order for the function n_0:

$$(9.7)\quad \frac{2\beta_0\eta(1+\varkappa^2\xi_0^2\eta^2)}{\varkappa\xi_0(2\beta_0-\alpha_0)}\cdot\frac{d^2}{d\eta^2}(y'n_0)+$$

$$+\left\{\frac{2\eta}{\varkappa\xi_0}\left[\frac{\beta_0(1+\varkappa^2\xi_0^2\eta^2)}{2\beta_0-\alpha_0}\right]'+\frac{2\varkappa\xi_0\beta_0\eta^2}{2\beta_0-\alpha_0}+\frac{2\beta_0\nu_0\eta^2}{\xi_0(2\beta_0-\alpha_0)}-\frac{1+\varkappa^2\xi_0^2\eta^2}{\varkappa(2\beta_0-\alpha_0)}+\right.$$

$$\left.+\varkappa\xi_0\eta^2\right\}\frac{d}{d\eta}(y'n_0)+$$

$$+\left[2\varkappa\xi_0\eta\left(\frac{\beta_0\eta}{2\beta_0-\alpha_0}\right)'+\frac{2\eta}{\xi_0}\left(\frac{\beta_0\eta\nu_0}{2\beta_0-\alpha_0}\right)+\varkappa\xi_0\eta-\frac{\varkappa\xi_0^2\eta+\nu_0\eta}{2\beta_0-\alpha_0}\right]y'n_0+\eta Q_0=0.$$

As in § 8 we choose for the still undetermined function $\nu_0(\eta)$, introduced in § 6 (formula (6.4)), the following one:

$$(9.8)\quad \nu_0(\eta)=-\frac{4\beta_0-\alpha_0+\xi_0}{\xi_0+2\beta_0}\varkappa\xi_0^2+\frac{2\beta_0-\alpha_0}{\varkappa\eta}\left(\frac{1+\varkappa^2\xi_0^2\eta^2}{2\beta_0-\alpha_0}\right)'+\frac{2\beta_0'(1+\varkappa^2\xi_0^2\eta^2)}{\varkappa\eta(\xi_0+2\beta_0)}.$$

Equation (9.7) can then be reduced by integration to a differential equation of the first order which is easy to solve. After some calculations we get

$$(9.9)\quad y'n_0(\eta)=\frac{2\beta_0-\alpha_0}{1+\varkappa^2\xi_0^2\eta^2}\exp(-\omega)\left[-\frac{\varkappa\xi_0}{2}\int\frac{\int\eta Q_0 d\eta+c_1}{\beta_0\eta}\exp\omega\, d\eta+c_2\right],$$

where

$$\omega=\int\left[-\frac{4\beta_0+\xi_0}{2\beta_0(\xi_0+2\beta_0)\eta}\xi_0+\frac{\varkappa^2\xi_0^3(2\beta_0-\alpha_0)\eta}{2\beta_0(\xi_0+2\beta_0)(1+\varkappa^2\xi_0^2\eta^2)}\right]d\eta,$$

and c_1, c_2 are constants of integration.

The unknown functions $\sigma_{II}(\eta)$ and $\tau_0(\eta)$ are expressed by n_0 as follows:

$$(9.10)\quad \sigma_{II}(\eta)=\left[\varkappa\xi_0^2-\frac{(\xi_0+2\beta_0)(1+\varkappa^2\xi_0^2\eta^2)}{2\beta_0\varkappa(2\beta_0-\alpha_0)\eta^2}\right]y'n_0+\frac{\xi_0}{2\beta_0\eta^2}\left(\int\eta Q_0 d\eta+c_1\right)+$$

$$+\frac{B_0\bar{Z}_0}{\varkappa\eta(2\beta_0-\alpha_0)}-\frac{\beta_0 Z}{\varkappa\eta(2\beta_0-\alpha_0)}-\frac{\xi_0 B_0\bar{Y}_0}{2\beta_0-\alpha_0}-\frac{fB_0\bar{X}_0}{\varkappa x_1(2\beta_0-\alpha_0)\eta},$$

$$\tau_0(\eta)=\frac{d}{d\eta}\left(\frac{\alpha_0\sigma_{II}}{y'}-\nu_0 n_0\right)+\frac{B_0\bar{X}_0}{x_1}.$$

If we supposed that the back bar did not carry the stresses T_1 from the shell, i. e. that $\tau_0=0$, then it would be necessary to solve system

(9.5) by a different method. Namely let us eliminate a_0 and n_0 from the system (9.5), assuming moreover that $\bar{X}_0 = 0$. After computations and by substituting

$$\sigma_{II}(\eta) = u''(\eta) \tag{9.11}$$

we obtain a differential equation of the second order where the unknown function is $u(\eta)$:

$$2\beta_0\eta(1+\varkappa^2\xi_0^2\eta^2)u'' - \xi_0(1+\varkappa^2\xi_0^2\eta^2)u' + \varkappa^2\xi_0^3\eta u = q_0, \tag{9.12}$$

where

$$q_0 = -\frac{\beta_0}{\varkappa}(1+\varkappa^2\xi_0^2\eta^2)Z(\xi_0, \eta) + \varkappa\xi_0^2\eta\int B_0\bar{Z}_0 d\eta + \xi_0\int B_0\bar{Y}_0 d\eta + c_1.$$

If we set

$$\beta_0 = \frac{\lambda - \varkappa^2\xi_0^2\eta^2}{4(1+\varkappa^2\xi_0^2\eta^2)}\xi_0, \quad \text{where} \quad \lambda = \text{const}, \tag{9.13}$$

then equation (9.2) can be solved. Namely

$$u(\eta) = \frac{\eta^{1+2/\lambda}}{(\lambda-\varkappa^2\xi_0^2\eta^2)^{1/\lambda}}\left[\int\int q_0 d\eta\cdot(\lambda-\varkappa^2\xi_0^2\eta^2)^{1/\lambda-1}\eta^{-2-2/\lambda}d\eta + c_2\right]. \tag{9.14}$$

The method of determining the constants c_1, c_2 will be shown at the end of the section.

