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On weak differential inequalities

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A. Pliś has proved a theorem concerning weak partial differential inequalities of first order by means of the theory of ordinary differential inequalities (the results are not yet published). Some theorems of this kind for partial differential second order inequalities of parabolic type have been established by J. Szarski ([6]). In proving of these theorems the authors make no use of well-known similar theorems concerning strong partial differential inequalities obtained earlier by J. Szarski ([3], [4], [5]) and by W. Mlak ([2]).

In this paper we show in a simple manner that the theorems on strong inequalities imply suitable theorems on weak ones for a wide class of differential inequalities (Theorem 1). The proof is not based on the theory of ordinary differential inequalities. From Theorem 1 and from the theorems on strong differential inequalities we derive Pliś's and Szarski's theorems concerning weak inequalities under slightly weaker assumptions.

§ 1. Let D be a set (possibly unbounded) of the $(n+1)$-dimensional Euclidean space of the variables $t, x_1, \ldots, x_n$, situated in a zone $0 < t < t_0$ ($t_0 \leqslant +\infty$). Denote by Γ an arbitrary fixed subset of the closure $\bar{D}$. In particular Γ may be a part of the boundary of D.

Let us consider the following system of m differential second order operators:

$$(1)\quad T_i[z_1, \ldots, z_m] = \frac{\partial z_i}{\partial t} - f_i\left(t, X, z_1, \ldots, z_m, \frac{\partial z_1}{\partial x_j}, \ldots, \frac{\partial z_m}{\partial x_j}, \frac{\partial^2 z_1}{\partial x_j \partial x_k}, \ldots, \frac{\partial^2 z_m}{\partial x_j \partial x_k}\right),$$

$$i = 1, \ldots, m;\ j, k = 1, \ldots, n,\ X = (x_1, \ldots, x_n).$$

The functions $f_i(t, X, z_1, \ldots, z_m, p_j^1, \ldots, p_j^m, p_{jk}^1, \ldots, p_{jk}^m)$ are supposed to be defined for $(t, X) \in D$, the other variables being arbitrary.

D_ε and Γ_ε will denote the parts of D and Γ, respectively, contained in the zone $0 \leqslant t \leqslant t_0 - \varepsilon$ $(0 < \varepsilon < t_0)$.

We introduce the following hypothesis.

(H). Suppose that the functions $u_i(t, X), v_i(t, X)$ $(i = 1, \ldots, m)$ are defined in $D+\Gamma$ and have all the derivatives occurring in operotors (1) in D and that for every (small) $\varepsilon > 0$ the inequalities

(2) $T_i[u_1, \ldots, u_m] < T_i[v_1, \ldots, v_m] \quad (i = 1, \ldots, m) \quad \text{for} \quad (t, X) \in D_\varepsilon$

and

(3) $$u_i(t, X) < v_i(t, X) \quad (i = 1, \ldots, m) \quad \text{for} \quad (t, X) \in \Gamma_\varepsilon$$

imply

(4) $$u_i(t, X) \leqslant v_i(t, X) \quad (i = 1, \ldots, m) \quad \text{for} \quad (t, X) \in D^\varepsilon .$$

We say that *the functions* $f_i(t, X, z_1, \ldots, z_m, \overset{1}{p}_j, \ldots, \overset{m}{p}_j, \overset{1}{p}_{jk}, \ldots, \overset{m}{p}_{jk})$ *satisfy the condition* $K[z_1, \ldots, z_m]$ if for $z_l \geqslant \bar{z}_l$ $(l = 1, \ldots, m)$ we have

(5) $$f_i(t, X, z_1, \ldots, z_m, \overset{1}{p}_j, \ldots, \overset{m}{p}_j, \overset{1}{p}_{jk}, \ldots, \overset{m}{p}_{jk}) - \\ - f_i(t, X, \bar{z}_1, \ldots, \bar{z}_m, \overset{1}{p}_j, \ldots, \overset{m}{p}_j, \overset{1}{p}_{jk}, \ldots, \overset{m}{p}_{jk}) \\ \leqslant \sigma_i(t, z_1 - \bar{z}_1, \ldots, z_m - \bar{z}_m) \quad (i = 1, \ldots, m),$$

where the functions $\sigma_i(t, y_1, \ldots, y_m)$ $(i = 1, \ldots, m)$ are continuous and non-negative for $0 \leqslant t < t_0$, $y_i \geqslant 0$ and if the functions $y_i(t) \equiv 0$ $(i = 1, \ldots, m)$, $0 \leqslant t < t_0$, constitute the unique solution, issuing from the origin, of the system of ordinary differential equations

