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On local derivatives

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Abstract. This paper deals with local derivatives of functions of several variables having values in a fixed Banach space. If the local derivative of a function of one real variable vanishes almost everywhere, then so does its first difference. This theorem can be generalized to any number of variables and to any order of the derivative, and this is the subject of this note. It is also shown that the local derivative (of order $m = (1,\dots,1)$) of an integral equals to the integrand.

Local derivatives of functions of one real variable having values in a Banach space were considered by J. Mikusiński in [3]. Such derivatives have an advantage as compared with the concept of almost everywhere derivatives. Namely, if the local derivative of a real function vanishes in some interval, we may conclude that the function itself is constant almost everywhere in that interval. This property is of crucial significance in the investigation of differential equations. It can be equivalently formulated as follows: if the local derivative of a function vanishes almost everywhere, then so does its first difference. In this formulation the theorem can be easily generalized to any number of dimensions and any order of derivation, and this is the subject of this note.

Local derivatives of functions of q real variables having values in a Hilbert space have been considered in [6].

1. Notation and basic definitions. In this paper we are concerned with distributions in the q-dimensional Euclidean space which admit their values in a fixed Banach space $\mathscr{X}$. The q-dimensional Euclidean space is denoted by R^q and its points by $x = (\xi_1,\dots,\xi_q)$. The set of all non-negative integer points of R^q will be denoted by P^q. Moreover, we shall use the following notation: $x+y = (\xi_1+\eta_1,\dots,\xi_q+\eta_q)$, $x-y = (\xi_1-\eta_1,\dots,\xi_q-\eta_q)$, $\lambda x = (\lambda\xi_1,\dots,\lambda\xi_q)$, $xy = (\xi_1\eta_1,\dots,\xi_q\eta_q)$, $x^m = \xi_1^{\mu_1}\dots\xi_q^{\mu_q}$, where $y = (\eta_1,\dots,\eta_q)$, $m = (\mu_1,\dots,\mu_q) \in P^q$ and λ is a real number. The symbol e_i denotes the point whose i-th coordinates is 1 and all the remaining ones are 0. If a point x has the coordinates $\xi_1,\dots,\xi_q$, then the point $x+e_i\chi_i$ differs from x only by the i-th coordinate, which is equal to $\xi_i+\chi_i$. It will be also convenient to write $e = (1,\dots,1)$.

Let $a = (\alpha_1, \ldots, \alpha_q)$ and $b = (\beta_1, \ldots, \beta_q)$. The set of all points $x \in R^q$ such that $\alpha_j < \xi_j < \beta_j$ $(j = 1, \ldots, q)$ will be called a *q-dimensional open interval* and denoted by $a < x < b$ or (a, b). Infinite values for α_j and β_j are admitted. If α_j and β_j are finite, then the set of all points $x \in R^q$ whose coordinates satisfy the inequality $\alpha_j \leqslant \xi_j \leqslant \beta_j$ $(j = 1, \ldots, q)$ will be called a *q-dimensional closed interval* and denoted by $a \leqslant x \leqslant b$ or $[a, b]$.

We say that a function defined in R^q is *smooth* if it is continuous in R^q, as are its partial derivatives of any order.

If $\varphi(x)$ is a smooth function and $k = (\varkappa_1, \ldots, \varkappa_q)$ is a system of non-negative integers, i.e., $k \in P^q$, then by its derivative of order k we shall mean the function

$$D^k \varphi(x) = \frac{\partial^{\varkappa_i + \ldots + \varkappa_q}}{\partial^{\varkappa_1} \xi_1 \ldots \partial^{\varkappa_q} \xi_q} \varphi(\xi_1, \ldots, \xi_q).$$

A function f is of class C^m in $[a, b]$ $(m = (\mu_1, \ldots, \mu_q) \in P^q)$, if all its derivatives of order $\leqslant m$ exist and are continuous functions in $[a, b]$.

It is known that if a function f is of class C^m and two mixed derivatives of order m differ only by the ordering of differentiation, then the two derivatives are equal (see [2], [3]).

