ANNALES
POLONICI MATHEMATICI
XLII (1983)

Semi-stability of motions and regular dependence of limit sets on points in general semi-systems

by A. Pelczar (Kraków)

Dedicated to the memory of Jacek Szarski

Abstract. Some conditions generalizing the classical stability (in the Lapunov sense) of motions are introduced and certain connections between them and regularity properties of limit sets are established.

Introduction. The purpose of the present paper is to study regular dependence of limit sets on points moving semi-stable in semi-systems and dynamical semi-systems on metric spaces. Some results presented below cover and generalize results given by the author in [7] and [9]. Certain results on upper semi-continuity of limit sets in so-called *generalized* (*multivalued*) *pseudo-dynamical systems* have been obtained independently by J. Kłapyta [4], [5]. There are given also some results about non-emptiness of limit sets under suitable assumptions; few remarks on such question are presented below in Section 5.

In order to present certain natural motivation for problems treated here we would like to make some general introductory remarks on stability and limit sets. The classical theory of stability, foundations of which are due to A. M. Lapunov (cf. [6]) has been begun originally with respect to solutions of differential equations and then extended by many authors for dynamical systems and generalizations of them. The fundamental question of the Lapunov stability of motions is now treated in very general various versions as well as several problems of Lapunov stability of trajectories and — in more general formulations — stability of sets. Some natural complements to stability problems are investigations of so-called *limit sets*. Stability theory, in a strict sense, examines mainly asymptotic behaviour of solutions of differential equations (or, generally, motions in abstract dynamical systems) when the independent variable, interpretated usually as time, tends to infinity; properties of limit sets, reached by corresponding points "at $t = \infty$" could be (and even — as one can expect —

should be) some consequences of stability properties, or other properties of motions "for large t". So in particular regular dependence of limit sets on given points should be expected to be established if motions of those points are regular. Since regularity properties of motions "near infinity" are those of stability type (continuity is here not quite adequate) and, on the other hand, a very natural regularity property for set-valued mappings is semi-continuity, it seems to be obvious to ask about connections between stability-like conditions assumed for motions and regular (semi-continuous) dependence of limit sets on stable-like moving points. Such a problem and some modification of it, are discussed in the present paper.

We limit ourselves to semi-systems generalizing the classical dynamical systems and dynamical semi-systems on metric spaces, but some results can easily be generalized and extended for Hausdorff topological spaces satisfying the first axiom of countability. Obvious modifications could be made in order to get similar results for systems (see Definition 1 below).

For a rich bibliography concerning stability of motions in various versions and limit sets we refer to (for example) [1]–[3] or [8].

We will use the classical form of logic notation (especially for implications). The sets of real numbers, real non-negative numbers and positive integers will be noted by $\boldsymbol{R}$, $\boldsymbol{R}_*$ and $\boldsymbol{N}$, respectively. The first two of them will be considered usually together with their natural topological group or semi-group structures. The sign "$:=$" stands for "equal by definition". The notation used for motions, trajectories and limit sets is generally in accordance with the classical one (see, for instance, [1]–[3], [8]). If (Y, ϱ) is a metric space, $y \in Y$ and $r > 0$ are given then by $B(y, r)$ we shall denote the open ball centered at the point y with the radius r.

1. General terminology, fundamental definitions, notation. We shall discuss dynamical systems and semi-systems and some generalizations of them. Let us recall fundamental definitions which will be given below together with some comments concerning the terminology simplifying slightly that used by the author in other papers.

Let Y be a non-empty set, $(G, +)$ an abelian semi-group with a neutral element 0, and let π be a mapping from the cartesian product $G \times Y$ into Y.

DEFINITION 1. The triple $(Y, G; \pi)$ is said to be a *semi-system* if and only if

$$\pi(0, y) = y \quad \text{for every } y \in Y, \tag{I}$$

$$\pi\big(t, \pi(s, y)\big) = \pi(t+s, y) \quad \text{for } t, s \in G,\ y \in Y. \tag{II}$$

A semi-system $(Y, G; \pi)$ is said to be a *dynamical semi-system* if and only if Y is a topological space, $(G, +)$ is a topological semi-group and

$$\pi \text{ is continuous.} \tag{III}$$

If $(G, +)$ is a group then any semi-system $(Y, G;\ \pi)$ is called a ***system***. Such a system which is simultaneously a dynamical semi-system is called a *dynamical system*.

Remark 1. The above terminology coincides fully with the general one used by many authors (compare, for instance, [1]–[3], [13]) in the case of dynamical systems (called also often *continuous flows*). Note that the classical theory is developed in particular with respect to systems of the type $(Y, \boldsymbol{R};\ \pi)$. In the case of semi-groups (in particular, if $G = \boldsymbol{R}_*$) some authors use other terminology slightly different from the above one. In particular, dynamical semi-systems in the meaning of our Definition 1 are called in [1] semi-dynamical systems. In the author's monographs [8]–[9] there was used some "richer" terminology: semi-systems (systems) were called there pseudo-dynamical semi-systems (respectively: pseudo-dynamical systems) in order to underline that they are generalizations of classical dynamical systems. In papers [10]–[12] there were considered only semi-systems in the sense of Definition 1 and they were called pseudo-dynamical systems. Here we propose for simplicity short names: systems and semi-systems, keeping the traditional names: dynamical systems and — consequently — dynamical semi-systems for regular (continuous) ones.

If $(Y, G;\ \pi)$ is a semi-systems then there are two families of mappings $\{\pi_t\}_{t\in G}$ and $\{\pi^y\}_{y\in Y}$ given by

$$\pi_t:\ Y \ni y \mapsto \pi_t(y) := \pi(t, y) \in Y \tag{1.1}$$

(for every fixed $t \in G$) and

$$\pi^y:\ G \ni t \mapsto \pi^y(t) := \pi(t, y) \in Y \tag{1.2}$$

(for every fixed $y \in Y$).

Let us recall the classical

DEFINITION 2. Let $y \in Y$ be given. The mapping (1.2) is called the *motion of the point* y or the *motion through the point* y.

Recall also the classical notation:

$$\pi(y) := \{\pi(t, y):\ t \in G\} \tag{1.3}$$

and if $G = \boldsymbol{R}$

$$\pi_+(y) := \{\pi(t, y):\ t \in \boldsymbol{R}_*\}, \tag{1.4}$$

$$\pi_-(y) := \{\pi(t, y):\ t \in \boldsymbol{R},\ t \leqslant 0\}. \tag{1.5}$$

The sets (1.3)–(1.5) are called : the *trajectory of* y, *positive semi-trajectory of* y and the *negative semi-trajectory of* y, respectively. If $G = \boldsymbol{R}_*$ and so we have a semi-system $(Y, \boldsymbol{R}_*;\ \pi)$, then the trajectory $\pi(y)$ coincides clearly with (1.4).