(6) $$\frac{dy_i}{dt} = \sigma_i(t, y_1, \ldots, y_m) \quad (i = 1, \ldots, m).$$

THEOREM 1. *We assume that the functions*

$$f_i(t, X, z_1, \ldots, z_m, \overset{1}{p}_j, \ldots, \overset{m}{p}_j, \overset{1}{p}_{jk}, \ldots, \overset{m}{p}_{jk}) \quad (i = 1, \ldots, m)$$

satisfy the condition $K[z_1, \ldots, z_m]$ *and that hypothesis* (H) *holds true. Furthermore we assume that*

(7) $$T_i[u_1, \ldots, u_m] \leqslant T_i[v_1, \ldots, v_m] \quad (i = 1, \ldots, m) \quad \textit{in } D$$

and

(8) $$u_i(t, X) \leqslant v_i(t, X) \quad (i = 1, \ldots, m) \quad \textit{in } \Gamma .$$

Under these assumptions the inequalities

(9) $$u_i(t, X) \leqslant v_i(t, X) \quad (i = 1, \ldots, m)$$

are satisfied in D.

Proof. Let $y_i = \omega_i^\delta(t)$ $(i = 1, \ldots, m)$, $\delta > 0$, be any solution of the system

(10) $$\frac{dy_i}{dt} = \sigma_i(t, y_1, \ldots, y_m) + \delta \quad (i = 1, \ldots, m),$$

satisfying the initial conditions $\omega_i^\delta(0) = \delta$. From the well-known theorem concerning a continuous dependence of solutions of ordinary differential equations on the parameter and on the initial data (see for instance [1], theorem 4.3, p. 72) and from our assumptions concerning system (6) it follows that for any $\varepsilon > 0$ there exists a $\delta_0 > 0$ such that for every $0 < \delta < \delta_0$ the integral $y_i = \omega_i^\delta(t)$ exists in the interval $0 \leqslant t \leqslant t_0 - \varepsilon$ and that $\omega_i^\delta(t) \to 0$ $(i = 1, \ldots, m)$ uniformly in this interval, as $\delta \to 0$.

According to assumption (5) we obtain

$$(11)\quad f_i\Big(t, X, v_1 + \omega_1^\delta, \ldots, v_m + \omega_m^\delta, \frac{\partial v_1}{\partial x_j}, \ldots, \frac{\partial v_m}{\partial x_j}, \frac{\partial^2 v_1}{\partial x_j \partial x_k}, \ldots, \frac{\partial^2 v_m}{\partial x_j \partial x_k}\Big) -$$
$$- f_i\Big(t, X, v_1, \ldots, v_m, \frac{\partial v_1}{\partial x_j}, \ldots, \frac{\partial v_m}{\partial x_j}, \frac{\partial^2 v_1}{\partial x_j \partial x_k}, \ldots, \frac{\partial^2 v_m}{\partial x_j \partial x_k}\Big) \leqslant \sigma_i(t, \omega_1^\delta, \ldots, \omega_m^\delta)$$

$(i = 1, \ldots, m)$. By (1), (10) and by the identities

$$\frac{\partial \omega_s^\delta}{\partial x_j} \equiv 0, \quad \frac{\partial^2 \omega_s^\delta}{\partial x_j \partial x_k} \equiv 0 \quad (s = 1, \ldots, m;\ j, k = 1, \ldots, n)$$

