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ON THE QUESTION OF STABILITY IN A HAMILTONIAN SYSTEM

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1. Let a real Hamiltonian system with n degrees of freedom

$$\dot{x}_j = \partial H/\partial y_j, \quad \dot{y}_j = -\partial H/\partial x_j, \quad j = 1, \ldots, n \tag{1}$$

have the Hamiltonian function $H(x, y)$ which is analytic at the point $x = y = 0$ and its Maclaurin series begins from quadratic members. Then zero is a fixpoint of the system (1). Let $\lambda_1, \ldots, \lambda_n, -\lambda_1, \ldots, -\lambda_n$ be eigenvalues of the matrix of the linear part of the system (1). We assume that all $\operatorname{Re}\lambda_j = 0$ and we consider the Liapounov stability of the fixpoint $x = y = 0$ of the system (1).

There exists [1, § 12, Theorem 12] a formal canonical transformation of coordinates $x, y \to w, z$ which reduces the Hamiltonian function to the normal form

$$h = \sum h_{pq} w^p z^q, \quad \langle p-q, \lambda \rangle = 0 \tag{2}$$

containing only reasonant members, for which the scalar product $\langle p-q, \lambda \rangle = 0$. Here we use the following notation: w, z, p, q, λ are n-vectors: $w = (w_1, \ldots, w_n)$, $p = (p_1, \ldots, p_n)$, $\lambda = (\lambda_1, \ldots, \lambda_n)$.

$$w^p = w_1^{p_1} w_2^{p_2} \ldots w_n^{p_n}, \quad \langle p, \lambda \rangle = p_1 \lambda_1 + \ldots + p_n \lambda_n.$$

If the coordinates x, y are real then the complex coordinates w, z satisfy a simple reality relation. As a rule the normalizing transformation $x, y \to w, z$ is given by divergent series. So the stability of the origin in the system with the Hamiltonian function (2) does not guarantee the stability of the origin in the system (1). Nevertheless the normal form (2) is convenient for a formulation of conditions which guarantee the stability of the origin in the system (1). If

$$\langle p, \lambda \rangle \neq 0 \text{ for all integral } p \text{ with } 0 < |p_1| + \ldots + |p_n| \leqslant 4 \tag{3}$$

then the normal form (2) has the form

$$h = \langle \alpha, \varrho \rangle + \langle \varrho, \beta\varrho \rangle + \cdots$$

where all occurring symbols are real: $\lambda_j = i\alpha_j$, $\varrho_j = |w_j z_j|$ is the square of the polar radius; $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $\varrho = (\varrho_1, \ldots, \varrho_n)$ are vectors; $\beta = (\beta_{jk})$ is a symmetric matrix.

If the equation $\langle \alpha, \varrho \rangle = 0$ has no solution $\varrho \geqslant 0$, $\varrho \neq 0$, then the fixpoint $x = y = 0$ is stable in the Liapounov sense. This is well known (see [1]).

Suppose that this condition is not fulfilled. Let us consider the condition:

(4) The system of equations $\langle \alpha, \varrho \rangle = 0$ and $\langle \varrho, \beta\varrho \rangle = 0$ does not have a solution $\varrho \geqslant 0$, $\varrho \neq 0$.

2. Now we shall consider the case $n = 2$. In 1968, Moser [2, Theorem 7] formulated the theorem: Conditions (3) and (4) are sufficient for stability of the origin in the system (1). However, other statements have preceded Moser's theorem. Thus Arnold [3] announced the stability if the number λ_1/λ_2 is irrational, is badly approached by rational numbers and condition (4) is fulfilled. Next, Moser [4] noticed that Arnold's condition for the number λ_1/λ_2 can be weakened up to condition (3). In the statement by Arnold [5] the stability is guaranteed by condition (3) and either condition (4) or the condition

$$\det \beta \neq 0 \tag{5}$$

But it was shown that conditions (3) and (5) are not sufficient for the stability (see [1, § 12, Section IV]). Then Arnold [6] demanded that the three conditions (3), (4), (5) be fulfilled.

Whatever be the exact formulations of the theorems on stability, their original proofs were incorrect because they used the following statement: A fixpoint is stable if invariant tori are in every small neighbourhood of it in each hypersurface $H = \text{const}$. A counterexample to this statement is given in [1, § 12, formulae (37)] and is considered in detail in [7, 8]. Namely, let

$$\begin{aligned} H &= \frac{1}{2} u [1 - u\varrho_1 + 2\mathrm{Re}(x_1 + iy_1)^{q_1} (x_2 + iy_2)^{q_2}], \\ u &= \alpha_1 \varrho_1 + \alpha_2 \varrho_2, \quad \varrho_j = x_j^2 + y_j^2, \quad \alpha_1 q_1 + \alpha_2 q_2 = 0, \end{aligned} \tag{6}$$

where q_j are integers.

