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Functions with bounded nth differences

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Abstract. Let A be a commutative semigroup under addition and let Y be a Banach space. For each natural number n, there exists $k_n > 0$ with the property: if $\delta > 0$ and $f\colon A \to Y$ such that

$$\Big|\sum_{k=0}^{n} (-1)^{n-k}\tbinom{n}{k} f(x+kh)\Big| \leqslant \delta \quad \text{for all } x,\ h \in A,$$

then there exists $g\colon A \to Y$ such that

$$|f(x)-g(x)| \leqslant k_n \delta$$

and

$$\sum_{k=0}^{n} (-1)^{n-k}\tbinom{n}{k} g(x+kh) = 0 \quad \text{for all } x,\ h \in A.$$

1. Introduction. Throughout this paper A denotes a commutative semigroup under addition and Y denotes a real Banach space with the norm of $y \in Y$ denoted by $|y|$. Let Y^A be the real vector space of all functions from A to Y. For $h \in A$, the linear difference operator $\underset{h}{\Delta}\colon Y^A \to Y^A$ is defined by $\underset{h}{\Delta} f(x) = f(x+h)-f(x)$ for $f \in Y^A$ and $x, h \in A$. The nth iterate of $\underset{h}{\Delta}$, $\underset{h}{\Delta}^n$, satisfies the identity

$$\underset{h}{\Delta}^n f(x) = \sum_{k=0}^{n} (-1)^{n-k}\binom{n}{k} f(x+kh) \quad \text{for } x,\ h \in A \text{ and } f \in Y^A.$$

Whitney [9] has shown that if $Y = R$ (the real numbers), $A = R$ (or

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$(0, +\infty)$), $\delta > 0$, n is a natural number, f: $A \to Y$ is bounded on an interval and $|\Delta_h^n f(x)| \leq \delta$ for all $x, h \in A$, then there is a polynomial g of degree at most $n-1$ such that

$$|f(x)-g(x)| \leq k_n \delta \quad \text{for all } x \in A,$$

where $k_n \leq 1$ if $A = (0, +\infty)$ and $k_n \leq 1/\sup_m \binom{n}{m}$ if $A = R$.

With no regularity assumptions on f, Hyers [4] showed that if $\delta > 0$, n is a natural number, A is a cone in a rational vector space and f: $A \to Y$ such that

$$|\Delta_{h_1} \ldots \Delta_{h_n} f(x)| \leq \delta \qquad \text{for all } x, h_1, \ldots, h_n \in A,$$

then there exists a unique g: $A \to Y$ such that $g(0) = 0$, $\Delta_{h_1} \ldots \Delta_{h_n} g(x) = 0$ and

$$|f(x)-f(0)-g(x)| \leq \delta \quad \text{for all } x, h_1, \ldots, h_n \in A.$$

The aim of this paper is to generalize and give a short proof of Hyer's theorem and to show how it can be used, together with a theorem of Djoković [2] to give a short proof of a generalization of Whitney's result. Whitney [9] also obtained a similar result in case the domain of f is a bounded interval but our methods are not applicable in this case.

The relationship between the theorems of Whitney and Hyers can be seen by noting the following well-known result (see for example [1] or [5]). If f: $R \to R$ is such that $\Delta_h^n f(x) = 0$ for all x, $h \in R$ and if f is bounded on a set of positive Lebesgue measure (for example, if f is Lebesgue measurable), then f is a polynomial of degree at most $n-1$.

2. Background. We will use the following results which were proved for mappings between vector spaces by Mazur and Orlicz [6] and [7] and, in greater generality than required here, by Djoković [2].

A function a: $A^n \to Y$ is called *n-additive* provided it is additive in each variable, i.e.

$$a(x_1+x_1', x_2, \ldots, x_n) = a(x_1, x_2, \ldots, x_n)+a(x_1', x_2, \ldots, x_n),$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$a(x_1, x_2, \ldots, x_n+x_n') = a(x_1, x_2, \ldots, x_n)+a(x_1, x_2, \ldots, x_n')$$

for all $x_1, x_1', \ldots, x_n$, $x_n' \in A$; it is said to be *symmetric* provided $a(x_1, \ldots, x_n) = a(y_1, \ldots, y_n)$ whenever $(x_1, \ldots, x_n) \in A^n$ and $(y_1, \ldots, y_n)$ is a permutation of $(x_1, \ldots, x_n)$. If a: $A^n \to Y$ is symmetric and n-additive we let $a^*(x) = a(x, x, \ldots, x)$ for all $x \in A$.