we get

$$(12)\quad T_i[v_1 + \omega_1^\delta, \ldots, v_m + \omega_m^\delta] = \frac{\partial v_i}{\partial t} + \sigma_i(t, \omega_1^\delta, \ldots, \omega_m^\delta) + \delta -$$
$$- f_i\Big(t, X, v_1 + \omega_1^\delta, \ldots, v_m + \omega_m^\delta, \frac{\partial v_1}{\partial x_j}, \ldots, \frac{\partial v_m}{\partial x_j}, \frac{\partial^2 v_1}{\partial x_j \partial x_k}, \ldots, \frac{\partial^2 v_m}{\partial x_j \partial x_k}\Big)$$
$$\geqslant \frac{\partial v_i}{\partial t} - f_i\Big(t, X, v_1, \ldots, v_m, \frac{\partial v_1}{\partial x_j}, \ldots, \frac{\partial v_m}{\partial x_j}, \frac{\partial^2 v_1}{\partial x_j \partial x_k}, \ldots, \frac{\partial^2 v_m}{\partial x_j \partial x_k}\Big) + \delta$$
$$= T_i[v_1, \ldots, v_m] + \delta \quad (i = 1, \ldots, m).$$

The last inequality is obtained from relation (11). Relations (12) and (7) yield the inequalities

$$(13)\quad T_i[u_1, \ldots, u_m] < T_i[v_1 + \omega_1^\delta, \ldots, v_m + \omega_m^\delta] \quad (i = 1, \ldots, m) \text{ in } D_\varepsilon.$$

Since $\omega_i^\delta(t) \geqslant \delta > 0$, we obtain by (8)

$$(14)\quad u_i(t, X) < v_i(t, X) + \omega_i^\delta(t) \quad (i = 1, \ldots, m) \quad \text{for } (t, X) \in \Gamma_\varepsilon.$$

Therefore, by hypothesis (H), the inequalities

$$(15)\quad u_i(t, X) \leqslant v_i(t, X) + \omega_i^\delta(t) \quad (i = 1, \ldots, m)$$

hold in D_ε. Now letting $\delta \to 0$, we infer that inequalities (9) are fulfilled in D_ε. Since ε is arbitrary, our assertion holds true in the whole set D.

§ 2. In this and in the next paragraphs we consider some particular cases of the set D and the operators T_i. We now assume that D satisfies the following conditions:

(a) D is an open set lying in the zone $0 < t < t_0$, where $t_0 \leqslant +\infty$. For every t', $0 < t' < t_0$, the part of D contained in the zone $0 \leqslant t \leqslant t'$ is bounded and the intersection of the closure of D with the plane $t = t'$ is non-empty. The projection of this intersection on the plane $t = 0$ is denoted by $S_{t'}$.

(b) Let $t' \in (0, t_0)$ and let X' be an arbitrary point of $S_{t'}$. To every sequence t_ν $(\nu = 1, 2, \ldots)$ such that $0 < t_\nu < t_0$ and $t_\nu \to t'$, there corresponds a sequence of points X_ν so that $X_\nu \in S_{t_\nu}$ and $X_\nu \to X'$.

Let Γ denote the set of all points of the boundary of D for which $t < t_0$.

Suppose that for any i $(i = 1, \ldots, m)$ the operator T_i does not contain the derivatives of the functions $z_1, \ldots, z_{i-1}, z_{i+1}, \ldots, z_m$, that is,

$$(16) \qquad T_i[z_1, \ldots, z_m] \equiv \frac{\partial z_i}{\partial t} - f_i\left(t, X, z_1, \ldots, z_m, \frac{\partial z_i}{\partial x_j}, \frac{\partial^2 z_i}{\partial x_j \partial x_k}\right)$$
$$(i = 1, \ldots, m;\ j, k = 1, \ldots, n).$$

We assume that the system of operators (16) is parabolic in the following very wide sense given by J. Szarski (cf. [6]): The system (16) is called *parabolic* with respect to a sequence of functions $z_1(t, X), \ldots, z_m(t, X)$ of class C^1 if for every system of numbers $p_{jk}, \bar{p}_{jk}$ $(p_{jk} = p_{kj}$, $\bar{p}_{jk} = \bar{p}_{kj}$; $j, k = 1, \ldots, n)$ such that $\sum_{j,k=1}^{n} (p_{jk} - \bar{p}_{jk})\lambda_j \lambda_k \leqslant 0$ for any real vector $(\lambda_1, \ldots, \lambda_n)$, the inequalities

$$f_i\left(t, X, z_1(t, X), \ldots, z_m(t, X), \frac{\partial z_i(t, X)}{\partial x_j}, p_{jk}\right) -$$
$$- f_i\left(t, X, z_1(t, X), \ldots, z_m(t, X), \frac{\partial z_i(t, X)}{\partial x_j}, \bar{p}_{jk}\right) \leqslant 0 \quad (i = 1, \ldots, m)$$

hold for $(t, X) \in D$.