Let us now consider the determination of $\sigma_I(\xi)$ and $\sigma_{II}(\xi)$. These functions, as already mentioned, will be determined from the way in which the shell is fastened along its right and left boundaries. We shall consider in detail only the computation of $\sigma_I(\xi)$ which is defined on the right boundary, since the method of computing $\sigma_{III}(\xi)$ is similar. Before calculating the σ_I and σ_{III} let us remark that on straight boundaries we have to determine only one function (σ_I or σ_{III}) on each boundary and not two, as along the back and front boundaries. This implies certain restrictions in the way in which the shell is fastened along the straight boundaries.

Let us suppose, as in § 6, that the shell is strengthened along the straight boundaries by bars. The unique determination of the functions σ_I and σ_{III} is not possible if we do not make some new assumptions. These assumptions can be of two kinds. Namely we can suppose that the bars are so constructed that they carry each stress T_2 which acts along the straight boundaries or that they carry each stress S acting along these boundaries (see fig. 6). We shall now consider the case when the first of these assumptions holds. In this case we shall calculate the function σ_{I} from the first equation of system (6.9), since the other equation holds by definition. Substituting in the first equation of system (6.9) $t_2(\xi)$ from the conditions of coincidence (formula (7.3)) and using formula (9.3) we get

$$(9.15)\quad \frac{d}{d\xi}\left\{\frac{b_2}{B(\xi,1)}\left[\xi\sigma_{\mathrm{I}}-\frac{3y'(1)+2y''(1)}{y'(1)}\int_{\xi_0}^{\xi}\sigma_{\mathrm{I}}d\xi-\right.\right.$$
$$\left.\left.-\int_{\xi_0}^{\xi}\xi^3P(\xi,1)d\xi+\frac{2y'(1)}{x_1}\int_{\xi_0}^{\xi}Xd\xi+c_1\right]\right\}+\sigma_{\mathrm{I}}-2y'(1)\overline{X}_2=0.$$

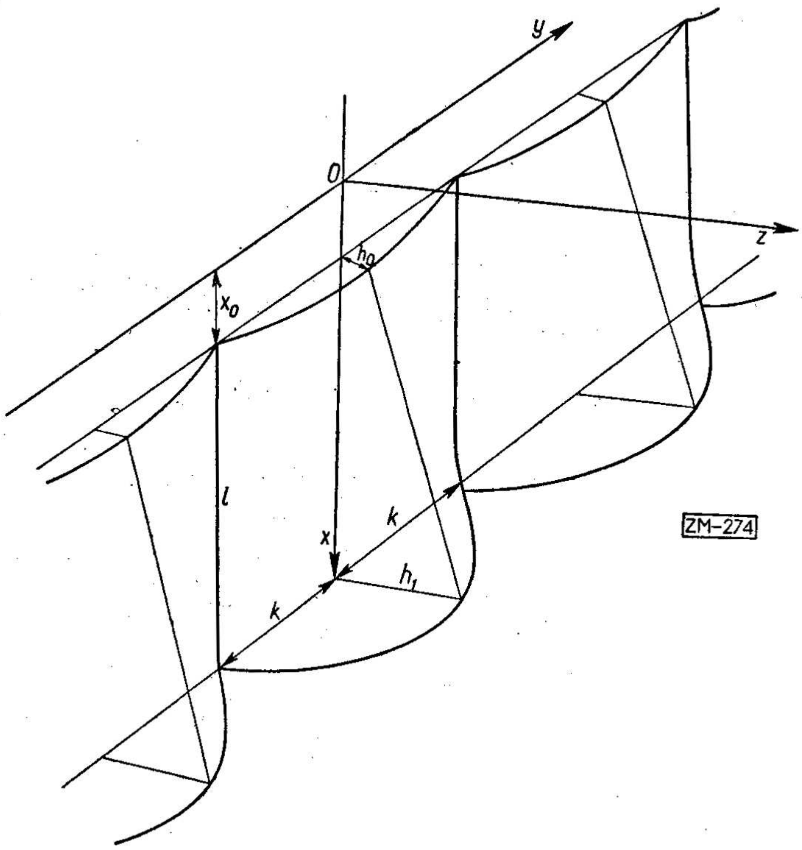

Fig. 6

Integrating both sides of this equation we obtain a differential equation of the first order which is linear in the unknown function $\int_{\xi_0}^{\xi}\sigma_{\mathrm{I}}d\xi$. Solving the last equation we get

$$(9.16)\qquad \int_{\xi_0}^{\xi}\sigma_{\mathrm{I}}d\xi = \xi^{(3y'(1)+2y''(1))/y'(1)}\exp\left(-2\int\frac{d\xi}{\beta_2\xi}\right)\times$$
$$\times\left\{\int\left[\int_{\xi_0}^{\xi}\xi^3P(\xi,1)d\xi+\frac{2y'(1)}{\beta_2}\left(\int_{\xi_0}^{\xi}\overline{X}_2d\xi-\frac{\beta_2}{x_1}\int_{\xi_0}^{\xi}Xd\xi\right)+c_1\right]\times\right.$$
$$\left.\times\,\xi^{-(4y'(1)+2y''(1))/y'(1)}\exp\left(2\int\frac{d\xi}{\beta_2\xi}\right)d\xi+c_2\right\},$$

where $\beta_2 = b_2/B(\xi, 1)$.

