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A COMMUTATIVITY THEOREM FOR BANACH ALGEBRAS

BY

DONALD Z. SPICER (POUGHKEEPSIE, NEW YORK)

The crucial property of commutative (complex) Banach algebras that allows the development of the Gelfand theory is that every irreducible representation of such an algebra is 1-dimensional (cf. [4], p. 108). We wish to study to what extent this property determines the commutativity of the algebra. Since, in general, a Banach algebra may have very few irreducible representations, one suspects that all such representations could be 1-dimensional, but the algebra still not be commutative. Consider as an example a non-commutative radical Banach algebra B, and let A be the Banach algebra resulting from adjoining an identity to B. Since B is a radical algebra, it is easily seen (cf. [3], Proposition 2, p. 206) that the only primitive ideal of A is B, which is in fact a maximal modular ideal of co-dimension 1. Thus $A \to A/B$ is the only irreducible representation of A, and it is of dimension 1. However, A is clearly not commutative.

Therefore, we would like to investigate under what conditions it is true that every irreducible representation of a Banach algebra is 1--dimensional and what additional conditions are necessary to insure commutativity. We would hope to find some such conditions that are common properties held by all commutative Banach algebras.

An algebraic characterization of the property that every irreducible representation of a Banach algebra A is 1-dimensional is easily determined. Let $\boldsymbol{R}$ be the Jacobson radical of A and $\mathscr{C}$ be the commutator of A (i. e., $\mathscr{C} = \{z\colon z = xy-yx;\ x, y \in A\}$). Then every irreducible representation of A is 1-dimensional if and only if every irreducible representation of $A/\boldsymbol{R}$ is 1-dimensional (cf. [3], p. 205). Since $A/\boldsymbol{R}$ is semi-simple, it is readily seen that if every irreducible representation of $A/\boldsymbol{R}$ is 1-dimensional, then $A/\boldsymbol{R}$ is commutative, i. e., $\mathscr{C} \subseteq \boldsymbol{R}$. Conversely, if $\mathscr{C} \subseteq \boldsymbol{R}$, then $A/\boldsymbol{R}$ is a commutative Banach algebra and, therefore, every irreducible representation of $A/\boldsymbol{R}$ is 1-dimensional. The desired criterion is thus that $\mathscr{C} \subseteq \boldsymbol{R}$. Actually, a somewhat finer statement can be made. If π is any irreducible representation of A, then π is 1-dimensional if and only if $\mathscr{C} \subseteq \ker \pi$. In fact, if $\mathscr{C} \subseteq \ker \pi$, then $\pi(A)$ is a commutative primitive Banach algebra, which must, therefore, be 1-dimensional (cf. [4], p. 61). The converse is obvious.

We have seen above that every irreducible representation of A is 1-dimensional if and only if $A/\boldsymbol{R}$ is commutative. Recently Aupetit [1] studied such algebras and called them *almost commutative Banach algebras.* Aupetit asserts the following topological characterization of almost commutativity.

Letting $\nu(x)$ be the spectral radius of $x \in A$, he says the following are equivalent:

(1) A is almost commutative;

(2) ν is subadditive (i. e., there exists an $a > 0$ such that, for all $x_1, \ldots, x_n \in A$, $\nu(x_1 + \ldots + x_n) \leqslant a(\nu(x_1) + \ldots + \nu(x_n))$);

(3) ν is submultiplicative (i. e., there is a $\beta > 0$ such that, for all x, $y \in A$, $\nu(xy) \leqslant \beta\nu(x)\nu(y)$).

The answer to our second question now follows.

THEOREM 1. *A Banach algebra A is commutative if and only if every irreducible representation of A is 1-dimensional and there exists a constant $k > 0$ such that $\|z\| \leqslant k\nu(z)$ for all $z \in \mathscr{C}$.*

Remark 1. Whereas the condition $\|z\| \leqslant k\nu(z)$ seems artificial, it does have an analogue in a purely algebraic context (see [2], p. 74).

Let R be a ring such that for every $x, y \in R$ there exists an integer $n(x, y) > 0$ such that $(xy - yx) = (xy - yx)^{n(x,y)}$. Then R is commutative.

Remark 2. It follows, by a standard argument, that the condition $\|z\| \leqslant k\nu(z)$ for all $z \in \mathscr{C}$ is satisfied in the event that $\|w^2\| \leqslant k\|w\|^2$ for all $w \in [\mathscr{C}]$, the ideal generated by $\mathscr{C}$.

Remark 3. It should be especially noted that Theorem 1 in conjunction with [1] provides an essentially topological characterization of commutativity for Banach algebras.

Proof of Theorem 1. Sufficiency is all that needs to be proven. The stated conditions imply that $\mathscr{C} \subseteq \boldsymbol{R}$. However, every element of $\boldsymbol{R}$ is topologically nilpotent (cf. [4], Theorem 2.3.5). Thus, in particular, for every $z \in \mathscr{C}$, $0 = k\nu(z) \geqslant \|z\|$, i. e., $\mathscr{C} = \{0\}$.

REFERENCES

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