To begin with we quote the theorem stated by J. Szarski ([6]).

THEOREM (J. Szarski). *Suppose that*

1° *the set D satisfies conditions* (a) *and* (b),

2° *the functions $u_i(t, X)$, $v_i(t, X)$ $(i = 1, \ldots, m)$ are defined and continuous in $D + \Gamma$, have the derivatives $\partial u_i/\partial t$, $\partial v_i/\partial t$ and continuous derivatives $\partial^2 u_i/\partial x_j \partial x_k$, $\partial^2 v_i/\partial x_j \partial x_k$ $(i = 1, \ldots, m;\ j, k = 1, \ldots, n)$ in D,*

3° *the following inequalities are fulfilled in D:*

$$(17) \qquad T_i[u_1, \ldots, u_m] \leqslant 0 \quad \text{and} \quad T_i[v_1, \ldots, v_m] \geqslant 0 \quad (i = 1, \ldots, m),$$

4° *$u_i(t, X) \leqslant v_i(t, X)$ for $(t, X) \in \Gamma$,*

5° *system* (16) *is parabolic with respect to the sequence $\{u_i(t, X)\}$ or with respect to the sequence $\{v_i(t, X)\}$ $(i = 1, \ldots, m)$,* 6° *the functions* $f_i(t, X, z_1, \ldots, z_m, p_j, p_{jk})$ $(i = 1, \ldots, m)$ *satisfy the condition* $K[z_1, \ldots, z_m]$ (see § 1),

7° *every function* $f_i(t, X, z_1, \ldots, z_m, p_j, p_{jk})$ $(i = 1, \ldots, m)$ *is non-decreasing in each of the variables* $z_1, \ldots, z_{i-1}, z_{i+1}, \ldots, z_m$,

8° *every function* $\sigma_i(t, y_1, \ldots, y_m)$ *occurring in* (5) *is non-decreasing in* $y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_m$.

Under these assumptions the inequalities

$$u_i(t, X) \leqslant v_i(t, X) \quad (i = 1, \ldots, m) \tag{18}$$

hold for $(t, X) \in D$.

We will show that in the theorem of J. Szarski the assumption (b) concerning the set D and the assumption 8° can be weakened. The point is we do not require the monotonicity condition 8°.

THEOREM 2. *Let* D *satisfy condition* (a) *and let the assumptions* 2°, 4°, 5°, 6° *and* 7° *of Szarski's theorem be fulfilled. We assume that*

$$T_i[u_1, \ldots, u_m] \leqslant T_i[v_1, \ldots, v_m] \quad (i = 1, \ldots, m) \quad \text{in } D.$$

Then inequalities (18) *are satisfied everywhere in* D.

The proof of Theorem 2 follows immediately from Theorem 1 and from the following theorem on strong inequalities proved by W. Mlak ([2]).

THEOREM (W. Mlak). *Suppose that* D *fulfils condition* (a) *and that assumptions* 2°, 5° *and* 7° *of the theorem of Szarski are satisfied. Further, suppose that*

$$T_i[u_1, \ldots, u_m] < T_i[v_1, \ldots, v_m] \quad (i = 1, \ldots, m) \quad \text{for} \quad (t, X) \in D \tag{19}$$

and

$$u_i(t, X) < v_i(t, X) \quad (i = 1, \ldots, m) \quad \text{for} \quad (t, X) \in \Gamma.$$

Under these assumptions we have

$$u_i(t, X) < v_i(t, X) \quad (i = 1, \ldots, m) \quad \text{in } D\ (^1).$$

By a similar reasoning one can obtain theorems similar to the other theorems of J. Szarski (see [6]) concerning weak differential inequalities of parabolic type in case where the boundary inequalities involve directional derivatives of the functions in question.