In a similar way we calculate σ_{III}. Namely

$$(9.17)\qquad \int_{\xi_0}^{\xi}\sigma_{\mathrm{III}}d\xi = \xi^{(3y'(-1)+2y''(-1))/y'(-1)}\exp\left(-2\int\frac{d\xi}{\beta_3\xi}\right)\times$$
$$\times\left\{\int\left[\frac{2y'(-1)}{\beta_3}\left(\int_{\xi_0}^{\xi}\overline{X}_3d\xi-\frac{\beta_3}{x_1}\int_{\xi_0}^{\xi}Xd\xi\right)-\int_{\xi_0}^{\xi}\xi^3P(\xi,-1)d\xi+c_1\right]\times\right.$$
$$\left.\times\,\xi^{-(4y'(-1)-2y''(-1))/y'(-1)}\exp\left(2\int\frac{d\xi}{\beta_3\xi}\right)d\xi+c_2\right\}.$$

Assuming that the bar carries each stress S which acts along the right boundary, we calculate σ_{I} from the second equation of system (6.9). The first equation holds by definition. Substituting s_2 from the condition of coincidence (7.3) in the second equation (6.9), we obtain from (9.3),

$$(9.18)\qquad \alpha_2\frac{d\sigma_{\mathrm{I}}}{d\xi} = -\left[\alpha_2'+\frac{2}{y'(1)\xi}\right]\sigma_{\mathrm{I}}-\frac{1}{\varkappa y'(1)\xi}Z-\frac{x_1}{y'(1)}\overline{Y}_2,$$

where $\alpha_2 = a_2/B(\xi, 1)$. Solving this equation we get

$$(9.19)\qquad \sigma_{\mathrm{I}}(\xi) = \frac{1}{\alpha_2}\exp\left(-\frac{2}{y'(1)}\int\frac{d\xi}{\alpha_2\xi}\right)\times$$
$$\times\left[-\frac{1}{y'(1)}\int\left(\frac{1}{\varkappa\xi}Z+x_1\overline{Y}_2\right)\exp\left(\frac{2}{y'(1)}\int\frac{d\xi}{\alpha_2\xi}\right)d\xi+c\right].$$

In a similar way we calculate σ_{III}:

$$(9.20)\qquad \sigma_{\mathrm{III}}(\xi) = \frac{1}{\alpha_3}\exp\left(-\frac{2}{y'(-1)}\int\frac{d\xi}{\alpha\,\xi}\right)\times$$
$$\times\left[\frac{1}{y'(-1)}\int\left(\frac{1}{\varkappa\xi}Z-x_1\overline{Y}_3\right)\exp\left(\frac{2}{y'(-1)}\int\frac{d\xi}{\alpha_3\xi}\right)d\xi+c\right].$$

If we assume that the bar does not carry any stresses T_2 produced by the shell, then both functions will be found from equation (9.18) where we substitute $a_2 = \bar{Y}_2 = 0$. Namely

$$\sigma_{\mathrm{I}}(\xi) = -\frac{Z(\xi, 1)}{2\varkappa}, \tag{9.21}$$

and in the case of the left bar

$$\sigma_{\mathrm{III}}(\xi) = \frac{Z(\xi, -1)}{2\varkappa}.$$

Hence, the stress in the shell, when determined by the second boundary problem, depends on the way in which the back, right and left boundaries are fastened. The constants of integration will be determined by the boundary conditions of the bars strengthening those boundaries and by the conditions of coincidence (5.2) which must be satisfied by the functions $\sigma_{\mathrm{I}}(\xi)$, $\sigma_{\mathrm{II}}(\eta)$, $\sigma_{\mathrm{III}}(\xi)$ in order to obtain the continuity of the stresses S, T_1, T_2 and of their derivatives in the whole rectangle $I_0 I_1 J_0 J_1$.

10. Examples. Let us here consider two cases of application of conoidal shells. They are: for roofing and for building dam walls (see [6], p. 202).

In the first case we shall restrict ourselves to shells which carry only a perpendicular exterior load such that on the unit area of the projection a constant force acts. We also suppose here that the stress depends on the boundary conditions along the front arc of the shell. In the second case we shall assume that the shell carries an exterior load acting in the direction of the x — axis (fig. 1) such that on the unit area of the projection a constant force acts and, moreover, that also a hydrostatic pressure, constant on a unit projection of the area, acts on the shell. In this case we shall assume that the stress depends on the boundary conditions given along the right and left boundaries and along the back arc.

In these examples we shall confine ourselves to a conoidal surface with a parabolical directrix (see (1.1)). Let this parabola be

$$Z = h_1(1 - y^2/k). \tag{10.1}$$

Hence the variables ξ, η introduced in § 1, are defined by

$$\xi = x/x_1, \quad \eta = y/k; \quad y' = k. \tag{10.2}$$

The coefficients of the metric form can be expressed as follows by the coordinates ξ and η:

$$(10.3)\qquad \begin{aligned} &A(\xi,\eta)=\sqrt{x_1^2+h_1^2(1-\eta^2)^2}, \qquad B(\xi,\eta)=k\sqrt{1+(\varkappa\xi\eta)^2},\\ &C(\xi,\eta)=k\sqrt{x_1^2+(x_1\varkappa\xi\eta)^2+h_1^2(1-\eta^2)^2}; \qquad \varkappa=-2h_1/k. \end{aligned}$$

According to what has been said above, we consider two kinds of exterior loads:

$$(10.4)\qquad X=0, \qquad Y=0, \qquad Z=\text{const},$$

$$(10.5)\qquad \begin{aligned} &X=R+\frac{h_1\gamma}{x_1}(\xi-\xi_0)(1-\eta^2), \qquad Y=\varkappa\gamma(\xi-\xi_0)\xi\eta,\\ &Z=-\gamma(\xi-\xi_0)-\frac{h_1R}{x_1}(1-\eta^2), \qquad R=\text{const}. \end{aligned}$$

Calculating the stresses in the shell we shall make use, in the first case, of the solution of the first boundary problem, and in the second case — of the solution of the second one (see §§ 8 and 9).