We first consider the 1-dimensional case, $q = 1$. We adopt the definition

$$\Delta^{(o,h)} f(x) = f(x), \qquad \Delta^{(1,h)} f(x) = f(x+h) - f(x),$$
$$\Delta^{(m,h)} f = \Delta^{(1,h)} (\Delta^{(m-1,h)} f) \qquad (m = 2, 3, \ldots).$$

The symbol $\Delta^{(1,h)}$ will be called the *difference operator of the first order*, and $f(x+h) - f(x)$ is usually called a *first difference* of a function $y = f(x)$ (see [5]).

It is easy to check, by induction, that the following equality

$$\Delta^{(m,h)} f(x) = \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} f(x + jh)$$

holds for each $m \in N$.

If f is a function defined in R^q, then we adopt the definition

$$\Delta_i^{(e_i, \chi_i)} f(x) = f(x + e_i \chi_i) - f(x), \qquad \Delta^{(e,h)} f = \Delta_1^{(e_1, \chi_1)} \ldots \Delta_q^{(e_q, \chi_q)} f,$$

where $h = (\chi_1, \ldots, \chi_q)$, and the symbol on the right-hand side denotes the composition of operators.

As regards functions of one variable, we extend the definition of the difference of order $e = (1, \ldots, 1)$ to the difference operator of order $m = (\mu_1, \ldots, \mu_q)$. Namely, we put

$$\Delta^{(m,h)} f = \Delta_1^{(\mu_1, \chi_1)} \ldots \Delta_q^{(\mu_q, \chi_q)} f,$$

where

$$\Delta_i^{(\mu_i,\chi_i)} f(x) = \sum_{j=0}^{\mu_i} (-1)^{(\mu_i-j)} \binom{\mu_i}{j} f(x+je_i\chi_i).$$

By the *m-th local derivative* of a function f in R^q we mean the local limit of the expression

$$\frac{1}{h^m}\Delta^{(m,h)} f(x)$$

as $h\to 0$. In other words, g is the m-th local derivative of f if

$$\lim_{h\to 0}\int_I \left|\frac{1}{h^m}\Delta^{(m,h)} f(x)-g(x)\right| dx = 0 \tag{1}$$

holds for every bounded interval I in R^q. In order that this definition should make sense, we always assume that the integrand in (1) is a locally integrable function of x. If f is locally integrable, then its local derivative is locally integrable, because the limit of a locally convergent sequence of locally integrable functions is locally integrable. The m-th local derivative of f will be denoted by $D^m_{\mathrm{loc}} f$.

This definition of the m-th local derivative does not just reduce to iterated derivatives, because it is more general. For instance, let f be the Weierstrass function, nowhere differentiable. Then the mixed derivative $D^{(e_1,e_2)}_{\mathrm{loc}}[f(x)+f(y)]$ does exist, whereas the first partial derivatives do not.

2. Theorems on convolutions. By the convolution of two functions f and g we mean the integral

$$\int_{R^q} f(x-t)g(t)\,dt. \tag{2}$$

The convolution exists at a point x, whenever the product $f(x-t)g(t)$ is Bochner integrable with respect to t. We assume that the values of f and g are in Banach spaces A and B, respectively, and the values of the product $f(x-t)g(t)$ are in a Banach space C. Therefore, if the convolution exists at some point x, its value is in C.

In order to have the convolution defined at as many points as possible, we adopt the following convention: if one of the factors $f(x-t)$ or $g(t)$ is 0 for some x and t, then the product $f(x-t)g(t)$ is taken to be 0, even if the second factor is not defined. We shall denote the convolution (2) by $f*g$ (see [3]).

For the values of the product ab, where $a\in A$ and $b\in B$, we admit elements of a third Banach space C.

We shall assume that the product of two vectors has the following basic properties:

1° $(a_1+a_2)(b_1+b_2) = a_1b_1+a_1b_2+a_2b_1+a_2b_2$,

2° $\lambda a\cdot\mu b = \lambda\mu\cdot ab$ (λ, μ — real numbers),

3° $|ab| \leqslant |a|\,|b|$.