§ 3. Now let D be the following pyramid:

$$0 < t < t_0, \quad |x_j| \leqslant b_j - Mt \quad (j = 1, \ldots, n,\ t_0 \leqslant \min(b_j/M)). \tag{20}$$

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(1) This theorem was stated by Mlak for the case where D was a cylindric domain and instead of (19) the inequalities $T_i[u_1, \ldots, u_m] \leqslant 0$ and $T_i[v_1, \ldots, v_m] > 0$ were supposed. But it is easy to see that the theorem is valid for the more general set D satisfying condition (a), as well as under assumption (19).

By Γ we denote the part of the boundary of D lying on the plane $t = 0$. Let us consider the following system of first order operators

$$(21)\quad T_i[z_1, \ldots, z_m] \equiv \frac{\partial z_i}{\partial t} - f_i\left(t, X, z_1, \ldots, z_m, \frac{\partial z_i}{\partial x_1}, \ldots, \frac{\partial z_i}{\partial x_n}\right) \quad (i = 1, \ldots, m).$$

We will prove

THEOREM 3 (²). *If:*

1° $u_i(t, X)$, $v_i(t, X)$ *are continuous in* $D+\Gamma$, *have first derivatives in* D *and possess Stolz differentials at the points of the side surface of the pyramid* (20),

2° $T_i[u_1, \ldots, u_m] \leqslant T_i[v_1, \ldots, v_m]$ $(i = 1, \ldots, m)$ *in* D,

3° $u_i(0, X) \leqslant v_i(0, X)$ $(i = 1, \ldots, m)$,

4° *the functions* $f_i(t, X, z_1, \ldots, z_m, p_1, \ldots, p_n)$ *satisfy the* $K[z_1, \ldots, z_m]$*-condition* (see § 1),

5° *the functions* f_i *satisfy the Lipschitz condition*

$$|f_i(t, X, z_1, \ldots, z_m, p_1, \ldots, p_n) - f_i(t, X, z_1, \ldots, z_m, \bar{p}_1, \ldots, \bar{p}_n)| \leqslant M \sum_{j=1}^{n} |p_j - \bar{p}_j| \quad (i = 1, \ldots, m),$$

M being the constant which occurs in the definition of the pyramid (20),

6° *for every* i $(i = 1, \ldots, m)$ *the function* $f_i(t, X, z_1, \ldots, z_m, p_1, \ldots, p_n)$ *is non-decreasing in each of the variables* $z_1, \ldots, z_{i-1}, z_{i+1}, \ldots, z_m$, *then the inequalities*

$$u_i(t, X) \leqslant v_i(t, X) \quad (i = 1, \ldots, m)$$

are satisfied in (20) (³).

The validity of Theorem 3 results from Theorem 1 and from the following theorem established by J. Szarski (cf. [5], Theorem 1.1) for strong inequalities:

THEOREM (J. Szarski). *Suppose that the assumptions* 1°, 5° *and* 6° *of Theorem* 3 *hold true. Suppose furthermore that*

$$T_i[u_1, \ldots, u_m] < T_i[v_1, \ldots, v_m] \quad (i = 1, \ldots, m) \quad \text{in } D$$

and

$$u_i(0, X) < v_i(0, X) \quad (i = 1, \ldots, m).$$

(²) This theorem was obtained earlier by A. Pliś by means of ordinary differential inequalities under the additional assumption that each of the functions $\sigma_i(t, y_1, \ldots, y_m)$ $(i = 1, \ldots, m)$ occurring in the definition of condition $K[z_1, \ldots, z_m]$ is non-decreasing with respect to the variables $y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_m$.

(³) I learned, after submitting this paper for publication, that Pliś had also applied a method similar to that of this paper.