THEOREM. *Solutions of the system* (1), (6) *have two properties*:

A. *Invariant tori are in every small neighbourhood of the fixpoint* $x = y = 0$ *in each hypersurface* $H = \text{const}$.

B. *The fixpoint* $x = y = 0$ *is not stable in Liapounov sense.*

The linear transformation

$$x_j - iy_j = w_j\sqrt{i}, \quad x_j + iy_j = z_j\sqrt{i}, \quad j = 1, 2$$

brings the Hamiltonian (6) into the normal form (2) with $h = 2H$. Hence the system (1), (6) has two independent integrals

$$u = \text{const}, \quad 2Hu^{-1} - 1 = \text{const},$$

and it is integrable. Thus, the solutions of the system (1), (6) can be studied by means of elementary analysis. Results can be explained as follows. The phase space consists of four domains S_j, U_j ($j = 1, 2$) such that S_j consist of invariant tori (stable domains), U_j consist of unbounded solutions (unstable domains) and their boundaries consist of bounded solutions. The projection of the phase space x, y onto the first quadrant $\varrho_1, \varrho_2 \geqslant 0$ is shown in Figure 1. Here the regions $P_j \cup Q_j$ are projections of the stable domains S_j, the regions $Q_j \cup R_j$ are projections of the unstable domains U_j. We see that unstable domains U_j approach the origin. Hence it is not stable in Liapounov sense. The level $H = 0$ consists of invariant tori and each level $H = \text{const} \neq 0$ intersects a stable domain S_j (i.e. it contains invariant tori). Projections of the levels $H = c$ and $H = -c$ are shaded in Figure 1.

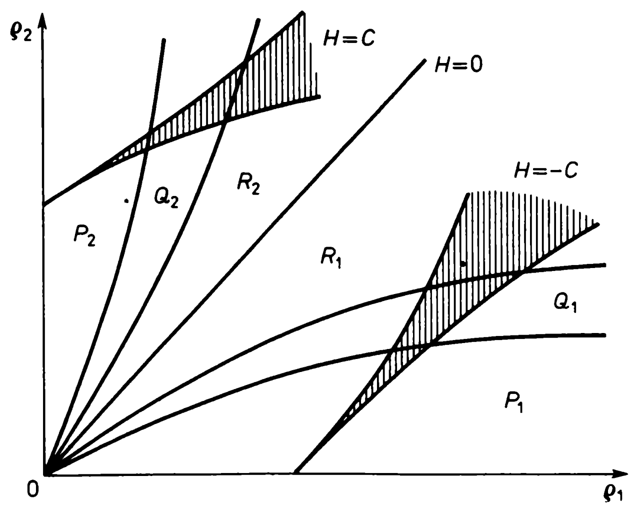

Fig. 1

The first correct proof of the stability of a fixpoint under conditions (3) and (4) was given by Moser [9].

3. Now let us consider the case $n > 2$. For that case, Thai [10] has published a proof of the Liapounov stability of a fixpoint of the system (1) under conditions (3) and (4). However, his proof considers only one case of a possible disposition of integral curves and does not consider other possible cases. For example, in Section 3 in the hypersurface $|H_4| = \gamma^4$, he considers a

curve $S_1(\gamma)$ and its dynamical image $S_2(\gamma)$. He thinks that $S_2(\gamma)$ must be a single curve, but it can be a more complicated set (such as a union of several curves). He also thinks that curves $S_1(\gamma)$ and $S_2(\gamma)$ must have a common end (the point B), but this is an additional restriction.

Note that under conditions (3), (4), there is formal stability [11] and conditions of steepness [12] are fulfilled. Hence if there is an instability it is very slow. Up to the present (1986) there are no examples of Liapounov instability for cases of formal stability.

4. Gavrilov [13] has proved a weak stability (not Liapounov one) of solutions of a perturbed Hamiltonian system with n degress of freedom. But his proof uses the statement: A real continuous function does not change its sign. Namely [7, Section 4], in the second step of the proof of Lemma, he says that in formula (16) the sign before the square root cannot change. If we denote

$$-w = 1+2(H_1-H_1^0)/\sigma$$

then (16) takes the form $w = \pm\sqrt{w^2}$, where $\sqrt{w^2} = |w|$. If real w varies continuously in an interval $(-\alpha, \alpha)$ with $\alpha > 0$, then the sign before the root must change. Arguments with complex branches in p. 226–227 are correct only if the point $w = 0$ is deleted. But here it is not deleted and a continuous passage from one branch to the other is possible.

As a consequence of that mistake, Lemma and Theorem in [13] are not proved. Moreover, they are a base for Theorem [14] on stability of Solar system, which is also not proved. Similar defects are in the first variant [15]. In [16] there is a counterexample to Theorem in [13].

For a general approach see in [17, 18].

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