THEOREM 1. *If the functions f and g are locally integrable and one of them vanishes outside a bounded interval, then the convolution $f*g$ is defined almost everywhere and it is a locally integrable function of x. Moreover, if at least one of the functions is locally bounded (i.e., bounded in every bounded interval), then the convolution $f*g$ is defined everywhere and is continuous.*

Proof. By Corollary 3.2.2 ([1], p. 125) the convolution $f*g$ exists almost everywhere and it is a locally integrable function. To prove the second part of the theorem, let $k(x) = (f*g)(x)$.

1° Consider the case where f is locally bounded and g has a bounded support.

For every fixed x the product $f(t)g(x-t)$ is integrable, since the function $f(t)$ is bounded on the set where $g(x-t) \neq 0$. Thus the convolution k exists everywhere. Moreover, we have

$$|k(x)-k(x_0)| \leqslant \int |f(t)|\,|g(x-t)-g(x_0-t)|dt.$$

There is a number $\varrho > 0$ such that $g(t) = 0$ for $|t| > \varrho$. If x_0 is fixed and $|x-x_0| < 1$, then the difference $g(x-t)-g(x_0-t)$ vanishes for t satisfying the inequality $|t-x_0| > \varrho-1$. Hence

$$|k(x)-k(x_0)| \leqslant M\int |g(x-t)-g(x_0-t)|dt = M\int \big|g\big(t+(x-x_0)\big)-g(t)\big|dt$$

for $|x-x_0| < 1$. By the Lebesgue Theorem (see [3]), the last integral tends to 0 as $x \to x_0$, which proves the continuity.

2° Consider the case where g has a bounded support and is locally bounded.

By the assumption it follows that there are a real number $M > 0$ and an integrable characteristic function $h(t)$ of the support of g such that

$$|g(t)| \leqslant Mh(t).$$

For every fixed x, the product $f(x-t)g(t)$ is a measurable function of t and bounded by the integrable function $|f(x-t)|Mh(t)$. Thus it is integrable and the convolution $k = f*g$ exists for every x. Moreover, we have

$$|k(x)-k(x_0)| \leqslant M\int |f(x-t)-f(x_0-t)|h(t)\,dt.$$

It is easy to see that

$$\begin{aligned}&|f(x-t)-f(x_0-t)|h(t)\\ &\quad\leqslant |f(x-t)\,h(t)-f(x_0-t)\,h(t+x-x_0)|+|f(x_0-t)|\,|h(t+x-x_0)-h(t)|.\end{aligned}$$

Let $u(t) = f(x-t)h(t)$; then $u(t+x-x_0) = f(x_0-t)h(t+x-x_0)$ and

$$|f(x-t)-f(x_0-t)|h(t) \leqslant |u(t+x-x_0)-u(t)|+|f(x_0-t)|\,|h(t+x-x_0)-h(t)|.$$

Thus

$$|k(x)-k(x_0)| \leqslant \int |u(t+x-x_0)-u(t)|dt + \int_{Z_x} |f(x_0-t)|dt,$$

where $Z_x = |h(t+x-x_0)-h(t)|$.

Let us remark that u and Z_x are integrable with respect to t and that all the sets Z_x with $|x-x_0| < 1$ are contained in a bounded interval I. Thus

$$\int_{Z_x} |f(x_0-t)|dt \to 0 \quad \text{as } x \to x_0,$$

by Theorem 1, Chapter XI (see [3], p. 106). Also,

$$\int |u(t+x-x_0)-u(t)|dt \to 0 \quad \text{as } x \to x_0,$$

by the Lebesgue Theorem (see [3], p. 165). This implies the continuity of $k = f*g$.

LEMMA 1. *If f_n and g_n are locally integrable functions with $f_n \to f$ loc. and $g_n \to g$ loc., and, moreover, if g_n vanish outside a common interval, then $f_n * g_n \to f*g$ loc.*

Proof. By Theorem 1 the convolution $f_n * g_n$ exists almost everywhere and is locally integrable.