THEOREM A. *If* $a\colon A^n \to Y$ *is symmetric and n-additive, then for all* $x, h_1, \ldots, h_k \in A$,

$$\mathop{\Delta}_{h_1} \ldots \mathop{\Delta}_{h_k} a^*(x) = \begin{cases} n!a(h_1, \ldots, h_n) & \text{if } k = n, \\ 0 & \text{if } k > n. \end{cases}$$

THEOREM B. *To each natural number n there correspond natural numbers s and p and integers* $m_1, \ldots, m_p$ *with the property: for every* $h_1, \ldots, h_n \in A$ *there exist* $u_1, \ldots, u_p, v_1, \ldots, v_p \in A$ *such that*

$$(n!)^{2^s} \mathop{\Delta}_{h_1} \ldots \mathop{\Delta}_{h_n} f(x) = \sum_{k=1}^{p} m_k \mathop{\Delta^n}_{u_k} f(x+v_k)$$

for all $x \in A$ *and* $f \in Y^A$.

For future reference, let $M_n = (n!)^{-2^s} \sum_{k=1}^{p} |m_k|$.

THEOREM C. *If n is natural number and* $f\colon A \to Y$, *then the following are equivalent*:

(i) $\mathop{\Delta^n}_{h} f(x) = 0$ *for all* $x, h \in A$,

(ii) $\mathop{\Delta}_{h_1} \ldots \mathop{\Delta}_{h_n} f(x) = 0$ *for all* $x, h_1, \ldots, h_n \in A$,

(iii) *there exist* $a_0 \in Y$ *and symmetric, k-additive* $a_k\colon A^k \to Y$, $1 \leqslant k \leqslant n-1$, *such that*

$$f(x) = a_0 + \sum_{k=1}^{n-1} a_k^*(x) \quad \text{for all } x \in A.$$

It is easy to check that $\mathop{\Delta}_{h+k} - \mathop{\Delta}_{h} - \mathop{\Delta}_{k} = \mathop{\Delta}_{h}\mathop{\Delta}_{k} = \mathop{\Delta}_{k}\mathop{\Delta}_{h}$ for all $h, k \in A$.

3. Main results. First we will generalize and give shorter proofs of the theorems of Hyers in [3] and [4]. The proof of the first theorem, on which our analysis rests, is essentially the same as the proof given by Hyers [3] but is short and included here for completeness.

THEOREM 1. *If* $\delta > 0$ *and* $f\colon A \to Y$ *such that* $|f(x+y) - f(x) - f(y)| \leqslant \delta$ *for all* $x, y \in A$, *then there exists a unique* $a\colon A \to Y$ *such that*

$$a(x+y) = a(x) + a(y)$$

and

$$|f(x) - a(x)| \leqslant \delta \quad \text{for all } x, y \in A.$$

Moreover, $a(x) = \lim_{k \to +\infty} f(kx)/k$ *for all* $x \in A$.

Proof. For every $x \in A$, $|\frac{1}{2}f(2x)-f(x)| = \frac{1}{2}|f(2x)-2f(x)| \leqslant \frac{1}{2}\delta$. If n is a natural number and $x \in A$, then

$$\left|\frac{f(2^{n+1}x)}{2^{n+1}}-f(x)\right| \leqslant \frac{1}{2^n}\left|\frac{f(2(2^n x))}{2}-f(2^n x)\right|+\left|\frac{f(2^n x)}{2^n}-f(x)\right|$$
$$\leqslant \delta/2^{n+1}+\left|\frac{f(2^n x)}{2^n}-f(x)\right|.$$

It follows by induction that for every natural number n and every $x \in A$,

$$\left|\frac{f(2^n x)}{2^n}-f(x)\right| \leqslant \left(\frac{1}{2}+\ldots+\frac{1}{2^n}\right)\delta.$$