In view of the symmetry of the exterior loads as well as of the boundary conditions here assumed we shall compute the stress in one half of the shell only.

EXAMPLE 1. Substituting the given exterior load (10.4) in formulas (4.2) and (2.4) we get

$$(10.6)\qquad \begin{aligned} &S(\xi,\eta)=\frac{\xi^2}{k}\sigma_1(\xi^2\eta), \qquad T_2(\xi,\eta)=-\frac{B}{A}\left[2\xi\eta\sigma_1(\xi^2\eta)+\frac{1}{\varkappa\xi}Z\right],\\ &T_1(\xi,\eta)=-\frac{A}{kB}\left[\int_1^{\xi}\xi^4\sigma_1'(\xi^2\eta)\,d\xi+k\tau_1(\eta)\right]. \end{aligned}$$

The functions $\sigma_1(\eta)$ and $\tau_1(\eta)$ are determined from the conditions on the front arc (see § 8).

Let us assume that the supports I_1, J_1 (fig. 1) of the front bar are flexible without possibility of translation, and that this bar is under an exterior vertical load, constant on the unit projection of the arc. Thus (see 6.3)

$$(10.7)\qquad \bar{X}_1=0, \qquad \bar{Y}_1=0, \qquad \bar{Z}_1B_1=\text{const}.$$

The area of the section of that bar will be denoted, as in § 7, by Δ_1. The area Δ_1 depends on η. We assume here that the stresses in the shell and in the bar are equal along the front boundaries, i. e. $b_1=\Delta_1/\delta$, $a_1=\Delta_1/\delta$.

We introduce the following notation: let Δ_1^* denote the area of the normal section of the front bar (fig. 7). Let φ be the angle between the sections Δ_1 and Δ_1^*. This angle is a function of η and besides we have $\varphi(0) = \varphi(1) = 0$. It is $\Delta_1^* = \Delta_1 \cos\varphi$. Formula (8.4) implies $\alpha_1 = a_1/A_1$, $\beta_1 = b_1/A_1$ and thus $\alpha_1 = \beta_1 = \Delta_1/\delta A_1 = \lambda$. Hence $\Delta_1^* = x_1\delta\lambda\sqrt{1+(h_1/x_1)^2(1-\eta^2)^2}\cos\varphi$. This formula can be written with an error which does not exceed $\frac{1}{2}h_1^2/x_1^2$ as follows:

$$\Delta_1^* = x_1\delta\lambda. \tag{10.8}$$

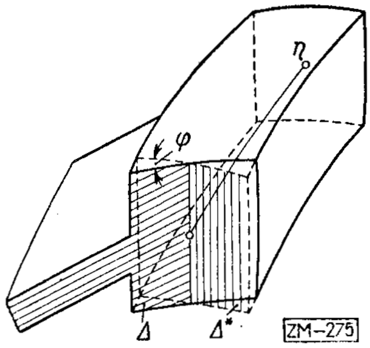

Fig. 7

In this formula λ is a parameter. We shall obtain the stress in the shell while the exterior load acting on it is (10.4) and the loads acting on the front bar are (10.7), by a superposition of the stresses which are produced by the load (10.4) and by the load (10.7).

Substituting in formula (8.7) and (8.8) the load acting on the front bar $\bar{X}_1 = Y_1 = \bar{Z}_1 = 0$ and the exterior load acting on the shell from formula (10.4), and putting $\lambda = 3/8$ in formula (10.8), we get

$$\begin{aligned} \frac{\sigma_1(\eta)}{Z} &= \frac{-0{,}175+0{,}05\varkappa^2\eta^2}{(1+\varkappa^2\eta^2)^2}\varkappa\eta, \\ \frac{k\tau_1(\eta)}{Z} &= \frac{0{,}947-1{,}527\varkappa^2\eta^2-0{,}438\varkappa^4\eta^4}{(1+\varkappa^2\eta^2)^2}\varkappa. \end{aligned} \tag{10.9}$$

We have determined the constants of integration c_1, c_2 from the condition $\sigma_1(0) = 0$.

Substituting in formulas (8.7) and (8.8) $X = Y = Z = 0$ for the exterior load acting on the shell and the values given by (10.7) for the exterior load acting on the bar, and putting $\lambda = 3/8$, we get

$$\begin{aligned} \frac{\sigma_1(\eta)}{B_1\bar{Z}_1} &= \frac{0{,}175-0{,}05\varkappa^2\eta^2}{(1+\varkappa^2\eta^2)^2}\cdot\frac{8}{3}\varkappa\eta, \\ \frac{k\tau_1(\eta)}{B_1\bar{Z}_1} &= \frac{-0{,}947+1{,}527\varkappa^2\eta^2+0{,}438\varkappa^4\eta^4}{(1+\varkappa^2\eta^2)^2}\cdot\frac{8}{3}\varkappa. \end{aligned} \tag{10.10}$$

We determine the constants of integration c_1, c_2 also from the condition $\sigma_1(0) = 0$.