2. Limit sets and generalized semi-stability of motions in semi-systems on metric spaces. Let us take under consideration a semi-system $(X, \boldsymbol{R}_*; \pi)$ where (X, ϱ) is a metric space. This semi-system we shall consider as fixed throughout this paper. Some particular suitable assumptions concerning the space X and the mapping π will be introduced and added in the sequel if it will be necessary in certain further particular problems. We shall admit the usual definition of limit sets (compare, for instance, [2], [3] or [8], [9]) with respect to semi-systems in metric spaces, namely:

DEFINITION 3. Let $x \in X$ be given. The set

$$\Lambda(x) := \{y \in X\colon \text{there is a sequence } \{t_m\} \text{ of elements of } \boldsymbol{R}_* \text{ such that } t_m \to \infty \text{ and } y = \lim \pi(t_m, x)\} \tag{2.1}$$

is the *limit set for* x.

Remark 2. In the general theory of semi-systems of the type $(Y, \boldsymbol{R}_*; \pi)$ (where Y is a topological space) by a limit set for x we mean the set

$$\bigcap \left\{\overline{\pi_t(\pi(x))}\colon t \in \boldsymbol{R}_*\right\}. \tag{2.2}$$

In systems $(Y, \boldsymbol{R}; \pi)$ there are considered positive and negative limit sets given by formulae

$$\Lambda^+(x) := \bigcap \left\{\overline{\pi_t(\pi_+(x))}\colon t \in \boldsymbol{R}\right\} \tag{2.3}$$

and

$$\Lambda^-(x) := \bigcap \left\{\overline{\pi_t(\pi_-(x))}\colon t \in \boldsymbol{R}\right\}, \tag{2.4}$$

respectively.

The set (2.2) is equal to (2.1) if $Y = X$ is a metric space provided with the natural topology induced by the metric (see, for instance, [2], [3]). More general statement says that the sets (2.1) and (2.2) (with X replaced by Y) are equal if the topology on Y is Hausdorff and satisfies the first countability axiom (for details see, for instance, [8]). It is obvious that in a system $(Y, \boldsymbol{R}; \pi)$ with a metric space (Y, r) we have

$$\Lambda^+(x) = \Lambda(x) \tag{2.5}$$

when $\Lambda(x)$ is given by (2.1) (with X replaced by Y) and

$$\Lambda^-(x) = \{y \in Y\colon \text{there is a sequence } \{t_m\} \subset \boldsymbol{R} \text{ such that } t_m \to -\infty \text{ and } y = \lim \pi(t_m, x)\}. \tag{2.6}$$

From the above observations it directly follows that results obtained for limit sets in semi-systems can be modified in such a natural way that parallel theorems for positive limit sets in systems will be obtainable by that modifications.

Remark 3. Directly from observations mentioned above in Remark 2 it follows that for every $x \in X$ the set $\Lambda(x)$ defined by (2.1) is closed. In order to make our paper reasonably self-contained we give, however, a short outline of a known direct proof of the closedness of $\Lambda(x)$ (compare, for instance, the method presented in [2], [3]).

Let $y_m \in \Lambda(x)$ for $m = 1, 2, \ldots$ and let $y = \lim y_m$. For every $m \in \boldsymbol{N}$ there is a sequence $\{t_k^m\}$ of elements of $\boldsymbol{R}_*$ such that

$$t_k^m \to \infty \quad \text{and} \quad \pi(t_k^m, x) \to y \quad \text{as} \quad k \to \infty \quad (\text{for every } m).$$

For each m there is k_m such that

$$\varrho\big(\pi(t_k^m, x), y_m\big) < 1/m;$$

without loss of generality we can assume that

$$t_{k_m}^m \to \infty \quad \text{as} \quad m \to \infty.$$

We have

$$\varrho\big(\pi(t_{k_m}^m, x), y\big) \leqslant \varrho(y, y_m) + \varrho\big(y_m, \pi(t_{k_m}^m, x)\big) \leqslant \varrho(y, y_m) + 1/m$$

which gives

$$\pi(t_{k_m}^m, x) \to y \quad \text{as} \quad m \to \infty$$

and proves that $y \in \Lambda(x)$.

The technics applied above (with a slight modification in some details) will be used below in the proof of Theorem 1.

We shall discuss some stability-like properties of motions, introducing first of all the following

DEFINITION 4. Let a be a non-negative real number and let $x \in X$ be given. The motion π^x is said to be *a-semi-stable* if and only if

(2.7) for every $\varepsilon > 0$ there exist $\delta > 0$ and $u \in \boldsymbol{R}_*$ such that
$$\varrho\big(\pi(t+u, x), \pi(t+u, y)\big) < a + \varepsilon \text{ for every } y \in B(x, \delta), t \geqslant 0.$$

The motion π^x which is 0-semi-stable is called *semi-stable*.

Remark 4. Let $a \geqslant 0$ be given. The following three conditions are equivalent:

(A) *π^x is a-semi-stable;*

(B) *for every sequence $\{u_n\}$ of elements of $\boldsymbol{R}_*$ such that $u_n \to \infty$ and for every $\varepsilon > 0$ there exist $\delta > 0$ and $k \in \boldsymbol{N}$ such that*
$$\varrho\big(\pi(u_n, y), \pi(u_n, x)\big) < a + \varepsilon \quad \textit{for every } y \in B(x, \delta) \textit{ and } n \geqslant k,$$

(C) *if $\{s_n\}$ is a sequence of elements of $\boldsymbol{R}_*$, $s_n \to \infty$ and $\{x_n\}$ is a sequence of elements of X such that $x_n \to x$, then*

$$\limsup_{n\to\infty} \varrho\big(\pi(s, x_n)\ \ , \pi(s_n, x)\big) \leqslant a.$$

Proof. I: (A) $\Rightarrow$ (B). Let $\{u_n\}$ be such that $u_n \to \infty$ and let $\varepsilon > 0$ be given. There are $\delta > 0$ and $u \in \boldsymbol{R}_*$ such that

$$\varrho\big(\pi(t+u, x), \pi(t+u, y)\big) < a+\varepsilon \quad \text{for } y \in B(x, \delta) \text{ and } t \geqslant 0$$

and then there is $k \in \boldsymbol{N}$ such that

$$\varrho\big(\pi(u_n, x), \pi(u_n, y)\big) < a+\varepsilon \quad \text{for } y \in B(x, \delta) \text{ and } n \geqslant k$$

(it is enough to take k such that $u_n \geqslant u$ for $n \geqslant k$).