If n and p are natural numbers and $x \in A$, then

$$\left|\frac{f(2^{n+p}x)}{2^{n+p}}-\frac{f(2^n x)}{2^n}\right| = \frac{1}{2^n}\left|\frac{f(2^p(2^n x))}{2^p}-f(2^n x)\right| \leqslant \delta/2^n.$$

Hence, for every $x \in A$, $\left\{\frac{f(2^n x)}{2^n}\right\}_{n=1}^{\infty}$ is a Cauchy sequence in Y and if we denote its limit by $a(x)$ we have

$$a(x+y) = a(x)+a(y) \quad \text{for all } x, y \in A$$

since

$$\left|\frac{f(2^n(x+y))}{2^n}-\frac{f(2^n x)}{2^n}-\frac{f(2^n y)}{2^n}\right| \leqslant \delta/2^n$$

for all $x, y \in A$ and every natural number n. Also, since

$$\left|\frac{f(2^n x)}{2^n}-f(x)\right| \leqslant \left(\frac{1}{2}+\ldots+\frac{1}{2^n}\right)\delta,$$

it follows that

$$|f(x)-a(x)| \leqslant \delta \quad \text{for all } x \in A.$$

Since a is additive, for every natural number k and each $x \in A$,

$$\left|\frac{f(kx)}{k}-a(x)\right| = \frac{1}{k}|f(kx)-a(kx)| \leqslant \delta/k,$$

so that

$$a(x) = \lim_{k \to +\infty} \frac{f(kx)}{k} \quad \text{for each } x \in A.$$

This calculation also proves the uniqueness of a and the proof is complete.

We will need

THEOREM 2. *Suppose $A_1, \ldots, A_n$ are additive commutative semigroups, $\delta_1, \ldots, \delta_n > 0$ and $f\colon A_1 \times \ldots \times A_n \to Y$ such that*

$$|f(x_1+x_1', \ldots, x_n) - f(x_1, \ldots, x_n) - f(x_1', \ldots, x_n)| \leqslant \delta_1,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$|f(x_1, \ldots, x_n+x_n') - f(x_1, \ldots, x_n) - f(x_1, \ldots, x_n')| \leqslant \delta_n$$

for all $x_i, x_i' \in A_i$, $1 \leqslant i \leqslant n$. Then there exists a unique $a\colon A_1 \times \ldots \times A_n \to Y$ which is additive in each variable and such that

$$|f(x_1, \ldots, x_n) - a(x_1, \ldots, x_n)| \leqslant \min(\delta_1, \ldots, \delta_n)$$

for all $(x_1, \ldots, x_n) \in A_1 \times \ldots \times A_n$. Moreover, if $A = A_1 = \ldots = A_n$ and f is symmetric, then a is symmetric.

Proof. Without loss of generality, assume $\delta_1 \leqslant \delta_k$ for $1 \leqslant k \leqslant n$. By Theorem 1 we may let

$$a(x_1, x_2, \ldots, x_n) = \lim_{k \to +\infty} \frac{f(kx_1, x_2, \ldots, x_n)}{k}$$

for $(x_1, x_2, \ldots, x_n) \in A_1 \times \ldots \times A_n$ and conclude that a is additive in the first variable and

$$|f(x_1, x_2, \ldots, x_n) - a(x_1, x_2, \ldots, x_n)| \leqslant \delta_1$$

for all $x_i \in A_i$, $1 \leqslant i \leqslant n$.

To show that a is additive in the second variable notice that

$$\left|\frac{f(kx_1, x_2+x_2', \ldots, x_n)}{k} - \frac{f(kx_1, x_2, \ldots, x_n)}{k} - \frac{f(kx_1, x_2', \ldots, x_n)}{k}\right| \leqslant \frac{\delta_2}{k}$$

for $x_1 \in A_1$, $x_2, x_2' \in A_2, \ldots, x_n \in A_n$ and every natural number k. Letting $k \to +\infty$ shows that a is additive in the second variable. Similarly, a is additive in each of the remaining variables.

