A CONVOLUTION PROPERTY
OF THE CANTOR-LEBESGUE MEASURE

BY

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Let $T$ be the circle group $\mathbb{R}/\mathbb{Z}$ and, for $1 \leq p < \infty$, let $L^p$ be the usual Lebesgue space formed with respect to normalized Lebesgue measure $m$ on $T$. It is well known that every complex Borel measure $\mu$ on $T$ acts as a convolution operator on any $L^p$-space: $\mu * L^p \subseteq L^p$. More interesting is the fact that there are probability measures $\mu$ on $T$ which are singular with respect to $m$ and yet have the property that $\mu * L^p \subseteq L^{p+\varepsilon}$ for some $\varepsilon = \varepsilon(p) > 0$ and all $p \in (1, \infty)$. For examples of such $\mu$ obtained using Riesz products see p. 393 in [1]. For another example and a discussion of this phenomenon see p. 120-122 in [2]. The purpose of this note* is to prove the following

**Theorem.** Let $\lambda$ be the Cantor-Lebesgue measure on $T$. For each $p \in (1, \infty)$ there is an $\varepsilon > 0$ such that $\|\lambda * f\|_{L^{p+\varepsilon}} \leq \|f\|_{L^p}$ for all $f \in L^p$.

This theorem is a consequence of the following two lemmas:

**Lemma 1.** Suppose the inequality

$$
\left\{ \frac{1}{3} \left[ \left( \frac{a+b}{2} \right)^q + \left( \frac{b+c}{2} \right)^q + \left( \frac{a+c}{2} \right)^q \right] \right\}^{1/q} \leq \left( \frac{a^p + b^p + c^p}{3} \right)^{1/p}
$$

holds for all positive numbers $a$, $b$, $c$. Then $\|\lambda * f\|_{L^q} \leq \|f\|_{L^p}$ for all $f \in L^p$.

**Lemma 2.** Inequality (1) is valid for $q = 2$ and $p = 2/(1+3^{-1/2}) \approx 1.2679$.

For $2/(1+3^{-1/2}) \leq p < 2$, the Theorem is a direct consequence of the lemmas. For other values of $p < 2$, our result follows from the Riesz-Thorin theorem and the fact that $\|\lambda * f\|_{L^1} \leq \|f\|_{L^1}$ ($f \in L^1$). Duality and another application of complex interpolation take care of the case $2 \leq p < \infty$. Thus it is enough to prove the lemmas.

* Partially supported by NSF Grant MCS-7827602.
Proof of Lemma 1. For $N = 1, 2, \ldots$, let $G_N$ be the cyclic group of $3^N$ elements realized as the set $\{0, 1, \ldots, 3^N-1\}$ with addition modulo $3^N$, and let $L^p(G_N)$ be the Lebesgue space formed with respect to normalized counting measure on $G_N$. The norm in $L^p(G_N)$ will be denoted by $\|\cdot\|_{p,N}$. Let $\mu_N$ be the probability measure uniformly distributed over the set

$$S_N = \left\{ \sum_{j=0}^{N-1} \epsilon_j \cdot 3^j : \epsilon_j = 0, 2 \right\}.$$ 

We will show that if (1) holds for $p$ and $q$, then

$$\|\mu_N \ast f\|_{q,N} \leq \|f\|_{p,N}, \quad f \in L^p(G_N), \quad N = 1, 2, \ldots.$$  \tag{2}$$

If we take the interval $[0, 1)$ as a model for $T$, then the Cantor-Lebesgue measure $\lambda$ is the limit (in an appropriate sense) of the sequence of measures $\left\{ \lambda_N \right\}_{N=1}^\infty$, where $\lambda_N$ is the probability measure uniformly distributed over the set

$$\left\{ \sum_{j=0}^{N-1} \epsilon_j \cdot 3^j : \epsilon_j = 0, 2 \right\}.$$ 

Thus the conclusion of Lemma 1 will follow from (2) and an elementary limit argument. We will establish (2) by induction on $N$.

For $N = 1$, inequality (2) is a direct consequence of (1). So suppose that (2) is valid with $N$ replaced by $N-1$ and let $f$ be a function on $G_N$.

