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On a criterion of uniqueness for periodic solutions of linear second order difference equations

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1. The purpose of this note is to establish a uniqueness criterion for periodic solutions of linear second order difference equations, which is a discrete analogue of the criterion for differential equations given by A. Lasota and Z. Opial in [3]. Section 2 contains the notations and preliminaries. In Section 3 three lemmas are given. The first of them provides formulae for the solution of the second order difference equation by means of a discrete analogue of Green's function. The second one is a discrete analogue of the well-known inequality of Beurling ([1], p. 124). In Section 4 these lemmas are used to state Theorem 1, from which immediately follows the above-mentioned criterion (Theorem 2). Finally, Section 5 contains an example which shows that this criterion is the best possible in a certain sense.

The results of this note will be utilized in [2].

2. By a net we will mean any sequence $\tau = \{t_i\}_{i\in Z}$ of real numbers satisfying the inequalities

$$t_{i-1} < t_i, \quad i \in \boldsymbol{Z},$$

where $\boldsymbol{Z}$ denotes the set of all integers. In the vector space $\boldsymbol{R}^{\boldsymbol{Z}}$ of all sequences of real numbers we define the difference operators $\Delta^{(k)}$, $\nabla^{(k)}$: $\boldsymbol{R}^{\boldsymbol{Z}} \to \boldsymbol{R}^{\boldsymbol{Z}}$, as follows:

$$\Delta^{(k)} v = (\ldots, \Delta^{(k)} v_{-1}, \Delta^{(k)} v_0, \Delta^{(k)} v_1, \ldots),$$
$$\nabla^{(k)} v = (\ldots, \nabla^{(k)} v_{-1}, \nabla^{(k)} v_0, \nabla^{(k)} v_1, \ldots),$$

where, for $i \in \boldsymbol{Z}$ we set

$$\Delta^{(0)} v_i = \nabla^{(0)} v_i = v_i,$$
$$\Delta^{(1)} v_i = \Delta v_i = v_{i+1} - v_i, \quad \nabla^{(1)} v_i = \nabla v_i = v_i - v_{i-1},$$

and for $k > 1$ we put

$$\Delta^{(k)} v_i = \Delta(\Delta^{(k-1)} v_i), \quad \nabla^{(k)} v_i = \nabla(\nabla^{(k-1)} v_i).$$

Similarly for a net τ we define difference operators $\Delta_\tau^{(k)}$, $\nabla_\tau^{(k)}$: $\boldsymbol{R}^{\boldsymbol{Z}} \to \boldsymbol{R}^{\boldsymbol{Z}}$ putting for the coordinates ($i \in \boldsymbol{Z}$)

$$\Delta_\tau^{(0)} v_i = \nabla_\tau^{(0)} v_i = v_i,$$

$$\Delta_\tau^{(1)} v_i = \Delta_\tau v_i = \frac{1}{t_{i+1}-t_i} \Delta v_i, \qquad \nabla_\tau^{(1)} v_i = \nabla_\tau v_i = \frac{1}{t_i - t_{i-1}} \nabla v_i,$$

and for $k > 1$

$$\Delta_\tau^{(k)} v_i = \Delta_\tau(\Delta_\tau^{(k-1)} v_i), \qquad \nabla_\tau^{(k)} v_i = \nabla_\tau(\nabla_\tau^{(k-1)} v_i).$$

Composing several times in arbitrary succession the difference operators of the same kind, we obtain so-called mixed difference operators.

From the above definition we have

$$\Delta_\tau v_i = \nabla_\tau v_{i+1}, \tag{2.1}$$

and in consequence (under the assumption that $\sum_{i=p}^{q} a_i = 0$ for $q < p$, $a_i \in \boldsymbol{R}$) we get for any $s \in \boldsymbol{Z}$ and $i \in N$ ($N = \{0, 1, 2, \ldots\}$) the following formulae:

$$v_{s+i} = \begin{cases} v_s + \sum_{j=s}^{s+i-1} \Delta_\tau v_j (t_{j+1} - t_j), \\ v_s + \sum_{j=s+1}^{s+i} \nabla_\tau v_j (t_j - t_{j-1}), \end{cases} \tag{2.2}$$

$$v_{s-i} = \begin{cases} v_s - \sum_{j=s-i}^{s-1} \Delta_\tau v_j (t_{j+1} - t_j), \\ v_s - \sum_{j=s-i+1}^{s} \nabla_\tau v_j (t_j - t_{j-1}). \end{cases} \tag{2.3}$$