Then we have

$$u_i(t, X) < v_i(t, X) \quad (i = 1, \ldots, m) \quad \text{in } D.$$

In particular, in Theorem 1 set D may be the one-dimensional interval $0 < t < t_0$ $(0 < t_0 \leqslant +\infty)$ and Γ may be the point $t = 0$. Considering the operators

$$T_i[z_1, \ldots, z_m] \equiv \frac{dz_i}{dt} - f_i(t, z_1, \ldots, z_m)$$

and applying the well-known theorem on strong ordinary differential inequalities one can obtain a theorem on weak ordinary differential inequalities which in the case of $f_i(t, z_1, \ldots, z_m)$ continuous in t is a particular case of the theorem of T. Ważewski (see [7], p. 124).

§ 4. Theorem 1 (and consequently Theorems 2 and 3) remain true if, in place of the condition $K[z_1, \ldots, z_m]$, the functions $f_i(t, X, z_1, \ldots, z_m, p_j^1, \ldots, p_j^m, p_{jk}^1, \ldots, p_{jk}^m)$ satisfy the following weaker condition $K_0[z_1, \ldots, z_m]$:

For $(t, X) \in D$, arbitrary $z_1, \ldots, p_{nn}^m$ and $y \geqslant 0$ we have

$$(22) \quad \begin{aligned} &f_i(t, X, z_1 + y, \ldots, z_m + y, p_j^1, \ldots, p_j^m, p_{jk}^1, \ldots, p_{jk}^m) - \\ &- f_i(t, X, z_1, \ldots, z_m, p_j^1, \ldots, p_j^m, p_{jk}^1, \ldots, p_{jk}^m) \leqslant \sigma(t, y) \quad (i = 1, \ldots, m), \end{aligned}$$

where the function $\sigma(t, y)$ is defined, continuous and non-negative for $0 \leqslant t < t_0$, $y \geqslant 0$. Moreover, we suppose that $y(t) \equiv 0$, $0 \leqslant t < t_0$, is the unique right-hand solution of the equation

$$\frac{dy}{dt} = \sigma(t, y)$$

passing through the point $(0, 0)$.

The proof of Theorem 1 does not require any essential changes. In particular, if we denote by $y = \omega^\delta(t)$, $\delta > 0$, any right-hand solution of the equation

$$\frac{dy}{dt} = \sigma(t, y) + \delta$$

fulfilling the initial condition $\omega^\delta(0) = \delta$, then, instead of (11), we obtain, by (22),

$$\begin{aligned} &f_i\left(t, X, v_1 + \omega^\delta, \ldots, v_m + \omega^\delta, \frac{\partial v_1}{\partial x_j}, \ldots, \frac{\partial v_m}{\partial x_j}, \frac{\partial^2 v_1}{\partial x_j \partial x_k}, \ldots, \frac{\partial^2 v_m}{\partial x_j \partial x_k}\right) - \\ &\quad - f_i\left(t, X, v_1, \ldots, v_m, \frac{\partial v_1}{\partial x_j}, \ldots, \frac{\partial v_m}{\partial x_j}, \frac{\partial^2 v_1}{\partial x_j \partial x_k}, \ldots, \frac{\partial^2 v_m}{\partial x_j \partial x_k}\right) \leqslant \sigma(t, \omega^\delta) \end{aligned}$$

for $(t, x) \in D$ $(i = 1, \ldots, m)$.

Under the assumption $K_0[z_1, \ldots, z_m]$ Theorem 1 involves the linear operators with unbounded coefficients at the unknown functions.

For example we consider the linear first order operators

$$(23)\quad T_i[z_1, \ldots, z_m] \equiv \frac{\partial z_i}{\partial t} - \Big(\sum_{j=1}^{n} a_j^i(t, X) \frac{\partial z_i}{\partial x_j} + \sum_{k=1}^{m} b_k^i(t, X) z_k \Big) \quad (i = 1, \ldots, m),$$

the coefficients being defined for

$$(24)\qquad 0 < t < t_0, \quad X \text{ being arbitrary}.$$

Then the suitable functions are

$$(25)\quad f_i(t, X, z_1, \ldots, z_m, p_1, \ldots, p_n) \equiv \sum_{j=1}^{n} a_j^i(t, X) p_j + \sum_{k=1}^{m} b_k^i(t, X) z_k .$$