We obtain the stress-system which is produced by the exterior load acting on the shell by substituting $\sigma_1(\eta)$ and $\tau_1(\eta)$ from formula (10.9) into formula (10.6). The stresses which are produced by the exterior load acting on the front bar can be obtained by substituting $\sigma_1(\eta)$ and $\tau_1(\eta)$ from formula (10.10) into formulas (10.6).

EXAMPLE 2. In this example we accept the extriror load acting on the shell given by formula (10.5). We shall apply here the solution of the second boundary problem. Substituting the known exterior load (10.5) in formulas (5.3) and (2.4), we get

$$kS(\xi,\eta)=\begin{cases}\dfrac{1}{\eta}\sigma_{\mathrm{I}}(\xi\sqrt{\eta})+\dfrac{kR}{4x_1}\left(1-\dfrac{1}{\eta^2}\right)\eta+\dfrac{\varkappa\gamma\xi^2}{2\sqrt{\eta}}\left[2\xi\eta(\sqrt{\eta}-1)+\xi_0\sqrt{\eta}(1-\eta)\right] & \text{in region I,}\\[2mm] \dfrac{\xi^2}{\xi_0^2}\sigma_{\mathrm{II}}\left(\dfrac{\xi^2\eta}{\xi_0^2}\right)+\dfrac{kR}{4x_1}\left(1-\dfrac{\xi^4}{\xi_0^4}\right)\eta-\dfrac{\varkappa\gamma\eta\xi^2}{2\xi_0}(\xi-\xi_0)^2 & \text{in region II,}\end{cases}$$

(10.11)

$$T_2(\xi,\eta)=-\frac{B}{A}\left[\frac{2\eta S(\xi,\eta)}{\xi}+\frac{R}{2x_1\xi}(1-\eta^2)-\frac{\gamma}{2\varkappa k\xi}(\xi-\xi_0)\right],$$

$$T_1(\xi,\eta)=-\frac{A}{x_1B}\left[\int_{\xi_0}^{\xi}\frac{\partial S(\xi,\eta)}{\partial\eta}d\xi+R(\xi-\xi_0)+\frac{h_1\gamma}{2x_1}(1-\eta^2)(\xi-\xi_0)^2+\tau_0(\eta)\right].$$

We shall compute the functions $\sigma_{\mathrm{I}}(\xi)$, $\sigma_{\mathrm{II}}(\eta)$, $\tau_0(\eta)$ from the boundary conditions on the back arc and on the right boundary (see § 9).

We suppose that the supports I_0, J_0 of the back bar are flexible with possibility of translation and that there is a force acting on this bar in the direction of the ξ-axis:

$$B_0\overline{X}_0=\text{const},\quad \overline{Y}_0=0,\quad \overline{Z}_0=0, \tag{10.12}$$

and also two concentrated forces which act at the points I_0, J_0 and have the direction of the η-axis. We suppose, moreover, that the last two forces have opposite senses and equal absolute values.

We also assume that the back bar does not carry either the stresses T_1 produced by the shell or any loads in the direction of the x-axis, i. e. that

$$\frac{B_0}{A_0}T_1(\xi_0,\eta)=\tau_0(\eta)=B_0\overline{X}_0.$$

As in the first example, we assume that the stresses in the shell and in the bar are equal along the back boundary, i. e. $b_0=\Delta_0/\delta$, $a_0=0$, where Δ_0 denotes, as in § 7, the area of the section of the back bar. Let us denote by Δ_0^* the area of the normal section of the bar and by φ the angle between sections Δ_0 and Δ_0^*. Hence $\Delta_0^* = \Delta_0 \cos\varphi$. We have assumed in formulas (9.11) and (9.14), from which $\sigma_{II}(\eta)$ will be computed, that

$$\beta_0 = \frac{\lambda - \varkappa^2 \xi_0^2 \eta^2}{4(1+\varkappa^2 \xi_0^2 \eta^2)}$$

(see (9.13)). Hence

$$\xi_0 \frac{\lambda - \varkappa^2 \xi_0^2 \eta^2}{4(1+\varkappa^2 \xi_0^2 \eta^2)} = \frac{\Delta_0}{\delta A_0},$$

and consequently

$$\Delta_0^* = \frac{x_0 \delta}{4} \cdot \frac{\lambda - \varkappa^2 \xi_0^2 \eta^2}{1+\varkappa^2 \xi_0^2 \eta^2} \sqrt{1+\left(\frac{h_1}{x_1}\right)^2 (1-\eta^2)^2} \cos\varphi .$$

This formula for $\lambda = 1$ can be expressed as follows:

$$\Delta_0^* = \frac{x_0 \delta}{4}. \tag{10.13}$$

Let us now consider the right boundary. We suppose that this boundary is strengthened by a bar which carries each stress S produced by the shell (see § 9). Let us denote the area of the normal section of this bar by $\Delta_2(\xi)$. Assuming that the stresses in the bar and those in the shell are equal along the boundary, we have $a_2 = \Delta_2(\xi)/\delta$. From formula (9.18) it follows that $a_2 = \Delta_2/\delta k\sqrt{1+\varkappa^2\xi^2}$, which implies

$$\Delta_2 = a_2 \delta k \sqrt{1+\varkappa^2 \xi^2}. \tag{10.14}$$

We suppose that $a_2 = \text{const}$. Furthermore, we assume that the loads acting on the bar are $\bar{Y}_2 = 0$, $\bar{X}_2$ and $\bar{Z}_2$ being arbitrary. If the exterior loads acting on it are expressed by (10.5), those which act on the back bar are expressed by (10.12), and a concentrated force H acts on I_0, we get the stresses in the shell by superposition of the stresses which are produced by loads (10.5), where $\gamma = 0$, with the stresses produced by load (10.12) and the concentrated force H.