The uniqueness of a is clear and the last assertion is trivial.

THEOREM 3. *Suppose n is a natural number, $\delta > 0$ and $f\colon A \to Y$ such that*

$$|\Delta_{h_1} \ldots \Delta_{h_n} f(x)| \leqslant \delta \quad \text{for all } x, h_1, \ldots, h_n \in A.$$

Then there exist symmetric, k-additive $a_k\colon A^k \to Y$, $1 \leqslant k \leqslant n-1$, such that

$$|\Delta_h (f - \sum_{k=1}^{n-1} a_k^*)(x)| \leqslant \delta \quad \text{for all } x, h \in A.$$

Proof. If $n = 1$, the conclusion is to be interpreted simply as $|\Delta_h f(x)| \leqslant \delta$ for all $x, h \in A$. Thus the assertion is trivially true if $n = 1$.

Suppose the theorem is true for $n = m \geqslant 1$. Let $f\colon A \to Y$ such that

$$|\underset{h_1}{\Delta} \cdots \underset{h_{m+1}}{\Delta} f(x)| \leqslant \delta \quad \text{for all } x, h_1, \ldots, h_{m+1} \in A.$$

For each $h_1, \ldots, h_m \in A$,

$$|\underset{h}{\Delta}(\underset{h_1}{\Delta} \cdots \underset{h_m}{\Delta}) f(x)| \leqslant \delta$$

or

$$\text{(1)} \qquad |\underset{h_1}{\Delta} \cdots \underset{h_m}{\Delta} f(x+h) - \underset{h_1}{\Delta} \cdots \underset{h_m}{\Delta} f(x)| \leqslant \delta \quad \text{for all } x, h \in A.$$

But, for any $x, h_1', h_1, h_2, \ldots, h_m \in A$,

$$\begin{aligned} &|\underset{h_1+h_1'}{\Delta} \underset{h_2}{\Delta} \cdots \underset{h_m}{\Delta} f(x) - \underset{h_1'}{\Delta} \underset{h_2}{\Delta} \cdots \underset{h_m}{\Delta} f(x) - \underset{h_1'}{\Delta} \underset{h_2}{\Delta} \cdots \underset{h_m}{\Delta} f(x)| \\ &\qquad = |(\underset{h_1+h_1'}{\Delta} - \underset{h_1}{\Delta} - \underset{h_1'}{\Delta})(\underset{h_2}{\Delta} \cdots \underset{h_m}{\Delta}) f(x)| = |\underset{h_1'}{\Delta} \underset{h_1}{\Delta} \underset{h_2}{\Delta} \cdots \underset{h_m}{\Delta} f(x)|\, \delta. \end{aligned}$$

Thus, for each $x \in A$, the mapping $(h_1, h_2, \ldots, h_m) \to \underset{h_1}{\Delta}\underset{h_2}{\Delta} \cdots \underset{h_m}{\Delta} f(x)$ of A into Y is "almost" additive in the first variable. But, since difference operators commute, for each $x \in A$ the mapping is symmetric and thus "almost" additive in each variable. Applying Theorem 2 we find that for each $x \in A$ there is a unique $a_x\colon A^m \to Y$ which is symmetric, m-additive and such that

$$\text{(2)} \qquad |\underset{h_1}{\Delta} \cdots \underset{h_m}{\Delta} f(x) - a_x(h_1, \ldots, h_m)| \leqslant \delta$$

for all $h_1, \ldots, h_m \in A$.

From (1) and (2) we find

$$\text{(3)} \qquad |\underset{h_1}{\Delta} \cdots \underset{h_m}{\Delta} f(x+h) - a_x(h_1, \ldots, h_m)| \leqslant 2\delta$$

for all $x, h, h_1, \ldots, h_m \in A$.