We can find an interval $-a \leqslant x \leqslant a$ such that the functions g_n vanish outside it. Let $f_{\nu n}$ denote a function which coincides with f_n on the interval $-\nu a \leqslant x \leqslant \nu a$ and vanishes outside it. Then the equality $f_{\nu n} * g_n = f_n * g_n$ holds on the interval

$$-(\nu+1)a \leqslant x \leqslant (\nu+1)a. \tag{3}$$

Since

$$\int |f_{\nu n} * g_n - f_\nu * g| \leqslant \int |f_{\nu n}-f| \cdot \int |g_n - g| + \int |f_{\nu n}-f_\nu| \cdot \int |g| + \int |f_\nu| \cdot \int |g_n - g|$$

and

$$\int |f_{\nu n}-f| \to 0, \quad \int |g_n-g| \to 0 \quad \text{as } n \to \infty,$$

we have

$$\int |f_{\nu n} * g_n - f_\nu * g| \to 0 \quad \text{as } n \to \infty,$$

i.e., $f_{\nu n} * g_n \to f_\nu * g$ in norm as $n \to \infty$. This implies that $f_n * g_n \to f*g$ loc.

3. Local derivatives of convolutions.

THEOREM 2. *Let f and g be locally integrable functions and let one of them have a bounded support. If the m-th local derivative of f and the k-th local* *derivative of g exist, then there also exists the $(m+k)$-th local derivative of the convolution $f*g$ and we have*

$$D_{\mathrm{loc}}^{m+k}(f*g) = (D_{\mathrm{loc}}^{m} f)*(D_{\mathrm{loc}}^{k} g).$$

Proof. By Theorem 1 it follows that $f*g$ exists almost everywhere and is a locally integrable function.

Assume that g has a bounded support. Observe that

$$\frac{1}{h^{m+k}}\Delta^{(m+k,h)}(f*g) = \left(\frac{1}{h^m}\Delta^{(m,h)} f\right)*\left(\frac{1}{h^k}\Delta^{(k,h)} g\right).$$

Let $h_n \to 0$ as $n \to \infty$ and let

$$g_n(x) = \frac{1}{h_n^k}\Delta^{(k,h_n)} g(x), \quad f_n(x) = \frac{1}{h_n^m}\Delta^{(m,h_n)} f(x).$$

By the assumption we have $f_n \to D_{\mathrm{loc}}^m f$ loc. and $g_n \to D_{\mathrm{loc}}^k g$ loc., as $n \to \infty$. Since g has a bounded support, so do the g_n. By Lemma 1 we have

$$(f_n * g_n) \to (D_{\mathrm{loc}}^m f)*(D_{\mathrm{loc}}^k g) \text{ loc.} \quad \text{as } n \to \infty,$$

which proves that

$$\frac{1}{h_n^{m+k}}\Delta^{(m+k,h_n)}(f*g) \to (D_{\mathrm{loc}}^m f)*(D_{\mathrm{loc}}^k g) \text{ loc.} \quad \text{as } n \to \infty.$$

This means that $D_{\mathrm{loc}}^{m+k}(f*g)$ exists and is equal to $(D_{\mathrm{loc}}^m f)*(D_{\mathrm{loc}}^k g)$.

From Theorems 1 and 2 follows

COROLLARY. *Let f and g be locally integrable functions and let g be bounded and with bounded support. If f has its m-th local derivative, then the convolution $f*g$ has its m-th local derivative, which is continuous.*

LEMMA 2 (see [6]). *If a real valued function φ defined in R^q is of class C^m in the interval $[x, x+mh]$ and if $g = D^m\varphi$, then there exists a point y in $[x, x+mh]$ such that*

$$g(y) = \frac{1}{h^m}\Delta^{(m,h)}\varphi(x),$$

LEMMA 3. *If a vector valued function φ of class C^m is such that $D^m\varphi = 0$, then*

$$\Delta^{(m,h)}\varphi(x) = 0$$

for each h and x in R^q.

Proof. Let h and x be arbitrary fixed points in R^q. Without loss of generality we may assume that $h > 0$.