Substituting in formulas (9.11) and (9.14) the load (10.5), where $\gamma = 0$ and the load acting on the back bar is $\bar{X}_0 = \bar{Y}_0 = \bar{Z}_0 = 0$, we obtain the function $\sigma_{II}(\eta)$, and substituting the load (10.5), where $\gamma = 0$

and $\bar{Y}_2 = 0$, in formulas (9.19) and assuming that $a_2 = \text{const}$, we obtain $\sigma_{\mathrm{I}}(\xi)$. Namely

$$(10.15) \qquad \sigma_{\mathrm{II}}(\eta) = -\frac{R\xi_0 k}{8x_1}\Big\{\frac{1}{3}(5-4\varkappa^2\xi_0^2)+3(c_2+\varkappa^2\xi_0^2 c_1)+ + [20+3\varkappa^2\xi_0^2+20\varkappa^2\xi_0^2 c_2+10\varkappa^4\xi_0^4 c_1]\eta^2- -18\varkappa^2\xi_0^2\eta^4+\frac{2}{3}(1+\varkappa^2\xi_0^2)(3+10\varkappa^2\xi_0^2\eta^2)\eta\ln\eta\Big\}\eta,$$

$$\sigma_{\mathrm{I}}(\xi) = -\frac{R\xi_0 k c_0}{8x_1}\xi^{-2/ka_2}.$$

We shall determine the constants c_0, c_1, c_2 from the conditions of supporting the back bar and from the conditions of coincidence (5.2) (see § 9). From the assumption that the supports of the back bar are flexible with the possibility of translation it follows that the component on the η-axis of the resultant of the interior force in the bar vanishes at the point of supporting. This and the second equation (9.5) imply $[\int \sigma_{\mathrm{II}}(\eta)d\eta]_{\eta=1}+\text{const} = 0$. Hence we can find the constants c_0, c_1, c_2 by solving the system of equations

$$\left(\frac{3}{2}\varkappa^2\xi_0^2+\frac{10}{3}\varkappa^4\xi_0^4-1\right)c_1+\left(\frac{3}{2}+\frac{20}{3}\varkappa^2\xi_0^2\right)c_2 = -\frac{1}{9}(68-29\varkappa^2\xi_0^2),$$

$$-\xi_0^{-2/ka_2}c_0+(3+10\varkappa^2\xi_0^2)\varkappa^2\xi_0^2 c_1+(3+20\varkappa^2\xi_0^2)c_2 = -\frac{1}{3}(65-49\varkappa^2\xi_0^2),$$

$$\left(1+\frac{1}{ka}\right)\xi_0^{-2/ka_2}c_0+(3+30\varkappa^2\xi_0^2)\varkappa^2\xi_0^2 c_1+3(1+20\varkappa^2\xi_0^2)c_2 = -\frac{1}{3}(191-221\varkappa^2\xi_0^2)-\frac{4}{\xi_0}.$$

The stresses S, T_1, T_2 are now produced by the exterior load acting on the shell (10.5) where $\gamma = 0$, i. e. by a load acting in the direction of the ξ-axis and such that on the unit area of projection a constant force acts. We shall obtain these stresses by substituting $\sigma_{\mathrm{II}}(\eta)$ and $\sigma_{\mathrm{I}}(\xi)$ from formulas (10.15) and $\gamma = 0$, $\tau_0 = 0$ in formulas (10.11).

After substituting in formulas (9.11) and (9.14) the load (10.5), where $R = 0$, and the load acting on the back bar $\bar{X}_0 = \bar{Y}_0 = \bar{Z}_0 = 0$, we obtain the functions $\sigma_{II}(\eta)$ and after substituting that load and $\overline{Y}_2 = 0$ in formula (9.19), we obtain the functions $\sigma_I(\xi)$. Namely

$$
\begin{aligned}
\sigma_{II}(\eta) &= [6(2\varkappa^2\xi_0^2c_1+6c_2)\eta+(12\varkappa^2\xi_0^2c_1+20c_2)\varkappa^2\xi_0^2\eta^3]\frac{\gamma}{2\varkappa}, \\
\sigma_I(\xi) &= \frac{\gamma}{2\varkappa}\left(\frac{2\xi}{2+ka_2}-\xi_0+c_0\xi^{-2/ka_2}\right).
\end{aligned}
\tag{10.16}
$$

We shall determine the constants from the system of equations

$$(\varkappa^2\xi_0^2+3\varkappa^4\xi_0^4-1)c_1+(3+5\varkappa^2\xi_0^2)c_2 = 0,$$

$$-\xi_0^{-2/ka_2}c_0+2\varkappa^2\xi_0^2(1+6\varkappa^2\xi_0^2)c_1+2(3+10\varkappa^2\xi_0^2)c_2 = -\frac{ka_2\xi_0}{2+ka_2},$$

$$\left(1+\frac{1}{ka_2}\right)\xi_0^{-2/ka_2}c_0+2\varkappa^2\xi_0^2(1+18\varkappa^2\xi_0^2)c_1+6(1+10\varkappa^2\xi_0^2)c_2 = \frac{1+ka_2}{2+ka_2}\xi_0.$$

We obtain the stresses S, T_1, T_2, which are produced by the hydrostatic load acting on the shell by substituting $\sigma_{II}(\eta)$ and $\sigma_I(\xi)$ from formulas (10.16) and $R = 0$, $\tau_0 = 0$ in formulas (10.11).