II: (B) $\Rightarrow$ (C). Let $\{s_n\}$ and $\{x_n\}$ be such that $s_n \to \infty$, $x_n \to x$ and let $\varepsilon > 0$ be given. We can find $\delta > 0$ and $k \in \boldsymbol{N}$ such that

$$[y \in B(x, \delta), n \geqslant k] \Rightarrow \varrho\big(\pi(s_n, x), \pi(s_n, y)\big) < a+\varepsilon.$$

Take m so large that $m \geqslant k$ and $\varrho(x, x_n) < \delta$ for $n \geqslant m$; so $x_n \in B(x, \delta)$ and then $\varrho\big(\pi(s_n, x_n), \pi(s_n, x)\big) < a+\varepsilon$ for $n \geqslant m$. Thus

$$\limsup \varrho\big(\pi(s_n, x_n), \pi(s_n, x)\big) \leqslant a+\varepsilon.$$

This is true for every $\varepsilon > 0$ and then finally

$$\limsup \varrho\big(\pi(s_n, x_n), \pi(s_n, x)\big) \leqslant a.$$

III: (C) $\Rightarrow$ (A). Let (C) be satisfied. Suppose that (A) does not hold true. So there is $\varepsilon^* > 0$ such that for every $\delta > 0$ and every $u \in \boldsymbol{R}_*$ there exist $y \in B(x, \delta)$ and $s \geqslant 0$ such that

$$\varrho\big(\pi(s+u, x), \pi(s+u, y)\big) \geqslant a+\varepsilon^*.$$

Hence there exist sequences $\{x_n\}$ of points of X and $\{s_n\}$ of elements of $\boldsymbol{R}_*$ such that $x_n \to x$, $s_n \to \infty$ and

$$\varrho\big(\pi(s_n, x_n), \pi(s_n, x)\big) \geqslant a+\varepsilon^* \quad \text{for } n = 1, 2, \ldots \tag{2.8}$$

(it is enough to consider $\delta = 1, \frac{1}{2}, \frac{1}{3}, \ldots, u = 1, 2, \ldots$).

The last condition (2.8) contradicts (C). Thus (A) must be satisfied. The proof is completed.

Remark 5. The a-semi-stability defined above is a special case of the general (β, ξ, E)-semi-stability of motions considered in [12]. If u can be taken as equal to zero, so if the following condition

(2.7') for every $\varepsilon > 0$ there exists $\delta > 0$ such that
$\varrho\big(\pi(t, y), \pi(t, x)\big) < a+\varepsilon$ for every $y \in B(x, \delta)$ and each $t \geqslant 0$

is satisfied, then we get a special case of so-called (β, ξ, E)-stability of motions considered in [10]. If $a = 0$, then condition (2.7') coincides with the usual stability of π^x (see, for instance, [2], [3], [8], [9]).

3. Some regularity conditions for set-valued mappings and certain compactness conditions for the space. Recall that a mapping F defined in X and ranged in the family $\mathrm{CL}(X)$ of all closed subsets of X is said to be *upper semi-continuous* at a given point $x \in X$ if and only if

$$\begin{Bmatrix} x_n \in X, \ y_n \in F(x_n) \\ x_n \to x, \ y_n \to y \end{Bmatrix} \Rightarrow y \in F(x). \tag{3.1}$$

It is clear that applying the usual notation

$$d(z, A) := \inf\{\varrho(z, w) \colon w \in A\} \tag{3.2}$$

for $z \in X$ and $A \subset X$, $A = \bar{A}$ and admitting the convention

$$d(z, \emptyset) := \infty, \tag{3.3}$$

we can replace in (3.1) the relation

$$y \in F(x)$$

by the condition

$$d\big(y, F(x)\big) = 0.$$

This suggests the following natural generalization of upper semi-continuity of set-valued mappings:

DEFINITION 5. Let F be a mapping from X into $\mathrm{CL}(X)$, let a be a non-negative real number and let $x \in X$ be given. The mapping F is said to be *a-upper semi-continuous* at the point x if and only if

$$\begin{Bmatrix} x_n \in X, \ y_n \in F(x_n) \\ y_n \to y, \ x_n \to x \end{Bmatrix} \Rightarrow d\big(y, F(x)\big) \leqslant a. \tag{3.4}$$

Remark 6. Usually it is supposed that F about which we are talking above takes values being non-empty closed subsets of X. We do not assume it however, admitting formally also the case $F(z) = \emptyset$ for some $z \in X$. Hence in particular it is possible that $F(z) = \emptyset$ for $z \in B(x, \delta)$ with some $\delta > 0$; in such a case F is (trivially) a-upper semi-continuous at x for every $a \geqslant 0$ (so we have also the trivial upper semi-continuity corresponding to the case $a = 0$). This is true in the both cases: $F(x) = \emptyset$ and $F(x) \neq \emptyset$. In the present paper we discuss F being the mapping $x \mapsto \Lambda(x)$ which can take in general also empty values. In some cases, however, it will be excluded by some additional assumptions. Moreover, we shall observe that semi-stability of motions will imply (under suitable additional assumptions on the space X) that in a neighbourhood of such a point x for which $\Lambda(x) \neq \emptyset$ all points have the same property: limit sets are non-empty (see Sec. 5 below).

We will use in the sequel some conditions assumed with respect to the space X which are weaker than compactness but stronger than local compactness. We shall introduce them formally giving the following

DEFINITION 6. Let $a \geqslant 0$ be given. We say that X satisfies at a point y the *condition* Comp(a) if and only if there is a positive number β such that the ball $B(y, a+\beta)$ is relatively compact.

If $a > 0$, then X satisfies the *condition* Comp$[a]$ at the point y if and only if the ball $B(y, a)$ is relatively compact

Remark 7. It is clear that every compact space satisfies the conditions Comp(a) and Comp$[a]$ for all $a \geqslant 0$. Local compactness is equivalent to the existence of a function $x \mapsto a(x)$ such that the space satisfies at every point x the condition Comp$\big(a(x)\big)$. We will need in some particular problems the condition Comp(a) (or Comp$[a]$) for certain fixed (and given previously) a; this cannot be assured by local compactness. In particular, the assumption that X satisfies at every point y the condition Comp(a) (a is the same for all y) is essentially stronger than local compactness.