Now replace x by $x+y$ in (3) and then replace h by $y+h$ in (3) and compare the resulting inequalities to conclude that

$$\text{(4)} \qquad |a_x(h_1, \ldots, h_m) - a_{x+y}(h_1, \ldots, h_m)| \leqslant 4\delta$$

for all $x, y, h_1, \ldots, h_m \in A$. Thus, for every $x, y \in A$, $a_x - a_{x+y}$ is bounded and m-additive. It easily follows that $a_x = a_{x+y}$. Hence $a_x = a_y$ for all $x, y \in A$. Let $a_x = m!a_m$, $x \in A$, to conclude from (2) that

$$\text{(5)} \qquad |\underset{h_1}{\Delta} \cdots \underset{h_m}{\Delta} f(x) - m!a_m(h_1, \ldots, h_m)| \leqslant \delta$$

for all $x, h_1, \ldots, h_m \in A$.

Let $f_1 = f - a_m^*$. Then, by Theorem A,

$$|\underset{h_1}{\Delta} \ldots \underset{h_m}{\Delta} f_1(x)| = |\underset{h_1}{\Delta} \ldots \underset{h_m}{\Delta} f(x) - \underset{h_1}{\Delta} \ldots \underset{h_m}{\Delta} a_m^*(x)|$$
$$= |\underset{h_1}{\Delta} \ldots \underset{h_m}{\Delta} f(x) - m!a_m(h_1, \ldots, h_m)| \leqslant \delta$$

for all $x, h_1, \ldots, h_m \in A$. By our inductive hypothesis, there exist symmetric k-additive a_k: $A^k \to Y$, $1 \leqslant k \leqslant m-1$, such that $|\underset{h}{\Delta}(f_1 - \sum_{k=1}^{m-1} a_k^*)(x)| \leqslant \delta$ for all $x, h \in A$. But $f - \sum_{k=1}^{m} a_k^* = f_1 - \sum_{k=1}^{m-1} a_k^*$ so we are done.

Theorem II of [4] follows from Theorem 3 by applying (b) of

LEMMA 1. *Suppose* f: $A \to Y$ *and* $\delta > 0$ *such that* $|\underset{h}{\Delta} f(x)| \leqslant \delta$ *for all* $x, h \in A$. *Then*

(a) *there exists* $a_0 \in Y$ *such that* $|f(x) - a_0| \leqslant 2\delta$ *for all* $x \in A$;

(b) *if* A *has a zero* (*a member* 0 *of* A *such that* $0 + x = x + 0 = x$ *for all* $x \in A$), *then* $|f(x) - f(0)| \leqslant \delta$ *for all* $x \in A$;

(c) *if for* $x, y \in A$ *there exists* $h \in A$ *such that either* $y = x + h$ *or* $x = y + h$ *and if* $Y = R$, *then there exists* $a_0 \in Y$ *such that* $|f(x) - a_0| \leqslant \delta/2$ *for all* $x \in A$.

Proof. (a) Fix $y_0 \in A$ and let $a_0 = f(y_0)$. For any $x \in A$

$$|f(x + y_0) - f(x)| = |\underset{y_0}{\Delta} f(x)| \leqslant \delta$$

and

$$|f(x + y_0) - f(y_0)| = |\underset{x}{\Delta} f(y_0)| \leqslant \delta$$

so

$$|f(x) - a_0| \leqslant 2\delta.$$

(b) For any $x \in A$, $|f(x) - f(0)| = |\underset{x}{\Delta} f(0)| \leqslant \delta$.

(c) Let $x, y \in A$ and suppose that $y = x + h$ for some $h \in A$. Then

$$|f(y) - f(x)| = |\underset{h}{\Delta} f(x)| \leqslant \delta.$$

Then f is bounded and we can let

$$a_0 = \{\sup_{x \in A} f(x) + \inf_{x \in A} f(x)\}/2.$$

We now use Theorem 3 to generalize the theorem of Whitney referred to in the introduction.