Substituting the load acting on the back bar (10.12) in formulas (9.11) and (9.14) and the load acting on the shell $X = Y = Z = 0$ and $\overline{Y}_2 = 0$ in formula (9.19), we get

$$
\begin{aligned}
\sigma_{II}(\eta) = h_1B_0\overline{Z}_0\Bigg[2\left(3c_2+3\varkappa^2\xi_0^2c_1+\frac{1}{3}\right)&+2(1+10c_2+8\varkappa^2\xi_0^2c_1)\varkappa^2\xi_0^2\eta^2+ \\
&+4\varkappa^2\xi_0^2\eta^4+\frac{4}{3}(3+14\varkappa^2\xi_0^2\eta^2)\ln\eta\Bigg]\eta, \\
\sigma_I(\xi) = h_1B_0\overline{X}_0c_0\xi^{-2/ka_2}.&
\end{aligned}
\tag{10.17}
$$

We shall determine the constants from the system of equations:

$$(3\varkappa^2\xi_0^2+4\varkappa^4\xi_0^4-1)c_1+(3+5\varkappa^2\xi_0^2)c_2 = \frac{2}{3},$$

$$-\xi_0^{-2/ka_2}c_0+(3+8\varkappa^2\xi_0^2)2\varkappa^2\xi_0^2c_1+2(3+10\varkappa^2\xi_0^2)c_2 = -\frac{2}{3}-6\varkappa^2\xi^2,$$

$$\left(1+\frac{1}{ka_2}\right)\xi_0^{-2/ka_2}c_0+6\varkappa^2\xi_0^2(1+8\varkappa^2\xi_0^2)c_1+6(1+10\varkappa^2\xi_0^2)c_2 = -\frac{14}{3}-\frac{134}{3}\varkappa^2\xi_0^2.$$

We obtain the stresses S, T_1, T_2, which are produced by the load of the back bar (10.12), by substituting $\sigma_{II}(\eta)$ and $\sigma_I(\xi)$ from formulas (10.17), $R = \gamma = 0$ and $\tau_0(\eta) = B_0\overline{X}_0$ in formulas (10.11).

Substituting $X = Y = Z = \overline{X}_0 = \overline{Y}_0 = \overline{Z}_0 = \overline{Y}_2 = 0$ in formulas (9.11), (9.14) and (9.19) we get

$$\begin{aligned} \sigma_{II}(\eta) &= 6(c_2 + \varkappa^2 \xi_0^2 c_1)\eta + 20(c_2 + \varkappa_2 \xi_0^2 c_1)\varkappa^2 \xi_0^2 \eta^3, \\ \sigma_I(\xi) &= c_0 \xi^{-2/ka_2}. \end{aligned} \tag{10.18}$$

We shall determine the constants c_0, c_1, c_2 from the system of equations

$$(3\varkappa^2\xi_0^2 + 5\varkappa^4\xi_0^4 - 1)c_1 + (3 + 5\varkappa^2\xi_0^2)c_2 = H,$$

$$-\xi_0^{-2/ka_2}c_0 + 2\varkappa^2\xi_0^2(3 + 10\varkappa^2\xi_0^2)c_1 + 2(3 + 10\varkappa^2\xi_0^2)c_2 = 0,$$

$$\left(1 + \frac{1}{ka_2}\right)\xi_0^{-2/ka_2}c_0 + 6\varkappa^2\xi_0^2(1 + 10\varkappa^2\xi_0^2)c_1 + 6(1 + 10\varkappa^2\xi_0^2)c_1 = 0.$$

We obtain the stresses S, T_1, T_2, produced by the concentrated force H, substituting σ_{II} and σ_I from formulas (10.18), $R = 0$, $\gamma = 0$, $\tau_0 = 0$ in formulas (10.11).

References

[1] M. Fauconnier, *Essais de rupture d'une voûte mince conoïde en beton armé*, Génie civil 1 (1933), p. 533.

[2] K. Hruban, *O nékterych novych druich tenkých kleneb*, Technický obzor 12, 13 (1942).

[3] Z. Leśniak, *Obliczanie i projektowanie cienkościennych sklepień wichrowatych*, Inżynieria i Budownictwo 11 (1952).

[4] H. Neuber, *Allgemeine Schalentheorie*, Zeitschr. f. angew. Math. u. Mech. 29 (1949), p. 97.

[5] D. Rüdiger, *Dehnungsspannungen und Verschiebungen der Konoidschalen*, Oesterreichisches Ingenieur-Archiv IX (1) (1955), p. 38.

[6] A. Jilek, *Specjální Železobetonove Konstrukcé*, Praha 1951.

[7] E. Kamke, *Differentialgleichungen reeller Funktionen*, Leipzig 1956.

[8] M. Soare, *Die Membrantheorie der Konoidschalen*, Der Bauingenieur 33 (7) (1958), p. 256-265.

[9] W. Flügge, *Statik und Dynamik der Schalen*, 1957, p. 127-131.

Praca wpłynęła 10. 4. 1958

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OBLICZANIE STATYCZNE POWŁOK KONOIDALNYCH

STRESZCZENIE

Konoida jest powierzchnią utworzoną przez prostą poruszającą się równolegle do ustalonej płaszczyzny po dwóch krzywych, leżących w płaszczyznach prostopadłych do tej płaszczyzny (rys. 1). W pracach omawiających obliczenia statyczne powłoki konoidalnej (zob. np. [2], [3], [8], [9]) nie uwzględnia się warunków brzegowych, ale przeważnie daje się jedynie jakieś rozwiązanie szczególne równań różniczkowych, którym czynią zadość napięcia w powłoce.