4. Regularity of the mapping $x \mapsto \Lambda(x)$. We shall discuss now some connections between semi-stability conditions for motions and semi-continuity properties of the mapping

$$X \ni y \mapsto \Lambda(y) \in \mathrm{CL}(X). \tag{4.1}$$

THEOREM 1. *Let $a \geqslant 0$ and $x \in X$ be given. Suppose that the motion π^x is a-semi-stable. Assume that $\{x_n\}$ and $\{y_n\}$ are sequences of elements of X such that*

$$x_n \to x \quad as \quad n \to \infty, \tag{4.2}$$

$$y_n \to y \quad as \quad n \to \infty, \tag{4.3}$$

$$y_n \in \Lambda(x_n) \quad for\ every\ n. \tag{4.4}$$

Then there exists a sequence $\{s_n\}$ of elements of $\boldsymbol{R}_*$ such that

$$s_n \to \infty \quad as \quad n \to \infty \tag{4.5}$$

and

$$\limsup_{n\to\infty} \varrho\big(\pi(s_n, x), y\big) \leqslant a. \tag{4.6}$$

Proof. Relations (4.4) mean that for every $m \in \boldsymbol{N}$ there is a sequence $\{t_k^m\}_{k=1,2,\ldots}$ of elements of $\boldsymbol{R}_*$ such that

$$t_k^m \to \infty \quad \text{as} \quad k \to \infty \quad (m = 1, 2, \ldots) \tag{4.7}$$

and

$$\pi(t_k^m, x) \to y_m \quad \text{as} \quad k \to \infty \quad (m = 1, 2, \ldots). \tag{4.8}$$

We have (see (4.2))

$$\varrho(x_m, x) \to 0 \quad \text{as} \quad m \to \infty \tag{4.9}$$

and (see (4.8))

$$\varrho\big(\pi(t_k^m, x_m), y\big) \to 0 \quad \text{as} \quad k \to \infty \quad (m = 1, 2, \ldots). \tag{4.10}$$

On the other hand, we get by (4.3)

(4.11) $$\varrho(y_m, y) \to 0 \quad \text{as} \quad m \to \infty.$$

By virtue of (4.7) we can find a strictly increasing sequence of positive integers $\{k_m\}$ such that

(4.12) $$m \leqslant t^m_{k_m} \quad \text{for every } m.$$

From (4.10) and (4.12) it follows that for every $m \in \boldsymbol{N}$ there is $p(m) \in \boldsymbol{N}$ such that

(4.13) $$\varrho\big(\pi(t^m_{k_{p(m)}}, x_m), y_m\big) < 1/m \quad (m = 1, 2, \ldots).$$

Without loss of generality we can assume that

(4.14) $$m \leqslant p(m) < p(m+1) \quad (m = 1, 2, \ldots).$$

This gives in particular

(4.15) $$p(m) \to \infty \quad \text{as} \quad m \to \infty.$$

Now put

(4.16) $$s_m := t^m_{k_{p(m)}}, \quad m = 1, 2, \ldots$$

We have obviously

(4.17) $$s_m \to \infty \quad \text{as} \quad m \to \infty$$

and

$$\varrho\big(\pi(s_m, x_m), y_m\big) < 1/m \quad \text{for every } m,$$

which gives

(4.18) $$\varrho\big(\pi(s_m, x_m), y_m\big) \to 0 \quad \text{as} \quad m \to \infty.$$

Using the triangle inequality we get

(4.19) $$\varrho\big(\pi(s_m, x), y\big) \leqslant \varrho(y_m, y) + \varrho\big(y_m, \pi(s_m, x_m)\big) + \\ + \varrho\big(\pi(s_m, x_m), \pi(s_m, x)\big)$$

for every m.

By virtue of (4.11), (4.18) and a-semi-stability of π^x (compare (C) in Remark 4) we obtain from (4.19) our assertion

$$\limsup \varrho\big(\pi(s_n, x), y\big) \leqslant a.$$

The proof is completed.

THEOREM 2. *If π^x is semi-stable then the mapping* (4.1) *is upper semi-continuous at the point x.*

Proof. Let $\{x_n\}$ and $\{y_n\}$ be sequences of elements of X such that conditions (4.2)–(4.4) are satisfied. By virtue of Theorem 1 we can find a sequence $\{s_n\}$ of real non-negative numbers such that $s_n \to \infty$ and

(4.20) $$\limsup_{n\to\infty} \varrho\big(\pi(s_n, x), y\big) \leqslant 0$$

(compare Definition 4; semi-stability means 0-semi-stability).

Inequality (4.20) means however that

$$\lim \varrho\big(\pi(s_n, x), y\big) = 0$$

and then $y \in \Lambda(x)$.

This completes the proof.

COROLLARY. *If π^x stable then the mapping* (4.1) *is upper semi-continuous at x.*

Remark 8. The result established by the Corollary has been presented previously in [7], [9]. Similar results with respect to generalized pseudo-dynamical systems were obtained independently by J. Kłapyta [4], [5] (compare remarks in Introduction).

THEOREM 3. *Suppose that X satisfies at every point the condition* Comp(a). *If π^x is a-semi-stable then the mapping* (4.1) *is a-upper semi-continuous at the point x.*

Proof. Suppose that $\{x_n\}$ and $\{y_n\}$ are such that (4.2)–(4.4) hold true. Let $\eta > 0$ be such that the ball $B(y, a+\eta)$ is relatively compact; such η exists because of the condition Comp(a). Let $\{s_n\}$ be a sequence of real non-negative numbers such that $s_n \to \infty$ and (4.6) holds true. There is $k \in \boldsymbol{N}$ such that

$$\varrho\big(\pi(s_n, x), y\big) < a+\eta \quad \text{for } n > k.$$

Since $\overline{B(y, a+\eta)}$ is compact, we can assume without loss of generality that the sequence $\{\pi(s_n, x)\}$ is convergent to some y^*. Because of (4.6) we get

$$\varrho(y, y^*) \leqslant a. \tag{4.21}$$

On the other hand, we have

$$y^* \in \Lambda(x). \tag{4.22}$$

The relations (4.21) and (4.22) give directly

$$d\big(y, \Lambda(x)\big) \leqslant a$$

which finishes the proof.

COROLLARY. *If X is compact and π^x is a-semi-stable, then the mapping* (4.1) *is a-upper semi-continuous at x.*

THEOREM 4. *Suppose that X satisfies at every point the condition* Comp$[a]$. *If there is $\beta < a$ such that π^x is β-semi-stable then the mapping* (4.1) *is a-upper semi-continuous at x.*

Proof. Let $\{x_n\}$ and $\{y_n\}$ be such that (4.2)–(4.4) hold true and let $\{s_n\}$ be such that (4.5) is satisfied and

$$\limsup \varrho\big(\pi(s_n, x), y\big) \leqslant \beta. \tag{4.23}$$

Take $\eta = \frac{1}{2}(a-\beta)$ and find $k \in \boldsymbol{N}$ such that

$$\varrho\big(\pi(s_n, x), y\big) < \beta+\eta \quad \text{for } n \geqslant k. \tag{4.24}$$

From (4.24) it follows that

$$\pi(s_n, x) \in \overline{B(y, a)} \quad \text{for } n \geqslant k.$$

So, without loss of generality we can assume that the sequence $\{\pi(s_n, x)\}$ is convergent to some y^* which must belong necessarily to the limit set $\Lambda(x)$. So $d(y, \Lambda(x)) \leqslant a$ (see (4.23)) which finishes the proof.