For $j = 0, 1, 2$, let $E_j = \{n \in G_N : n \equiv j \pmod{3}\}$ and let

$$f_j(n) = \begin{cases} f(n) & \text{if } n \in E_j, \\ 0 & \text{if } n \notin E_j. \end{cases}$$

For $j = 0, 2$, let $\mu_N^j$ be the probability measure uniformly distributed over $S_N \cap E_j$. Thus $\mu_N = (\mu_N^0 + \mu_N^2)/2$. Now

$$\|\mu_N \ast f\|_{q,N}$$

$$= \frac{1}{3^N} \left[ \sum_{n \in E_0} \left( \frac{\mu_N^0 \ast f_0(n) + \mu_N^2 \ast f_1(n)}{2} \right)^q + \sum_{n \in E_1} \left( \frac{\mu_N^0 \ast f_1(n) + \mu_N^2 \ast f_2(n)}{2} \right)^q + \sum_{n \in E_2} \left( \frac{\mu_N^0 \ast f_2(n) + \mu_N^2 \ast f_0(n)}{2} \right)^q \right]^{1/q}$$

$$= \frac{1}{3} \left[ \left\| \mu_{N-1} \ast \frac{f_0 + f_1}{2} \right\|_{q,N-1}^q + \left\| \mu_{N-1} \ast \frac{f_1 + f_2}{2} \right\|_{q,N-1}^q + \left\| \mu_{N-1} \ast \frac{f_2 + f_0}{2} \right\|_{q,N-1}^q \right]^{1/q},$$
where \( \tilde{f}_j, \hat{f}_j \) are functions on \( G_{N-1} \) such that

\[
\| \tilde{f}_j \|_{p,N-1} = \| \hat{f}_j \|_{p,N-1} = 3 \| f_j \|_{p,N}.
\]

By way of example, we elaborate on the equality

\[
\frac{1}{3^N \sum_{n \in E_1}} \left( \frac{\mu_N * f_1(n) + \mu_N * f_2(n)}{2} \right)^q = \frac{1}{3} \left\| \mu_{N-1} * \frac{\tilde{f}_1 + \tilde{f}_2}{2} \right\|_{q,N-1}.
\]

Since \( E_1 = E_0 + 1 \) and \( \mu_N(j) = \mu_N(j-2) \), the LHS of the above is

\[
\frac{1}{3^N \sum_{n \in E_0}} \left( \frac{\mu_N * f_1(n+1) + \mu_N * f_2(n-1)}{2} \right)^q.
\]

Putting \( \tilde{f}_1(n) = f_1(n+1) \) and \( \tilde{f}_2(n) = f_2(n-1) \), we obtain

\[
\frac{1}{3^N \sum_{n \in E_0}} \left( \mu_N * \frac{\tilde{f}_1 + \tilde{f}_2}{2} (n) \right)^q,
\]

where \( \tilde{f}_1 \) and \( \tilde{f}_2 \) are supported on \( E_0 \). Now, identifying \( E_0 \) with \( G_{N-1} \) and \( \mu_N \) with \( \mu_{N-1} \), we get

\[
\frac{1}{3} \left\| \mu_{N-1} * \frac{\tilde{f}_1 + \tilde{f}_2}{2} \right\|_{q,N-1}.
\]

By (2) (with \( N - 1 \) instead of \( N \)) and the triangle inequality, the last term of (3) is not greater than

\[
\left\{ \frac{1}{3} \left[ \frac{\| f_0 \|_{p,N-1} + \| f_1 \|_{p,N-1}}{2} \right]^q + \frac{\| f_0 \|_{p,N-1} + \| f_2 \|_{p,N-1}}{2} \right] \right\}^{1/q} \leq \left\{ \frac{1}{3} \left[ 3 \| f_0 \|_{p,N} + 3 \| f_1 \|_{p,N} + 3 \| f_2 \|_{p,N} \right] \right\}^{1/p} = \| f \|_{p,N}.
\]

Here the inequality is a consequence of (1) and (4). Thus (2) is established and the lemma is proved.