Similarly, the equality

$$\Delta_\tau \nabla_\tau v_i (t_{i+1} - t_i) = \Delta_\tau \nabla_\tau v_i (t_i - t_{i-1}) \tag{2.4}$$

and formulae (2.2), (2.3) yield the following formulae:

$$v_{s+i} = v_s + \Delta_\tau v_s (t_{s+i} - t_s) + \begin{cases} \sum_{j=s+1}^{s+i-1} \Delta_\tau \nabla_\tau v_j (t_{j+1} - t_j)(t_{s+i} - t_j), \\ \sum_{j=s+1}^{s+i-1} \nabla_\tau \Delta_\tau v_j (t_j - t_{j-1})(t_{s+i} - t_j), \end{cases} \tag{2.5}$$

$$v_{s-i} = v_s - \nabla_\tau v_s (t_s - t_{s-i}) - \begin{cases} \sum_{j=s-i+1}^{s-1} \Delta_\tau \nabla_\tau v_j (t_{j+1} - t_j)(t_j - t_{s-i}), \\ \sum_{j=s-i+1}^{s-1} \nabla_\tau \Delta_\tau v_j (t_j - t_{j-1})(t_j - t_{s-i}). \end{cases} \tag{2.6}$$

Finally, one can easily obtain the summation by parts formula

$$(2.7)\qquad \sum_{j=s}^{s+n-1}(\Delta_\tau u_j)v_j(t_{i+1}-t_j)+\sum_{j=s+1}^{s+n}u_j(\nabla_\tau v_j)(t_j-t_{j-1}) = \sum_{j=s}^{s+n-1}(\Delta_\tau u_j v_j)(t_{j+1}-t_j),$$

which will be needed in the sequel.

Throughout the paper, by a solution of the difference equation

$$(2.8)\qquad \nabla_\tau \Delta_\tau v_i = g(i, v_i, \Delta_\tau v_i), \quad i = s+1, \ldots, s+n-1$$

(g is a real function defined on the set $\{s+1, \ldots, s+n-1\}\times \boldsymbol{R}^2$, n is an integer >1, and s is a index from $\boldsymbol{Z}$) we mean a vector $v \in \boldsymbol{R}^{\boldsymbol{Z}}$ with the coordinates $v_s, \ldots, v_{s+n}$ satisfying equations (2.8) and with the remaining coordinates equal to zero.

3. We start with a Lemma, which will be applied also in [2].

LEMMA 1. *If for a fixed τ-net, integer $n>1$, and $s\in \boldsymbol{Z}$, the vector $v\in \boldsymbol{R}^{\boldsymbol{Z}}$ is a solution of the difference equation*

$$(3.1)\qquad \nabla_\tau \Delta_\tau v_i + q_i = 0, \quad i = s+1, \ldots, s+n-1, \ q_i \in \boldsymbol{R},$$

satisfying the condition

$$(3.2)\qquad v_s = v_{s+n} = 0,$$

then the coordinates of the vector v are given by the formulae

$$(3.3)\qquad v_{s+i} = \sum_{j=1}^{n-1}\Gamma^n_{s+i,s+j}q_{s+j}(t_{s+j}-t_{s+j-1}), \quad i = 1, \ldots, n-1,$$

or

$$(3.4)\qquad v_{s+n-i} = \sum_{j=1}^{n-1}\Gamma^n_{s+n-j,s+n-i}q_{s+n-j}(t_{s+n-j}-t_{s+n-j-1}), \quad i = 1, \ldots, n-1,$$

where the function Γ^n_s: $\{1, \ldots, n-1\}^2 \to \boldsymbol{R}$ defined by

$$(3.5)\qquad \Gamma^n_{s+i,s+j} = \begin{cases} \dfrac{(t_{s+n}-t_{s+i})(t_{s+j}-t_s)}{t_{s+n}-t_s}, & 1\leqslant j\leqslant i-1,\\[2mm] \dfrac{(t_{s+n}-t_{s+j})(t_{s+i}-t_s)}{t_{s+n}-t_s}, & i\leqslant j\leqslant n-1,\end{cases}$$

is the discrete analogue of Green's function in the theory of differential equations.