5. Weak a-semi-stability of motions and non-emptiness of some limit sets.

DEFINITION 7. Let $a \geqslant 0$ and $x \in X$ be given. The motion π^x is said to be *weak a-semi-stable* if and only if

(5.1) *for every $\varepsilon > 0$ there is $\delta > 0$ such that for every $z \in B(x, \delta)$ there is $u \in \boldsymbol{R}_*$ for which*

$$\varrho\big(\pi(t+u, x), \pi(t+u, z)\big) < a+\varepsilon \quad \textit{for } t \geqslant 0.$$

If π^x is weak 0-semi-stable then it is said to be *weak semi-stable.*

THEOREM 5. *Assume that $x \in X$ is such that $\Lambda(x) \neq \emptyset$ and suppose that X satisfies the condition* Comp(a) *at every point of $\Lambda(x)$.*

If π^x is weak a-semi-stable then there is $r > 0$ such that $\Lambda(z) \neq \emptyset$ for every $z \in B(x, r)$.

Proof. Let us take some fixed $y \in \Lambda(x)$ and $\gamma > 0$ such that the ball $B(y, a+\gamma)$ is relatively compact. Let $\{s_n\}$ be a sequence of non-negative numbers such that $s_n \to \infty$ and $\pi(s_n, x) \to y$ as $n \to \infty$. We have

$$\varrho\big(\pi(s_n, x), y\big) < a+\tfrac{1}{2}\gamma \tag{5.2}$$

for n sufficiently large.

On the other hand, we can find $r \geqslant 0$ such that for every $z \in B(x, r)$ there exists $u \in \boldsymbol{R}_*$ for which

$$\varrho\big(\pi(t+u, x), \pi(t+u, z)\big) < a+\tfrac{1}{2}\gamma \quad \text{for } t \geqslant 0.$$

So for $z \in B(x, r)$ and n sufficiently large we have

$$\varrho\big(\pi(s_n, x), \pi(s_n, z)\big) < a+\tfrac{1}{2}\gamma. \tag{5.3}$$

From (5.2) and (5.3) we get for n sufficiently large:

$$\pi(s_n, z) \in B(y, a+\gamma) \quad \text{for } z \in B(x, r)$$

which permits us to assume that $\{\pi(s_n, z)\}$ is convergent (since $B(y, a+\gamma)$ is relatively compact). The limit belongs to $\Lambda(z)$; so $\Lambda(z) \neq \emptyset$ and the proof is completed.

COROLLARY. *If π^x is a-semi-stable, $\Lambda(x) \neq \emptyset$ and X satisfies* Comp(a) *at every point of $\Lambda(x)$, then there is $r > 0$ such that $\Lambda(z) \neq \emptyset$ for every $z \in B(x, r)$.*

In order to justify this statement it is enough to observe that a-semi-stability implies clearly the weak a-semi-stability.

THEOREM 6. *Assume that X is locally compact and suppose that $x \in X$ is such that $\Lambda(x) \neq \emptyset$. If π^x is weak semi-stable then there is $r > 0$ such that $\Lambda(z) \neq \emptyset$ for $z \in B(x, r)$.*

In order to prove this theorem we apply the same method as that used in the proof of Theorem 5; details concerning some little obvious modification will be omitted.

COROLLARY. *If X is locally compact, π^x is semi-stable and $\Lambda(x) \neq \emptyset$, then there exists $r > 0$ such that $\Lambda(z) \neq \emptyset$ for $z \in B(x, r)$.*

6. Regularity of the mapping $x \mapsto \Lambda(x)$ in dynamical semi-systems. In the present section we shall make some observations about the regularity of the mapping (4.1) if π is continuous. First of all we shall state and prove two lemmas.

LEMMA 1. *Let $x \in X$ be given. Suppose that*
(a) *for every $t \in \boldsymbol{R}_*$ the mapping π_t given by (1.1) is continuous at the point x;*
(b) *for every $\varepsilon > 0$ there exist $\delta > 0$ and $u \in \boldsymbol{R}_*$ such that if $z \in B(x, \delta)$, then*

$$d\big(\pi(t, z), \pi(x)\big) < \varepsilon \quad \text{for every } t \geqslant u. \tag{6.1}$$

If $\{x_n\}$ and $\{y_n\}$ are sequences of elements of X such that

$$x_n \to x, \tag{6.2}$$

$$y_n \to y, \tag{6.3}$$

$$y_n \in \Lambda(x_n), \tag{6.4}$$

then

$$y \in \overline{\pi(x)}.$$

Proof. There is a family $\big\{\{t_k^m\}_{k=1,2,\ldots}\big\}_{m=1,2,\ldots}$ of sequences of real non-negative numbers such that

$$t_k^m \to \infty \quad \text{as} \quad k \to \infty \quad \text{(for every } m)$$

and

$$\pi(t_k^m, x_m) \to y_m \quad \text{as} \quad k \to \infty \quad \text{(for every } m).$$

Using the method presented in the proof of Theorem 1, we construct a sequence $\{s_n\}$ such that

$$s_n \to \infty \tag{6.5}$$

and

$$\varrho\big(\pi(s_n, x_n), y_n\big) \to 0 \quad \text{as} \quad n \to \infty \tag{6.6}$$

(compare (4.16)–(4.18)). By virtue of the fact that

$$\varrho(y, y_n) \to 0$$

we get from (6.6)

$$\varrho\big(\pi(s_n, x_n), y\big) \to 0. \tag{6.7}$$

Let $\varepsilon > 0$ be arbitrary fixed and let $\delta > 0$ and $u \in \boldsymbol{R}_*$ be chosen in such a way that if $z \in B(x, \delta)$, then

$$d\big(\pi(t, z), \pi(x)\big) < \tfrac{1}{2}\varepsilon \quad \text{for } t \geqslant u \tag{6.8}$$

(see condition (b)). Let now $n^* \in \boldsymbol{N}$ be such that

$$\varrho(x_n, x) < \delta \quad \text{for } n > n^*, \tag{6.9}$$

$$u < s_n \quad \text{for } n > n^* \tag{6.10}$$

and

$$\varrho\big(\pi(s_n, x_n), y\big) < \tfrac{1}{2}\varepsilon \quad \text{for } n > n^*. \tag{6.11}$$

The relations (6.9) and (6.10) give (compare (6.8))

$$d\big(\pi(s_n, x_n), \pi(x)\big) < \tfrac{1}{2}\varepsilon \quad \text{for } n > n^* \tag{6.12}$$

and then there exists $w \in \pi(x)$ such that

$$\varrho\big(\pi(s_{n^*+1}, x_{n^*+1}), w\big) < \tfrac{1}{2}\varepsilon. \tag{6.13}$$