THEOREM 4. *Suppose n is a natural number, $\delta > 0$ and $f\colon A \to Y$ such that*

$$|\Delta_h^n f(x)| \leqslant \delta \quad \text{for all } x,\ h \in A.$$

Then

(i) *there exist symmetric, k-additive $a_k\colon A^k \to Y$, $1 \leqslant k \leqslant n-1$ such that*

$$|\Delta_h (f - \sum_{k=1}^{n-1} a_k^*)(x)| \leqslant M_n \delta \quad \text{for all } x,\ h \in A;$$

(ii) *if in addition A admits division by n! (for every $k \in A$ there exists $h \in A$ such that $k = n!h$), then*

$$|\Delta_h (f - \sum_{k=1}^{n-1} a_k^*)(x)| \leqslant 2\delta \quad \text{for all } x,\ h \in A;$$

(iii) *if in addition A is a group and admits division by n!, then*

$$|\Delta_h (f - \sum_{k=1}^{n-1} a_k^*)(x)| \leqslant 2\delta/\sup_m \binom{n}{m} \quad \text{for all } x, h \in A.$$

Proof. By Theorem B,

$$|\Delta_{h_1} \dots \Delta_{h_n} f(x)| \leqslant M_n \delta \qquad \text{for all } x,\ h_1, \dots, h_n \in A$$

and (i) follows from Theorem 3.

Let $f_1 = f - \sum_{k=1}^{n-1} a_k^*$ so that f_1 is bounded and $|\Delta_h^n f_1(x)| = |\Delta_h^n f(x)| \leqslant \delta$ for all x, $h \in A$. Let x, $k \in A$. Choose $h \in A$ such that $k = n!h$. Then the argument used by Whitney on pages 83 and 84 of [8], and attributed to A. Beurling (see [9] also) shows that

$$|\Delta_k f_1(x)| = |f_1(x + n!h) - f_1(x)| \leqslant 2\delta$$

in case (ii) and

$$|\Delta_k f_1(x)| \leqslant 2\delta/\sup_m \binom{n}{m}$$

in case (iii).

Notice that the assumptions of Theorem 4 are weaker than those of Theorem 3 but the estimate is not as good in case (i). Using Lemma 1, the result can be sharpened.

The next theorem generalizes Whitney's result.

THEOREM 5. *In addition to the assumptions of Theorem 4, suppose A is a cone in a normed linear space X with nonvoid interior and suppose f is bounded on a nonvoid open subset of A. Then $a_1, \dots, a_{n-1}$ are continuous.*

Proof. Let $g = \sum_{k=1}^{n-1} a_k^*$. Then $\Delta_h^n g(x) = 0$ for all $x, h \in A$ and, by Theorem 4 (i), $\Delta_h (f-g)(x)$ is bounded for $x, h \in A$. By Lemma 1 (a), $f-g$ is bounded and so g is bounded on a nonvoid open subset of A.

Now $a_1 \colon A \to Y$ is additive. Since A has nonvoid interior, $X = A-A = \{x-y \mid x, y \in A\}$. If $x_1, x_2, y_1, y_2 \in A$ and $x_1 - y_1 = x_2 - y_2$, then $x_1 + y_2 = x_2 + y_1$ so $a_1(x_1) + a_1(y_2) = a_1(x_2) + a_1(y_1)$ or $a_1(x_1) - a_1(y_1) = a_1(x_2) - a_1(y_2)$. Thus we may define $\tilde{a}_1 \colon X \to Y$ by letting $\tilde{a}_1(x-y) = a_1(x) - a_1(y)$ for all $x, y \in A$. It is easy to check that $\tilde{a}_1$ is the unique additive extension of a_1 from A to X. Similarly for each $k = 1, 2, \ldots, n-1$, there is a unique symmetric k-additive $\tilde{a}_k \colon X \to Y$ such that $\tilde{a}_k(x_1, \ldots, x_n) = a_k(x_1, \ldots, x_n)$ for all $x_1, \ldots, x_n \in A$. Let $\tilde{g} = \sum_{k=1}^{n-1} \tilde{a}_k^*$ so that $\tilde{g}$ extends g, $\tilde{g}$ is bounded on a nonvoid open subset of X and

$$\Delta_h^n \tilde{g}(x) = 0 \quad \text{for all } x, h \in X.$$

From a theorem of Mazur and Orlicz [7] it follows that a_k is continuous for $1 \leqslant k \leqslant n-1$.

A similar argument, using a theorem of Kemperman [5] instead of the theorem of Mazur and Orlicz, can be applied to prove the following theorem, thereby partially answering a question raised in [9].

