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MULTIPLE RECURRENCE FOR DISCRETE TIME MARKOV PROCESSES. II

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Introduction. This note is a continuation of [2]. By constructing an invariant distribution for the 2-parameter Markov process we are able to improve upon previous "double recurrence" result (Theorem 3 in [2]) by ridding it of unnecessary topological restrictions.

Throughout this paper, X is a Polish space and $\mathscr{B}$ denotes the σ-algebra of its Borel sets. A *transition probability* on X is a function $P: X \times \mathscr{B} \to [0, 1]$ such that $P(x, \cdot)$ is a probability measure for every $x \in X$ and $P(\cdot, A)$ is Borel measurable for every $A \in \mathscr{B}$. It follows that for every bounded Borel function f the function

$$Pf(x) = \int f(y) P(x, dy)$$

is also Borel. If P_1 and P_2 are transition probabilities, then

$$P_1 P_2(x, A) = \int P_2(y, A) P_1(x, dy)$$

is another transition probability and $(P_1 P_2)f = P_1(P_2 f)$.

1. Distribution of a 2-parameter process. Let $N_0 = \{0, 1, 2, \ldots\}$ and consider the partial order in $N_0 \times N_0$ defined by

$$(i, j) \leqslant (k, l) \Leftrightarrow i \leqslant k \text{ and } j \leqslant l.$$

A finite subset $\pi = \{\alpha_0, \alpha_1, \ldots, \alpha_n\}$ of $N_0 \times N_0$ will be called a *path* if $\alpha_0 = (0, 0)$ and $\alpha_k - \alpha_{k-1} = (1, 0)$ or $(0, 1)$ for $k = 1, 2, \ldots, n$. Let $\Omega = X^{N_0 \times N_0}$ and Σ be the product σ-algebra in Ω. For a given path π and Borel sets $A_0, A_1, \ldots, A_n$ the set

$$C(\alpha_k, A_k; k = 0, 1, \ldots, n) = \{\omega \in \Omega: \omega(\alpha_k) \in A_k;\ k = 0, 1, \ldots, n\}$$

is called a *path cylinder* based on π.

THEOREM 1. *Let P_1 and P_2 be transition probabilities on X. If $P_1 P_2 = P_2 P_1$, then for every $x_0 \in X$ there exists a probability measure Q_{x_0} on (Ω, Σ) such that for every path cylinder*

$$C = C(\alpha_k, A_k; k = 0, 1, \ldots, n)$$

we have

$$(1)\qquad Q_{x_0}(C) = \chi_{A_0}(x_0) \int_{A_1} \cdots \int_{A_{n-1}} \int_{A_n} P_{i_n}(x_{n-1}, dx_n) P_{i_{n-1}}(x_{n-2}, dx_{n-1}) \cdots \\ \cdots P_{i_1}(x_0, dx_1),$$

where $i_k = 1$ *or* 2 *depending on whether* $\alpha_k - \alpha_{k-1} = (1, 0)$ *or* $(0, 1)$.

Proof. Observe that the right-hand side of (1) is the same as the Markov measure P_{x_0} of C for a one-parameter Markov chain with transition probabilities $P_{i_1}, P_{i_2}, \ldots, P_{i_n}$ (the Ionescu–Tulcea theorem; see, e.g., [3], V.2.1). Now P_{x_0} is defined on the σ-algebra generated by the path cylinders on $\pi = \{\alpha_0, \alpha_1, \ldots, \alpha_n\}$. In particular, for a jointly measurable function $f(x_1, x_2)$ we have

$$\begin{aligned} \int\int f(x_1, x_2) P_{i_2}(x_1, dx_2) P_{i_1}(x_0, dx_1) &= \mathrm{E}_{x_0} f(\omega(\alpha_1), \omega(\alpha_2)) \\ &= \mathrm{E}_{x_0}\big(\mathrm{E}_{x_0}(f(\omega(\alpha_1), \omega(\alpha_2)) | \omega(\alpha_2))\big) \\ &= \int \mathrm{E}_{x_0}(f(\omega(\alpha_1), \omega(\alpha_2)) |\ \omega(\alpha_2) = x_2) P_{i_1} P_{i_2}(x_0, dx_2), \end{aligned}$$