**Proof of Lemma 2.** To study inequality (1) is essentially to examine the maxima of the quantity \( (a+b)^q + (b+c)^q + (a+c)^q \) subject to the constraint \( a^p + b^p + c^p = 1 \). When \( q = 2 \), the method of Lagrange shows that if such a maximum occurs for a triple \( (a, b, c) \), then there is a constant
λ < 0 such that

\[ a + \lambda a^{p-1} = b + \lambda b^{p-1} = c + \lambda c^{p-1}. \]

Since an equation \( x + \lambda x^{p-1} = \text{const} \) (\( \lambda < 0, 1 < p < 2 \)) can have at most two solutions \( x \geq 0 \), at least two of the values \( a, b, c \) are equal. Thus, setting \( a = b = t \) and \( c = 1 \), it suffices to show that for \( p > 2/(1+3^{-1/2}) \), the maximum of

\[ f(t) = \frac{[(2t)^2 + 2(1+t)^2]^{1/2}}{(2t^p + 1)^{1/p}} \]

for \( t \geq 0 \) occurs when \( t = 1 \).

Now \( f'(t) \) has the same sign as \( s(t) = -2t^p + 3t - 2t^{p-1} + 1 \). Since \( s''(t) < 0 \) for \( t > (2-p)/p \) and since \( s'(1) = 5 - 4p \), it follows that, for \( p > 5/4 \), \( f(t) \) is decreasing for \( t \geq 1 \). A computation shows that \( f(0) \leq f(1) \) if

\[ p \geq 2 \left( 1 + \frac{\log 2}{\log 3} \right)^{-1} \approx 1.2263. \]

Thus, for \( p > 5/4 \), it follows that if \( f(t) > f(1) \) for any \( t \geq 0 \), then there exists \( t_1 \) with \( 0 < t_1 < 1 \) and \( s(t_1) = 0 \), or

\[ 2t_1^{p-1} = 1 + \frac{2t_1}{1+t_1}. \tag{5} \]

Let \( y_1(t) = 2t^{p-1} \) and \( y_2(t) = 1 + 2t/(1+t) \). If \( 5/4 < p_0 < 2 \), it is easy to see that there exists \( t_0 \in (0, 1) \) and \( \varepsilon > 0 \) such that, for \( t_0 < t < 1 \) and \( p_0 \leq p \leq 2 \), we have \( y_1(t) - y_2(t) \geq \varepsilon \), so \( y_2(t) - y_1(t) \geq (1-t)\varepsilon > 0 \). Let \( S \) be the set of all \( p \in [1.2561, 2] \) for which there exists \( t_1 \in (0, 1) \) such that (5) holds. It follows from the preceding remark that \( S \) is closed. Let \( p_1 \) be the greatest element of \( S \). Then \( p_1 < 2 \). (If \( S = \emptyset \), the lemma is proved.) We will show that

\[ p_1 \leq 2/(1+3^{-1/2}), \tag{6} \]

which will complete the proof of the lemma.

For \( p = p_1 \), let \( t_1 = \sup \{ t \in [0, 1]: y_1(t) = y_2(t) \} \). Since \( y_1(t) < y_2(t) \) for \( t < 1 \) and \( |1-t| \) small, we have \( t_1 < 1 \). It then follows that \( y_1'(t_1) = y_2'(t_1) \), and so

\[ 2(p_1 - 1)t_1^{p_1-1} = \frac{2t_1}{(1+t_1)^2}. \]

Since also

\[ 2t_1^{p_1-1} = y_1(t_1) = y_2(t_1) = 1 + \frac{2t_1}{1+t_1}, \]

(6) holds.
we have

$$p_1 = 1 + \frac{2t_1}{1 + 4t_1 + 3t_1^2}.$$

But the function

$$g(t) = 1 + \frac{2t}{1 + 4t + 3t^2}$$

satisfies $g(t) \leq 2/(1 + 3^{-1/2})$ for $0 \leq t \leq 1$. This establishes (6) and completes the proof of the lemma.

It would be interesting to determine the precise range of values $p$ and $q$ for which $\lambda \ast L^p \subseteq L^q$ (P 1267) and also to determine the range of values for which inequality (1) holds (P 1268). The only additional information we have concerning these problems is the following: if $\lambda \ast L^p \subseteq L^q$, then

$$\frac{1}{p} + \left(1 - \frac{\log 2}{\log 3}\right)\left(1 - \frac{1}{q}\right) \leq 1.$$

(Thus if $\lambda \ast L^p \subseteq L^q$, then $p \geq 2(1 + \log 2/\log 3)^{-1} \approx 1.2263$.)

added in proof. W. Beckner has shown that (1) holds with $q = 2$ precisely when $p > \log 4/\log 3 \approx 1.2619$.

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Reçu par la Rédaction le 20.10.1979; en version modifiée le 20.2.1980