The theorem of J. Szarski concerning first order strong inequalities, quoted in § 3 of this paper, may be transfered to the unbounded zone (24) (see [5], corollary 1.1). A corollary to that theorem formulated for the operators of form (23) reads as follows:

Suppose that in (24) $a_j^i(t, X)$ *are bounded and* $b_k^i(t, X) \geqslant 0$ *for* $k \neq i$. *Let* $u_i(t, X)$, $v_i(t, X)$ $(i = 1, \ldots, m)$ *be continuous for* $0 \leqslant t < t_0$, X *being arbitrary, have Stolz differentials in* (24) *and satisfy the initial inequalities*

$$u_i(0, X) < v_i(0, X) \quad (i = 1, \ldots, m).$$

Further, if

$$T_i[u_1, \ldots, u_m] < T_i[v_1, \ldots, v_m] \quad (i = 1, \ldots, m) \quad \textit{in } (24),$$

then

$$u_i(t, X) < v_i(t, X) \quad (i = 1, \ldots, m) \quad \textit{in } (24).$$

Applying this theorem and Theorem 1 with condition $K[z_1, \ldots, z_m]$ replaced by condition $K_0[z_1, \ldots, z_m]$ we will prove

THEOREM 4. *Let* $u_i(t, X)$, $v_i(t, X)$ *be continuous for* $0 \leqslant t < t_0$ *and an arbitrary* X, *possess Stolz differentials in* (24) *and satisfy the inequalities*

$$u_i(0, X) \leqslant v_i(0, X) \quad (i = 1, \ldots, m),$$

$$T_i[u_1, \ldots, u_m] \leqslant T_i[v_1, \ldots, v_m] \quad (i = 1, \ldots, m) \quad \textit{in } (24).$$

We assume that the coefficients $a_j^i(t, X)$ *are bounded in* (24), $b_k^i(t, X) \geqslant 0$ *for* $k \neq i$ *and*

$$(26)\qquad \sum_{k=1}^{m} b_k^i(t, X) \leqslant C = \text{const} \quad (i = 1, \ldots, m).$$

Then the inequalities

$$u_i(t, X) \leqslant v_i(t, X) \quad (i = 1, \ldots, m)$$

hold true in (24).

Proof. We will show that functions (25) fulfil the condition $K_0[z_1, \ldots, z_m]$. Indeed, for $y \geqslant 0$ we have

$$f_i(t, X, z_1+y, \ldots, z_m+y, p_1, \ldots, p_n) - f_i(t, X, z_1, \ldots, z_m, p_1, \ldots, p_n)$$
$$= \sum_{k=1}^{n} b_k^i(t, X) y \leqslant Cy .$$

The unique solution of the equation $dy/dt = Cy$ issuing from the origin is $y(t) \equiv 0$, q.e.d.

It is easy to verify that the remaining assumptions of Theorem 1 are satisfied.

Inequality (26) can be satisfied also for $b_k^i(t, X)$ unbounded in (24), e.g. for the coefficients

$$b_k^i = A \exp(p|X|^q), \quad k \neq i ,$$
$$b_i^i = -(m-1) A \exp(p|X|^q) + C ,$$

A, p, q being constants, $A \geqslant 0$, $|X| = \left(\sum_{i=1}^{n} x_i^2\right)^{1/2}$.

The assumption $K[z_1, \ldots, z_m]$ does not embrace such cases because the inequalities

$$f_i(t, X, z_1, \ldots, z_m, p_1, \ldots, p_n) - f_i(t, X, \bar{z}_1, \ldots, \bar{z}_m, p_1, \ldots, p_n)$$
$$= \sum_{k=1}^{m} b_k^i(t, X)(z_k - \bar{z}_k) \leqslant \sigma_i(t, z_1 - \bar{z}_1, \ldots, z_m - \bar{z}_m) \quad (i = 1, \ldots, m) ,$$

with arbitrary $z_k, \bar{z}_k, z_k \geqslant \bar{z}_k$ $(k = 1, \ldots, m)$ are satisfied if and only if each of the coefficients $b_k^i(t, X)$ is bounded from above by a quantity which is independent of X in (24).

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