where E_{x_0} is the expectation with respect to P_{x_0}. Choosing a regular version of the conditional probability $P_{i_1}(x_0, dx_1 | x_2)$ defined by

$$\int g(x_1) P_{i_1}(x_0, dx_1 | x_2) = \mathrm{E}_{x_0}(g(\omega(\alpha_1)) |\ \omega(\alpha_2) = x_2)$$

we obtain

$$\int\int f(x_1, x_2) P_{i_2}(x_1, dx_2) P_{i_1}(x_0, dx_1) \\ = \int\int f(x_1, x_2) P_{i_1}(x_0, dx_1 | x_2) P_{i_1} P_{i_2}(x_0, dx_2).$$

In particular, for $A, B, C \in \mathscr{B}$ the following formula holds:

$$(2)\qquad \int_A\int_B\int_C P_2(x_{00}, dx_{01} | x_{11}) P_2(x_{10}, dx_{11}) P_1(x_{00}, dx_{10}) \\ = \int_B\int_A\int_C P_2(x_{00}, dx_{01} | x_{11}) P_1(x_{00}, dx_{10} | x_{11}) P_1 P_2(x_{00}, dx_{11}).$$

Now we are in a position to define Q_{x_0} on the cylinders based on the initial square

$$\pi_m = \{\alpha \in N_0 \times N_0;\ \alpha \leqslant (m, m)\}.$$

First choose an arbitrary path $\pi = \{\alpha_0, \alpha_1, \ldots, \alpha_n\}$ $(n = 2m)$ joining $(0, 0)$ with (m, m). For the σ-algebra based on π, Q_{x_0} is defined by (1). If $(i, j) \in \pi_m \setminus \pi$ and, say, $(i, j-1) = \alpha_k \in \pi$ and $(i+1, j) = \alpha_{k+2} \in \pi$ (so that π has a corner at α_{k+1}), then we extend Q_{x_0} to the cylinders based on $\pi \cup \{(i, j)\}$ by letting

(3) $$Q_{x_0}(C) = \chi_{A_{\alpha_0}}(x_0) \int \dots \int P_{i_n}(x_{\alpha_{n-1}}, dx_{\alpha_n}) \dots P_{i_{k+3}}(x_{\alpha_{k+2}}, dx_{\alpha_{k+3}}) \times P_2(x_{\alpha_k}, dx_{(i,j)} \mid x_{\alpha_{k+2}}) P_2(x_{\alpha_{k+1}}, dx_{\alpha_{k+2}}) P_1(x_{\alpha_k}, dx_{\alpha_{k+1}}) \times P_{i_k}(x_{\alpha_{k-1}}, dx_{\alpha_k}) \dots P_{i_1}(x_0, dx_{\alpha_1}),$$

where $C = C(\alpha, A_\alpha; \alpha \in \pi \cup \{(i, j)\})$ and $\int \dots \int$ stands for

$$\int_{A_{\alpha_1}} \cdots \int_{A_{\alpha_k}} \int_{A_{\alpha_{k+1}}} \int_{A_{\alpha_{k+2}}} \int_{A_{(i,j)}} \int_{A_{\alpha_{k+3}}} \cdots \int_{A_{\alpha_n}} .$$

Note that $A_{(i,j)}$ and $A_{\alpha_{k+1}}$ are declared conditionally independent given x_{α_k} and $x_{\alpha_{k+2}}$. An analogous extension formula applies to (i', j') if $(i', j'+1)$ and $(i'-1, j')$ are in π. Since $P_1 P_2 = P_2 P_1$, we have

$$\int_{A_{\alpha_{k+2}}} \int_{A_{\alpha_{k+1}}} \int_{A_{(i,j)}} P_2(x_{\alpha_k}, dx_{(i,j)} \mid x_{\alpha_{k+2}}) P_1(x_{\alpha_k}, dx_{\alpha_{k+1}} \mid x_{\alpha_{k+2}}) \times P_1 P_2(x_{\alpha_k}, dx_{\alpha_{k+2}})$$