From (6.11) and (6.13) we get

$$\varrho(y, w) < \varepsilon$$

which gives

$$B(y, \varepsilon) \cap \pi(x) \neq \emptyset. \tag{6.14}$$

Since (6.14) holds true for every $\varepsilon > 0$, we have our assertion

$$y \in \overline{\pi(x)}.$$

LEMMA 2. *Let $x \in X$ be such that π^x is continuous. If $\{x_n\}$ and $\{y_n\}$ satisfy* (6.2), (6.3) *and*

$$y_n \in \pi(x_n), \tag{6.15}$$

then

$$y \in \overline{\pi(x)}.$$

Proof. For every n there is t_n such that $y_n = \pi(t_n, x)$. There are two possible cases: (1) $\{t_n\}$ contains a subsequence $\{t_{p_n}\}$ convergent to some $t^* \in \boldsymbol{R}_*$, (2) $\{t_n\}$ contains a subsequence $\{t_{q_n}\}$ tending to infinity. In the first case we have $y = \lim \pi(t_{p_n}, x) = \pi(t^*, x) \in \pi(x)$, while in the second one we obtain $y = \lim \pi(t_{q_n}, x) \in \Lambda(x)$. This means that everywhen

$$y \in \pi(x) \cup \Lambda(x).$$

It is enough to apply now a well-known formula

$$\overline{\pi(x)} = \pi(x) \cup \Lambda(x)$$

which is true under our assumptions (see, for instance, [2], [3] or [8], [9]), from which we get immediately our assertion.

Remark 8. In [11] there is introduced the so-called semi-stability of sets; the general definition restricted to our case will have the form:

A set M is *semi-stable* if and only if

(6.16) for every $y \in M$ and every $\varepsilon > 0$ there are $\delta > 0$ and $u \in \boldsymbol{R}_*$ such that if $z \in B(y, \delta)$ then $d\big(\pi(t, z), M\big) < \varepsilon$ for $t \geqslant u$.

It is obvious that condition (b) considered above in Lemma 1 is satisfied if the trajectory $\pi(x)$ is semi-stable in the sense of definition (6.16). The implication (b) $\Rightarrow$ (6.16) is not true in general (even if (a) is satisfied). We have however the following easy

PROPOSITION 0. *If for every t the mapping π_t is open then for $M = \pi(x)$:* (b) $\Rightarrow$ (6.16).

COROLLARY. *If $(X, \boldsymbol{R};\ \pi)$ is a system such that for every t the mapping π_t is continuous then:*

$(*)$ $\{$(b) *is fulfilled for $\pi(x)$ replaced by $\pi_+(x)$*$\}$

$$\Rightarrow \{(6.16) \textit{ is fulfilled for } M = \pi_+(x)\}.$$

In particular, $(*)$ holds true if $(X, \boldsymbol{R};\ \pi)$ is a dynamical system.

THEOREM 7. *Assume that X is locally compact and $(X, \boldsymbol{R}_*;\ \pi)$ is a dynamical semi-system. Let $x \in X$ be such that condition* (b) *from the assumptions of Lemma* 1 *holds true and moreover*

(c) *for every $\varepsilon > 0$ there is $\delta > 0$ such that for every $z \in B(x, \delta)$ and every $t \in \boldsymbol{R}_*$ there is $u \in \boldsymbol{R}_*$ for which*

$$\varrho\big(\pi(t+u, x), \pi(t+u, z)\big) < \varepsilon.$$

If $\{x_n\}$ is a sequence of elements of X such that

$$x_n \to x, \tag{i}$$

$$\Lambda(x_n) = \{y_n\} \quad \textit{for every } n, \tag{ii}$$

where

$$y_n \to y, \tag{iii}$$

then

$$y \in \Lambda(x). \tag{iv}$$

Proof. By virtue of Lemma 1 we get

$$y \in \overline{\pi(x)}$$

and so

$$y \in \pi(x) \cup \Lambda(x). \tag{6.17}$$

We have to prove that $y \in \Lambda(x)$. Assume the contrary; so

(6.18) $$y \in \pi(x) \setminus \Lambda(x).$$

Since $y \in \pi(x)$, there is $s \in \boldsymbol{R}_*$ such that

(6.19) $$y = \pi(s, x).$$

We shall prove now the following

PROPOSITION 1. *If $\{r_n\}$ is a sequence of non-negative real numbers tending to infinity, then there exist $\beta > 0$ and $n^* \in \boldsymbol{N}$ such that*

$$(r_n, y) \notin \overline{B(y, \beta)} \quad \textit{for } n > n^*.$$

Proof. Suppose the contrary. So for every β there is a subsequence $\{r_{n_k}\}$ of $\{r_n\}$ (the sequence $\{n_k\}$ depends on β) such that

$$\pi(r_{n_k}, y) \in \overline{B(y, \beta)} \quad \text{for every } k.$$

Let us take $\beta = 1, \frac{1}{2}, \frac{1}{3}, \ldots$ For every m we have a subsequence $\{r_{n_{m,k}}\}$ of $\{r_n\}$ such that

$$\pi(r_{n_{m,k}}, y) \in \overline{B\left(y, \frac{1}{m}\right)}, \quad k, m = 1, 2, \ldots$$

Take now $w_1 := r_{n_{1,1}}$ and find w_2 among the members of the sequence $\{r_{n_{2,k}}\}_{k=1,2,\ldots}$ in such a way that $w_1 < w_2$; then find w_3 among the members of the sequence $\{r_{n_{3,k}}\}_{k=1,2,\ldots}$ in such a way that $w_2 < w_3, \ldots$ etc.; having w_{p-1} we can find w_p among the members of the sequence $\{r_{n_{p,k}}\}_{k=1,2,\ldots}$ in such a way that $w_{p-1} < w_p$.

We have

$$\pi(w_n, y) \to y \quad \text{as} \quad n \to \infty$$

and, on the other hand,

$$w_n \to \infty.$$

This means that (see (6.19))

(6.20) $$\pi(w_n + s, x) \to y \quad \text{as} \quad n \to \infty$$

and

(6.21) $$w_n + s \to \infty \quad \text{as} \quad n \to \infty.$$

Relations (6.20) and (6.21) give directly

$$y \in \Lambda(x)$$

which contradicts (6.18). The proof of Proposition 1 is completed.