1° Consider the case where φ is a real valued function. By Lemma 2 there exists $y \in [x, x+mh]$ such that

$$g(y) = \frac{1}{h^m} \Delta^{(m,h)} \varphi(x), \qquad \text{where } g = D^m \varphi.$$

Hence, by assumption, it follows that

$$\Delta^{(m,h)} \varphi(x) = 0.$$

2° Let φ be a vector function from R^q to a Banach space $\mathscr{X}$ and let F be any continuous linear functional on $\mathscr{X}$. Then

$$F(0) = F(D^m \varphi) = D^m(F\varphi) = 0.$$

Hence, by the first part of this proof, it follows that

$$\Delta^{(m,h)} F(\varphi(x)) = 0.$$

This implies that

$$F[\Delta^{(m,h)} \varphi(x)] = 0.$$

The last equality is true for any F; thus $\Delta^{(m,h)} \varphi(x) = 0$.

Since x and h were arbitrary fixed, this proves the lemma.

LEMMA 4 (see [6]). *If a real valued function f defined in R^q is of class C^m in R^q, then there exists its local derivative of order m and it is equal to the ordinary derivative, $D^m_{\mathrm{loc}} f = D^m f$.*

LEMMA 5. *If f is a locally integrable vector function and g is a real function of class C^m with bounded support, then the convolution $f * g$ is also of class C^m and the equality holds*

$$D^m(f*g) = f*D^m g.$$

The proof of this lemma is similar to the proof of Theorem 2.4 in [6].

4. Theorems on local derivatives. By a delta-sequence in R^q we mean any sequence of smooth functions δ_n with the following properties:

1° There is a sequence of positive numbers α_n tending to 0 such that $\delta_n(x) = 0$ for $|x| \geqslant \alpha_n$, $n \in N$;

2° $\int\limits_{R^q} \delta_n(x)dx = 1$ for $n \in N$;

3° For every $k \in P^q$ there is a positive integer M_k such that $\alpha_n^k \int\limits_{R^q} |\delta_n^k(x)|\,dx \leqslant M_k$ for $n \in N$.

THEOREM 3. *A locally integrable function f has its m-th local derivative equal to 0, iff, for each fixed h, the equation*

$$\Delta^{(m,h)} f(x) = 0$$

holds for almost all $x \in R^q$.

Proof. If $\Delta^{(m,h)}f$ is the null function, then so is $\frac{1}{h^m}\Delta^{(m,h)}f$. Hence

$$\lim_{h\to 0}\int_I \left|\frac{1}{h^m}\Delta^{(m,h)}f(x)\right| dx$$

is null. This means that the null function is the m-th local derivative of f.

Assume now that the function f has its m-th local derivative equal to 0. Take a real delta-sequence δ_n. In view of Lemma 5, Lemma 4 and Theorem 2

$$D^m\varphi_n = f*(D^m\delta_n) = f*(D^m_{\text{loc}}\delta_n) = (D^m_{\text{loc}}f)*\delta_n,$$

where $\varphi_n = f*\delta_n$. Since $D^m_{\text{loc}}f = 0$, we have $D^m\varphi_n = 0$. Hence, by Lemma 3,

$$\Delta^{(m,h)}\varphi_n = 0. \tag{4}$$

Let us remark that

$$\Delta^{(m,h)}\varphi_n = \Delta^{(m,h)}(f*\delta_n) = (\Delta^{(m,h)}f)*\delta_n.$$

By Theorem 2.2.3 (see [1], p. 118)

$$\Delta^{(m,h)}\varphi_n \to \Delta^{(m,h)}f \text{ loc}.$$

Hence and from (4) it follows that $\Delta^{(m,h)}f(x) = 0$ for almost all $x\in R^q$ and any fixed $h\in R^q$.

Lemma 6 (see [6]). *If f is a locally integrable function, then*

$$\Delta^{(e,h)}\int_{x_0}^{x} f(t)dt = \int_{x}^{x+h} f(t)dt,$$

where $e = (1,\ldots,1)$, $h = (\chi_1,\ldots,\chi_q)$ and $x\in R^q$.