$$= \int_{A_{\alpha_{k+2}}} \int_{A_{(i,j)}} \int_{A_{\alpha_{k+1}}} P_1(x_{\alpha_k}, dx_{\alpha_{k+1}} \mid x_{\alpha_{k+2}}) P_2(x_{\alpha_k}, dx_{(i,j)} \mid x_{\alpha_{k+2}}) \times P_2 P_1(x_{\alpha_k}, dx_{\alpha_{k+2}}),$$

so, by (2), we could have started from $\pi' = (\pi \cup \{(i, j)\}) \setminus \{\alpha_k\}$, and the extension to $\pi' \cup \{\alpha_k\}$ would have produced the same measure on cylinders based on $\pi' \cup \{\alpha_k\} = \pi \cup \{(i, j)\}$. We also note that formula (3) is obtained from the integral formula (1) by simply inserting the integral

$$\int_{A_{(i,j)}} P_2(x_{\alpha_k}, dx_{(i,j)} \mid x_{\alpha_{k+2}}).$$

In the same manner we fill out all the remaining corners of the path π and, subsequently, of all the newly obtained paths π' contained in the so far constructed subset. Each time a new integral with an appropriate conditional transition probability is inserted. After a finite number of steps we cover π_m by the paths and obtain a formula for the measure Q_{x_0} of any cylinder based on π_m. The resulting set function is clearly independent of the initial choice of π. That Q_{x_0} is σ-additive on the algebra based on π_m follows from the fact that σ-additivity is preserved for each one-step extension. For a one-step extension, however, the proof is standard (see, e.g., [3], III.2.1). By letting $m \to \infty$ we obtain a consistent family of σ-additive measures, thus extending to a single probability measure Q_{x_0} on Σ. This completes the proof of the theorem.

For every probability measure μ on X we define a measure Q_μ on Ω with initial distribution μ by the formula

(4) $$Q_\mu = \int Q_x \, d\mu(x).$$

It is not difficult to see that μ invariant for P_1 and P_2 (i.e., $\int P_i f\, d\mu = \int f\, d\mu$) implies Q_μ invariant for the shift transformations

$$S_1(\omega)(i,j) = \omega(i+1,j), \qquad S_2(\omega)(i,j) = \omega(i,j+1)$$

in Ω.

Remark. The above construction fails for $X^{N_0 \times N_0 \times N_0}$. In fact, let $X = \{1, 2, \ldots, 8\}$ and consider the stochastic matrices

$$P_1 = \begin{bmatrix} 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \qquad P_2 = \begin{bmatrix} 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$P_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

It is easy to see that P_1, P_2, and P_3 commute. We show that for $x_0 = 1$ there is no measure Q_{x_0} on $\Omega = X^{N_0 \times N_0 \times N_0}$ coinciding with the usual Markov measures on each path.

Suppose Q_1 is such a measure. Consider the initial cube

$$\pi_1 = \{(i,j,k);\ 0 \leqslant i,j,k \leqslant 1\}.$$

Suppose $Q_1(U) > 0$ for

$$U = \{\omega\colon\ \omega(i,j,k) = a_{ijk};\ (i,j,k) \in \pi_1\}.$$

We have $a_{000} = 1$, because Q_1 is supported by trajectories starting from the state 1. Since U is contained in every path cylinder of length 3 determined by the values a_{ijk}, all the one-step transitions must be positive. Thus, glancing at P_2 we obtain $a_{010} = 3$ or 4. Suppose it is 3. Then

$$a_{011} = 1, \quad a_{001} = 7, \quad a_{101} = 1, \quad a_{100} = 3, \quad a_{110} = 5, \quad a_{010} = 4,$$

by P_3, P_2, P_1, P_3, P_2 and P_1, respectively (each time looking only at the previous position). We have obtained a contradiction at a_{010}. The case $a_{010} = 4$ is dealt with in the same manner. Thus $Q_1(U) = 0$ for every such U, implying $Q_1 \equiv 0$, a contradiction.