THEOREM 6. *In Theorem 4, suppose A is a cone in R^p with nonvoid interior and $Y = R$. If f is bounded on a subset of A having positive Lebesgue measure (in particular, if f is Lebesgue measurable), then a_k is continuous for $1 \leqslant k \leqslant n-1$.*

4. Related functional inequalities. For real valued functions on abelian groups, many of our results hold assuming only one-sided boundedness.

LEMMA 2. *Suppose n is a natural number, A is an abelian group and $f \colon A \to Y$ and $\delta > 0$. If $\Delta_{h_1} \ldots \Delta_{h_n} f(x) \leqslant \delta$ for all $x, h_1, \ldots, h_n \in A$, then*

$$|\Delta_{h_1} \ldots \Delta_{h_n} f(x)| \leqslant \delta \quad \text{for all } x, h_1, \ldots, h_n \in A.$$

If n is odd and $\Delta_h^n f(x) \leqslant \delta$ for all $x, h \in A$, then $|\Delta_h^n f(x)| \leqslant \delta$ for all $x, h \in A$.

Proof. For any $x, h \in A$ and any $g \colon A \to R$, $\Delta_h g(x) = -\Delta_{-h} g(x+h)$. Thus, in the first case

$$\Delta_{h_1} \Delta_{h_2} \ldots \Delta_{h_n} f(x) = -\Delta_{-h_1} \Delta_{h_2} \ldots \Delta_{h_n} f(x+h_1) \geqslant -\delta \quad \text{for all } x, h_1, \ldots, h_n \in A.$$

The second assertion follows from the fact that if n is odd, and $g \colon A \to R$, then $\Delta_h^n g(x) = -\Delta_{-h}^n g(x+nh)$ for any $x, h \in A$.

We now turn to inequalities associated with equations considered, for example, in [2].

THEOREM 7. *Suppose n is a natural number, $\delta > 0$ and f, g: $A \to Y$ such that*

$$|\Delta_h^n f(x) - g(h)| \leqslant \delta \quad \text{for all } x, h \in A.$$

Then there exist symmetric, k-additive a_k: $A^k \to Y$, $1 \leqslant k \leqslant n$, such that

$$|\Delta_h (f - \sum_{k=1}^{n} a_k^*)(x)| \leqslant 2M_{n+1}\delta$$

and

$$|g(h) - n!a_n^*(h)| \leqslant (1 + 2^n M_{n+1})\delta \quad \text{for all } x, h \in A.$$

Proof. For all $x, h \in A$,

$$|\Delta_h^{n+1} f(x)| = |\{\Delta_h^n f(x+h) - g(h)\} - \{\Delta_h^n f(x) - g(h)\}| \leqslant 2\delta.$$

Hence, by Theorem 4 (i), there exist symmetric, k-additive a_k: $A^k \to Y$, $1 \leqslant k \leqslant n$ such that, for all $x, h \in A$,

$$|\Delta_h (f - \sum_{k=1}^{n} a_k^*)(x)| \leqslant 2M_{n+1}\delta$$

and hence

$$|\Delta_h^n (f - \sum_{k=1}^{n} a_k^*)(x)| \leqslant 2^n M_{n+1}\delta.$$

But

$$\Delta_h^n (\sum_{k=1}^{n} a_k^*)(x) = n!a_n^*(h)$$

so that

$$|\Delta_h^n f(x) - n!a_n^*(h)| \leqslant 2^n M_{n+1}\delta$$

and hence

$$|g(h) - n!a_n^*(h)| \leqslant (1 + 2^n M_{n+1})\delta \quad \text{for all } x, h \in A.$$

COROLLARY. *If n is a natural number, $\delta > 0$ and f: $A \to Y$ such that*

$$|\Delta_h^n f(x) - n!f(h)| \leqslant \delta \quad \text{for all } x$$

there exists a symmetric, n-additive a_n: $A^n \to Y$ such that

$$|f(h) - a_n^*(h)| \leqslant (1 + 2^n M_{n+1})\delta/n! \quad \text{for all } h \in A.$$

In Theorem 7 and the corollary, the estimates could be improved, if stronger assumption are made on A and Y, by applying (ii) or (iii) of Theorem 4.

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