Now we have to repeat the begining of the proof of Theorem 1; let us recall that

(j) *for every m there is a sequence $\{t_k^m\}_{k=1,2,\ldots}$ such that $t_k^m \to \infty$ as $k \to \infty$ (for every m),*

(jj) *selecting in a suitable way a subsequence from the double sequence $\{t_k^m\}_{k=1,2,\ldots,m=1,2,\ldots}$ we obtain a sequence $\{s_n\}$ such that for $n \to \infty$:*

$$s_n \to \infty, \ \pi(s_n, x_n) \to y, \quad \varrho\big(\pi(s_n, x_n), y_n\big) \to 0.$$

Let $\gamma > 0$ be such that $B(y, \gamma)$ is relatively compact and let $\beta > 0$ and n_1 be such that

$$\pi(s_n, y) \notin \overline{B(y, \beta)} \quad \text{for } n > n_1 \tag{6.22}$$

(with $\{s_n\}$ as in (jj) above). Let us put

$$\eta := \min(\tfrac{1}{2}\beta, \gamma). \tag{6.23}$$

By virtue of assumption (c) we have

(jjj) *there is $\delta > 0$ (chosen for $\varepsilon = \frac{1}{2}\beta$) such that for every $z \in B(x, \delta)$ and $t \in \boldsymbol{R}_*$ there is $u \in \boldsymbol{R}_*$ for which*

$$\varrho\big(\pi(t+u, x), \pi(t+u, z)\big) < \tfrac{1}{2}\beta.$$

We can choose now n_2 such that

$$x_n \in B(x, \delta) \quad \text{for } n > n_2. \tag{6.24}$$

Since $\{y_n\}$ tends to y, there is n_3 such that

$$y_n \in B(y, \eta) \quad \text{for } n > n_3. \tag{6.25}$$

Let us now put

$$n^* := \max(n_1, n_2, n_3) \tag{6.26}$$

and fix some $m = m^*$ greater than n^*.

From (6.22)–(6.26) it follows directly that

$$\pi(s_n, y) \notin B(y, \beta) \quad \text{for } n > n^*, \tag{6.27}$$

$$x_{m^*} \in B(x, \delta), \tag{6.28}$$

$$y_{m^*} \notin B(y, \eta). \tag{6.29}$$

According to (j) we have

$$y_{m^*} = \lim \pi(t_k^{m^*}, x_{m^*}), \quad t_k^{m^*} \to \infty \quad \text{as} \quad k \to \infty,$$

and so there is k^* such that

$$\pi(t_k^{m^*}, x_{m^*}) \in B(y, \eta) \quad \text{for} \quad k > k^* \tag{6.30}$$

(recall that $m^* > n^*$). Let us take now

$$r_1 := t_{k^*+1}^{m^*}, \quad m_1 = n^* + 1 \tag{6.31}$$

and choose q_1 in such a way that

$$\varrho\big(\pi(s_{m_1}+r_1+s+q_1, x_{m^*}), \pi(s_{m_1}+r_1+s+q_1, x)\big) < \tfrac{1}{2}\beta; \tag{6.32}$$

such a q_1 exists because of (c) (see (jjj) above) applied for $\varepsilon = \frac{1}{2}\beta$ and $t = s_{m_1}+r_1+s$.

Let us now take such an integer $k_2 > m_1$ that $t^{m^*}_{k^*+k_2} > q_1$; put

$$r_2 := t^{m^*}_{k^*+k_2} \tag{6.33}$$

and find $m_2 > m_1$ such that

$$s_{m_2} > s_{m_1}+r_1+q_1. \tag{6.34}$$

We can then find q_2 such that $q_2 > q_1$ and

$$\varrho\big(\pi(s_{m_2}+r_2+s+q_2, x_{m^*}), \pi(s_{m_2}+r_2+s+q_2, x)\big) < \tfrac{1}{2}\beta. \tag{6.35}$$

Generally, if r_p, k_p, m_p and q_p are defined, we choose m_{p+1} in such a way that

$$m_{p+1} > m_p \quad \text{and} \quad s_{m_{p+1}} > s_{m_p}+r_p+q_p$$

and k_{p+1} such that

$$k_{p+1} > m_p \quad \text{and} \quad t^{m^*}_{k^*+k_{p+1}} > q_p$$

and define

$$r_{p+1} := t^{m^*}_{k^*+k_{p+1}}$$

and finally choose q_{p+1} such that $q_{p+1} > q_p$ and

$$\varrho\big(\pi(s_{m_{p+1}}+r_{p+1}+s+q_{p+1}, x_{m^*}), \pi(s_{m_{p+1}}+r_{p+1}+s+q_{p+1}, x)\big) < \tfrac{1}{2}\beta.$$

Putting

$$w_p := s_{m_p}+r_p+s+q_p, \tag{6.36}$$

we get a sequence $\{w_p\}$ such that

$$w_p \to \infty \quad \text{as} \quad p \to \infty \tag{6.37}$$

and

$$\varrho\big(\pi(w_p, x_{m^*}), \pi(w_p, x)\big) < \tfrac{1}{2}\beta, \quad p = 1, 2, \ldots \tag{6.38}$$

Relations (6.38) imply

$$\pi(w_p, x_{m^*}) \notin \overline{B(y, \eta)} \quad \text{for} \quad p = 1, 2, \ldots \tag{6.39}$$

On the other hand, putting for simplicity

$$v_p := t^{m^*}_p, \quad p = 1, 2, \ldots, \tag{6.40}$$

we obtain

$$v_p \to \infty \quad \text{as} \quad p \to \infty, \tag{6.41}$$

and

$$(6.42)\qquad \pi(v_p, x_{m^*}) \in \overline{B(y,\eta)}, \quad p = 1, 2, \ldots$$

From (6.37) and (6.41) it follows that we can assume without loss of generality that

$$(6.43)\qquad v_p \leqslant w_p < v_{p+1}, \quad p = 1, 2, \ldots$$

The mapping π is continuous, so because of (6.39), (6.42) and (6.43) we can find for every fixed p a real number c_p such that

$$(6.44)\qquad w_p \leqslant c_p < v_{p+1}$$

and

$$(6.45)\qquad \pi(c_p, x_{m^*}) \in \partial B(y, \eta).$$

It is clear that (compare (6.44)) for p tending to infinity we have

$$(6.46)\qquad c_p \to \infty.$$

Since $\eta \leqslant \gamma$ (compare (6.23)) we have

$$\partial B(y, \eta) \subset \overline{B(y, \gamma)}$$

and then, because of the compactness of $\overline{B(y,\gamma)}$, we can assume that the sequence $\{\pi(c_p, x_{m^*})\}$ is convergent to some $y^* \in \partial B(y, \eta)$. Relation (6.46) implies, however, that $y^* \in \Lambda(x_{m^*})$ which gives (see assumption (ii)) the equality

$$(6.47)\qquad y^* = y_{m^*}.$$

The last equality contradicts (6.29). Thus we have proved that condition (6.18) cannot be satisfied and then $y \in \Lambda(x)$; this completes the proof.

As a simple corollary of Theorem 7 we obtain the following

THEOREM 8. *Assume that X is locally compact, $(X, \boldsymbol{R}_+; \pi)$ is a dynamical semi-system, $x \in X$ is such that conditions* (b) *and* (c) *are satisfied and, moreover,*

(U) *there is a neighbourhood V of x such that for every $z \in V$ the limit set $\Lambda(z)$ has exactly one element.*

Then $\Lambda(x)$ has exactly one element and the mapping

$$(6.48)\qquad V \ni y \mapsto L(y) \in X,$$

where $L(y)$ denotes the unique element of $\Lambda(y)$, is continuous at the point x.