2. Multiple recurrence for Feller transition probabilities. A transition probability P on $X \times \mathscr{B}$ is called *Feller* if Pf is continuous whenever f is a bounded continuous function. Note that if P is Feller, then the function $P(\cdot, U)$ is l.s.c. for every open set U. By $U(x, \varepsilon)$ we denote the open ball of radius ε centered at x. In essentially the same manner as in Lemma 2 of [2] we prove that if $Q_1, Q_2, \ldots, Q_m$ are Feller transition probabilities, then the function

$$F(x) = \inf_{n \geqslant 1} \inf \{\varepsilon\colon Q_i^n(x, U(x, \varepsilon)) > 0 \text{ for } i = 1, 2, \ldots, m\}$$

is u.s.c.

As in [2], we say that a point x is *multiply recurrent* for a family Φ of Feller transition probabilities if there exists a sequence $n_k \to \infty$ such that for every $\varepsilon > 0$ and every $Q \in \Phi$ we have

$$Q^{n_k}(x, U(x, \varepsilon)) > 0$$

for all sufficiently large k.

THEOREM 2. *Let P_1, P_2 be commuting Feller transition probabilities on X. Denote by Φ the semigroup generated by P_1, P_2. If there exists a common invariant probability measure μ for P_1 and P_2, then the multiply recurrent points for Φ form a dense G_δ-subset of* $\operatorname{supp}\mu$.

Proof. We may assume $\operatorname{supp}\mu = X$. By Lemma 3 in [2], it suffices to prove that for any finite collection $Q_1, Q_2, \ldots, Q_m$ of transition probabilities of the form $P_1^i P_2^j$ the multiply recurrent points form a dense G_δ-set. Consider the space Ω of Theorem 1 and let Q_μ be the shift invariant measure on Ω defined by (4). Denote by S_k the shift transformation corresponding to Q_k, i.e., if $Q_k = P_1^i P_2^j$, then

$$S_k(\omega)(k, l) = \omega(k+i, l+j).$$

Clearly, Q_μ is invariant with respect to each S_k. Given $\varepsilon > 0$ and $x \in X$ consider the open cylinder

$$\tilde{U} = \{\omega \in \Omega\colon \omega(0, 0) \in U(x, \varepsilon)\}.$$

Since $\operatorname{supp}\mu = X$, we have $Q_\mu(\tilde{U}) = \mu(U(x, \varepsilon)) > 0$. Applying the Furstenberg–Szemeredi theorem ([1], Theorem 7.14) we obtain

$$Q_\mu(\tilde{U} \cap S_1^{-n}(\tilde{U}) \cap \ldots \cap S_m^{-n}(\tilde{U})) > 0$$

for some $n \geqslant 1$. In particular, there exists $y \in X$ with the positive measure Q_y of the same cylinder set. By the construction of Q_y we have $y \in U(x, \varepsilon)$ and

$$Q_k^n(y, U(x, \varepsilon)) > 0 \quad \text{for all } k = 1, 2, \ldots, m.$$

This clearly implies $F(y) < 2\varepsilon$. We have proved that F assumes arbitrarily small values in every neighbourhood. It follows that $F(x) = 0$ if x is a point of continuity. By the definition of F, this means that every point of continuity is multiply recurrent for $Q_1, Q_2, \ldots, Q_m$. Since the points of continuity of a u.s.c. function form a dense G_δ-set, this completes the proof of the theorem.

If X is compact, then there is a one-to-one correspondence between Feller transition probabilities and Markov operators on $C(X)$. Now Theorem 3 in [2] is covered by the following consequence of our Theorem 2:

COROLLARY. *Let T_1, T_2 be commuting Markov operators on $C(X)$, X compact metric. Then there exists a multiply recurrent point for the semigroup of operators generated by T_1 and T_2.*

Proof. It suffices to note that there exists a common invariant measure μ and apply Theorem 2. The existence of μ follows from the Markov–Kakutani fixed point theorem. An invariant measure can also be found directly as a weak* limit point of the sequence

$$n^{-2}\Big(\sum_{i=1}^{n}\sum_{j=1}^{n}(T_1^i)^*(T_2^j)^*\delta_x\Big)$$

in the dual space $C(X)^*$.

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