Using Lemma 6 we shall show

Theorem 4. *If f is a locally integrable function, then the indefinite integral*

$$F(x) = \int_{x_0}^{x} f(t)dt$$

is a local primative for f, i.e., $D^e_{\text{loc}}F = f$.

Proof. We have

$$\int_a^b \left|\frac{1}{h}\Delta^{(e,h)}F(x)-f(x)\right| dx$$

$$= \int_a^b \left|\frac{1}{h}\Delta^{(e,h)}\int_{x_0}^{x} f(x)\,dt - f(x)\right| dx = \int_a^b \left|\frac{1}{h}\int_{x}^{x+h} f(t)\,dt - f(x)\right| dx$$

$$= \int_a^b \left|\frac{1}{h}\int_0^h [f(x+t)-f(x)]\,dt\right| dx \leqslant \frac{1}{h}\int_0^h dt\int_a^b |f(x+t)-f(x)|\,dx.$$

The last expression tends to 0. Indeed, by Corollary 1.2 (see [3], p. 166), given any $\varepsilon > 0$, there exists an index h_0 such that

$$\int_a^b |f(x+t)-f(x)|\, dx < \varepsilon \quad \text{for } 0 < t < h_0.$$

Hence

$$\int_a^b \left| \frac{1}{h} \Delta^{(e,h)} F(x) - f(x) \right| dx \leqslant \frac{1}{h} \int_0^h \varepsilon dt = \varepsilon \quad \text{for } h \leqslant h_0.$$

This proves that the local derivative of F is f. Consequently F is a local primitive of f.

THEOREM 5. *If a function f has the locally integrable local derivative $D^e_{\text{loc}} f$, then*

$$\Delta^{(e,h)} f(x) = \int_x^{x+h} D^e_{\text{loc}} f(t)\, dt$$

for each fixed $h \in R^q$ and almost all $x \in R^q$.

Proof. The function

$$F(x) = \int_{x_0}^x D^e_{\text{loc}} f(t) dt$$

is continuous and $D^e_{\text{loc}} f$ is its e-th local derivative (see Theorem 4). Thus the difference $F-f$ has its e-th local derivative equal to 0. By Theorem 3 we have

$$\Delta^{(e,h)}(f-F) = 0$$

for each fixed $h \in R^q$ and for almost all $x \in R^q$. Hence

$$\Delta^{(e,h)} f(x) = \Delta^{(e,h)} F(x) = \Delta^{(e,h)} \int_{x_0}^x D^e_{\text{loc}} f(t) dt = \int_x^{x+h} D^e_{\text{loc}} f(t)\, dt$$

for almost all $x \in R^q$, which proves the theorem.

THEOREM 6. *If a function f and its local derivative $D^e_{\text{loc}} f$ are continuous, then*

$$\frac{1}{h} \Delta^{(e,h)} f(x) \to D^e_{\text{loc}} f(x)$$

almost uniformly as $h \to 0$.

Proof. By Theorem 5

$$\Delta^{(e,h)} f(x) = \int_x^{x+h} D^e_{\text{loc}} f(t)\, dt$$

for $x \in R^q$ and for each $h \in R^q$. Hence

$$\left|\frac{1}{h}\Delta^{(e,h)}f(x) - D^e_{\mathrm{loc}}f(x)\right| \leqslant \frac{1}{h}\int_0^h |D^e_{\mathrm{loc}}f(x+t) - D^e_{\mathrm{loc}}f(x)|\,dt. \tag{5}$$

Given any bounded interval I and a number $\varepsilon > 0$, we can choose, owing to the continuity of $D^e_{\mathrm{loc}}f$, a number $\eta > 0$ such that

$$|D^e_{\mathrm{loc}}f(x+t) - D^e_{\mathrm{loc}}f(t)| < \varepsilon \quad \text{for } x \in I \text{ and } |t| < \eta.$$

Thus we obtain from (5)

$$\left|\frac{1}{h}\Delta^{(e,h)}f(x) - D^e_{\mathrm{loc}}f(x)\right| < \varepsilon \quad \text{for } x \in I \text{ and } 0 < |h| < \eta,$$

which proves the theorem.

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