Celem niniejszej pracy jest przedstawienie metody matematycznej, służącej do obliczania statycznego powłoki konoidalnej przy danych jej obciążeniach zewnętrznych oraz warunkach brzegowych sprężystych i geometrycznych.

Metoda ta opiera się na następujących założeniach.

1. Przyjmujemy, że w przekroju normalnym powłoki panują napięcia, których wypadkowa leży w płaszczyźnie stycznej do powłoki, a nie ma napięć normalnych ani momentów.

2. Powłoka jest na brzegach swoich usztywniona prętami, w których mogą występować momenty. Zakładamy, że napięcia w powłoce są w pewien sposób (który omówimy później) związane z odpowiednimi siłami wewnętrznymi w prętach.

Zagadnienie, które rozwiązujemy w tej pracy, polega na tym, by przy tych założeniach otrzymać rozkład napięć w powłoce i w prętach, wywołany danym obciążeniem zewnętrznym.

Rozwiązanie układu równań równowagi powłoki konoidalnej (zob. (2.3)) o niewiadomych napięciach S, T_1, T_2 (rys. 2), sprowadza się do rozwiązania równania różniczkowego o pochodnych cząstkowych pierwszego rzędu o niewiadomej S (zob. (3.1), (3.2)). Niewiadome T_1, T_2 wyznaczają się bezpośrednio przez S (zob. (2.4)). Analizując przebieg rzutów charakterystyk równania (3.2) (rys. 3), postawiono i rozwiązano dwa zadania brzegowe Cauchy'ego dla tego równania. W pierwszym zadaniu (§ 4) stan napięć w powłoce rozpostartej nad prostokątem $I_0I_1J_1J_0$ (rys. 1 i 3) zależy od dwóch funkcji danych na boku I_1J_1 tego prostokąta, a w drugim (§ 5) od czterech funkcji, przy czym dwie z nich dane są na boku I_0J_0, a pozostałe dwie odpowiednio na bokach I_0I_1, J_0J_1. Inaczej mówiąc, stan napięć zależy w pierwszym zadaniu brzegowym od napięć S i T_1 danych na krawędzi czołowej, a w drugim od napięć S i T_1 danych na krawędzi tylnej oraz od napięcia S danego na krawędziach prostych. W §§ 8 i 9 podano sposób wyznaczania napięć na krawędziach powłoki dla obu wymienionych zadań brzegowych. W tym celu założono, że powłoka wzmocniona jest wzdłuż swoich krawędzi prętami. W § 6 wyprowadzono równania równowagi tych prętów przyjmując, że pręty te są obciążone daną siłą zewnętrzną oraz napięciami z powłoki, panującymi w miejscu styku powłoki z prętem (rys. 4 i 5). Zakładając równość naprężeń na krawędziach powłoki i w prętach wzmacniających (w tych samych przekrojach) otrzymano związki między napięciami na krawędziach powłoki oraz siłami wewnętrznymi w prętach (§ 7). Rozwiązując równanie różniczkowe równowagi prętów wzmacniających powłokę, przy założonych związkach między napięciami a siłami wewnętrznymi w prętach, wyznaczono w pierwszym i drugim zadaniu brzegowym napięcia na krawędziach powłoki (w §§ 8 i 9 podano rozwiązania przy ogólnych związkach między siłami wewnętrznymi w prętach a napięciami w powłoce, tzn. niekoniecznie wynikających z równości naprężeń).

W przykładach rozważono dwa przypadki zastosowania powłoki konoidalnej, mianowicie zastosowanie powłoki jako pokrycia dachowego oraz zastosowanie do konstrukcji zapory wodnej (lub muru oporowego), o konoidalnym kształcie ścian oporowych. W pierwszym przypadku ograniczono się do powłoki przy obciążeniu pionowym stałym na jednostkę rzutu pola, zakładając, że stan napięć w powłoce zależy od warunków brzegowych na krawędzi czołowej. W drugim przypadku założono, że powłoka jest obciążona w kierunku osi x (rys. 6) stałą siłą na jednostkę rzutu pola oraz parciem hydrostatycznym stałym na jednostkę rzutu pola, przy czym założono, że stan napięć w powłoce zależy od warunków brzegowych wzdłuż

krawędzi prostych oraz łuku tylnego. W przykładach ograniczono się do powłoki konoidalnej, której kierownicą jest parabola. Końcowe wzory podano w tzw. postaci zamkniętej, przy dowolnych wartościach liczbowych obciążeń i wymiarach geometrycznych powłoki.

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О СТАТИЧЕСКОМ РАССЧЕТЕ КОНОИДООБРАЗНЫХ ОБОЛОЧЕК

РЕЗЮМЕ

В этой статье определены напряжения в коноидообразной оболочке в зависимости от краевых условий. Для этого составлены и решены две краевых задачи Коши для системы дифференциальных уравнений, описывающей общую картину напряжений в коноидообразной оболочке, простирающейся над прямоугольником. Предполагается, что края оболочки сделаны жескими путем укрепления их прутьями. Предполагая некоторые зависимости между напряжениями на краях оболочки и внутренними силами в прутьях, автор получает краевые условия для обеих задач Коши. Решение первой и второй задачи Коши получено в квадратурах для любой внешней нагрузки оболочки.

Полученные общие результаты иллюстрируются примерами. Рассматривается применение коноидообразной оболочки для перекрытия крыш и для построения плотин с коноидообразными опорными стенками.