Proof. First we shall prove that the mapping

$$(6.49)\qquad X \ni y \mapsto \Lambda(y) \in \mathrm{CL}(X)$$

is upper semi-continuous at the point x.

Let $\{x_n\}$ and $\{y_n\}$ be such that $x_n \to x$, $y_n \to y$ and $y_n \in \Lambda(x_n)$. For n sufficiently large we have $\Lambda(x_n) = \{y_n\}$ for infinitely many n or $y_n \in \Lambda(x)$

for infinitely many n (the second case appears if $x_n = x$ for infinitely many n). In the first case we apply Theorem 7 for the subsequence $\{y_{n_k}\}$ of those members of $\{y_n\}$ which are single elements of $\Lambda(x_n)$, in the second case we obtain directly $y \in \Lambda(x)$ since y is the limit of a sequence of elements of $\Lambda(x)$. Thus $y \in \Lambda(x)$ in every case.

Now we shall prove that $\Lambda(x)$ has exactly one element. Since $\Lambda(x)$ is non-empty it is enough to show that $\Lambda(x)$ has at most one element. Suppose the contrary; let z, w be two distinct elements of $\Lambda(x)$. Put $a = \varrho(z, w)$ and $\varepsilon^0 = a/3$. Take ε so small that the balls $B(z, \varepsilon)$ and $B(w, \varepsilon)$ are relatively compact. There are two sequences $\{t_n\}$ and $\{s_n\}$ such that $t_n \to \infty$, $s_n \to \infty$ and $\pi(t_n, x) \to z$, while $\pi(s_n, x) \to w$, and so $\varrho(\pi(t_n, x), z) < \varepsilon/2$ and $\varrho(\pi(s_n, x), w) < \varepsilon/2$ for n sufficiently large. On the other hand there is $\delta > 0$ such that for $y \in V \setminus \{x\}$ for which $\varrho(y, x) < \delta$ we have $\varrho(\pi(t, x), \pi(t, y)) < \varepsilon/2$ for every $t \geqslant 0$ and so in particular for every n: $\varrho(\pi(t_n, y), \pi(t_n, x)) < \varepsilon/2$, $\varrho(\pi(s_n, y), \pi(s_n, x)) < \varepsilon/2$. Thus $\varrho(\pi(t_n, y), z) < \varepsilon$ and $\varrho(\pi(s_n, y), w) < \varepsilon$. Since $B(z, \varepsilon)$ and $B(w, \varepsilon)$ are relatively compact we can assume without loss of generality that $\{\pi(t_n, y)\}$ and $\{\pi(s_n, y)\}$ are convergent. The limits of these sequences, say u and v respectively, belong obviously to $\Lambda(y)$ and so $u = v$. This is, however, impossible since $\varrho(u, v) > 0$. The contradiction proves that $\Lambda(x)$ has exactly one element.

In order to finish the proof observe that the upper semi-continuity of a set valued mapping which takes values being one-element sets is equivalent to the usual continuity of the induced single-valued mapping. So the upper semi-continuity of (6.49) implies the continuity of the function (6.48). The proof is completed.

COROLLARY. *If X is locally compact, $(X, \boldsymbol{R}_*; \pi)$ is a dynamical semi-system, $\pi(x)$ is semi-stable, π^x is weakly semi-stable and condition* (U) *is satisfied, then the mapping* (6.49) *is continuous at x.*

Remark 9. Condition (b) does not imply (c), (c) does not imply (b); semi-stability of $\pi(x)$ does not imply weak semi-stability of π^x, weak semi-stability does not imply semi-stability of $\pi(x)$.

As a simple and direct corollary of Lemmas 1 and 2 we get the following

THEOREM 9. *Assume that $(X, \boldsymbol{R}_*; \pi)$ is a dynamical semi-system, $x \in X$ is such that condition* (b) *is fulfilled. Then the mapping*

$$X \ni y \mapsto \overline{\pi(y)} \in \mathrm{CL}(X) \tag{6.50}$$

is upper semi-continuous at x.

Proof. Let $\{x_n\}$ and $\{y_n\}$ be such that $y_n \in \overline{\pi(x_n)}$ for every n and $x_n \to x$, $y_n \to y$. There is a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ such that $y_{n_k} \in \Lambda(x_{n_k})$ or there is a subsequence $\{y_{m_p}\}$ of (y_n) such that $y_{m_p} \in \pi(x_{m_p})$; in the first case we apply Lemma 1, in the second one Lemma 2, getting in the both cases: $y \in \overline{\pi(x)}$, which finishes the proof.

References

[1] N. P. Bhatia and O. Hajek, *Local semi-dynamical systems*, Lecture Notes in Mathematics, 90, Berlin–Heidelberg–New York 1969.

[2] —, G. P. Szegö, *Dynamical systems: Stability theory and applications*, Lecture Notes in Mathematics, 35, Berlin–Heidelberg–New York 1967.

[3] —, —, *Stability theory of dynamical systems*, Berlin–Heidelberg–New York 1970.

[4] J. Kłapyta, *Efekty asymptotyczne w teorii uogólnionych układów pseudo-dynamicznych* (*Asymptotic phenomenons in the theory of generalized pseudo-dynamical systems*, in Polish), Ph. D. Thesis, Jagiellonian University, Kraków 1978 (typescript).

[5] —, *Limit sets in generalized dynamical systems*, to be published.

[6] A. M. Lapunov, *General problem of motion stability* (in Russian), Com. Soc. Math. Kharkov 3 (1893), p. 205–272; see also a book titled as above, Gosehizdat, Moscow 1950.

[7] A. Pelczar, *Stability of motions and limit sets in pseudo-dynamical systems*, A report during the international conference EQUADIFF in Prague, August 1977.

[8] —, *Ogólne układy dynamiczne*, Uniwersytet Jagielloński, skrypty uczelniane, 293 (*General Dynamical Systems*, Monographs of Jagiellonian University, 293; in Polish), Kraków 1978.

[9] —, *Wstęp do teorii układów dynamicznych*, Akademia Górniczo-Hutnicza, skrypty uczelniane, 694 (*Introduction to the theory of dynamical systems*, Academy of Mining and Metallurgy, Monographs, 694, in Polish), Kraków 1980.

[10] —, *Stability of motions in pseudo-dynamical systems*, Bull. Acad. Polon. Sci., Ser. Sci. Math., Astronom. et Phys. 24 (1976), p. 409–418.

[11] —, *Semi-stability in pseudo-dynamical systems*, ibidem, 24 (1976), p. 419–428.

[12] —, *Semi-stability of sets and motions in pseudo-dynamical systems*, ibidem, 25 (1977), p. 633–642.

[13] G. R. Sell, *Topological dynamics and ordinary differential equations*, Van Nostrand Reinhold Comp., London–New York–Cincinnati–Toronto–Melbourne 1971.

Reçu par la Rédaction le 19.01.81