Formulae (3.3) and (3.4) are simple consequences of formulae (2.5) and (2.6), respectively, and of the assumptions of our lemma. Notice that the function Γ^n_s is positive and bounded,

$$(3.6)\qquad \Gamma^n_{s+i,s+j}\leqslant \frac{t_{s+n}-t_s}{4}.$$

LEMMA 2. *If for a fixed net* τ, *integer* $n > 1$ *and index* $s \in \mathbf{Z}$, *the vector* $v \in \mathbf{R}^{\mathbf{Z}}$ *is a non-trivial solution of difference equation*

$$\nabla_\tau \Delta_\tau v_i + p_i v_i = 0, \quad i = s+1, \ldots, s+n-1, \tag{3.7}$$

then the real numbers p_i *fulfil the inequality*

$$\sum_{j=1}^{n-1} |p_{s+j}|(t_{s+j} - t_{s+j-1}) \geqslant \frac{4}{t_{s+n} - t_s}. \tag{3.8}$$

Proof. Supposing that $|v_{s+i_0}| = \max_{s \leqslant i \leqslant s+n} |v_{s+i}|$ (where $i_0 \in \{1, \ldots, n-1\}$), we can write, owing to (3.3), the following inequality:

$$|v_{s+i}| \leqslant |v_{s+i_0}| \max_{1 \leqslant i,j \leqslant n-1} \Gamma^n_{s+i,s+j} \sum_{j=1}^{n-1} |p_{s+j}|(t_{s+j} - t_{s+j-1}), \quad i = 1, \ldots, n-1.$$

Now, to complete the proof, it is sufficient to make use of the assumption that $|v_{s+i_0}| > 0$ and to apply inequality (3.6).

In order to formulate Lemma 3 we need some additional notions. A division $\tau' = \{t'_i\}_{i \in \mathbf{Z}}$ will be called an *extended net* for a net $\tau = \{t_i\}_{i \in \mathbf{Z}}$ if $\{t'_i\}_{i \in \mathbf{Z}} \subset \{t_i\}_{i \in \mathbf{Z}}$ and if the set $\{t'_i\}_{i \in \mathbf{Z}} \setminus \{t_i\}_{i \in \mathbf{Z}}$ is finite.

For a τ-net and a vector $v \in \mathbf{R}^{\mathbf{Z}}$, let the mapping φ: $\mathbf{R} \to \mathbf{R}$ denote the piece-linear function whose graph is the polygonal with points (t_i, v_i) (for $i \in \mathbf{Z}$) as vertices.

LEMMA 3. *If, for a* τ-*net*, τ' *is an extended net such that*

$$t'_r = t_s, \quad t'_{r+m} = t_{s+n}, \quad m > n > 1$$

and a vector v *is a solution of difference equation* (3.7), *then the vector* v' *with the coordinates*

$$v'_i = \begin{cases} v_j & \text{if } t'_i = t_j, \\ \varphi(t'_i) & \text{if } t'_i \in \tau' \setminus \tau \end{cases} \tag{3.9}$$

satisfies the equation

$$\nabla_{\tau'} \Delta_{\tau'} v'_i + p'_i v'_i = 0, \quad i = r+1, \ldots, r+m-1, \tag{3.10}$$

where

$$p'_i = \begin{cases} -\dfrac{\nabla_{\tau'} \Delta_{\tau'} v'_i}{v'_i} & \text{if } v'_i \neq 0, \\ 0 & \text{if } v'_i = 0. \end{cases} \tag{3.11}$$

Moreover, the inequality

$$\sum_{j=1}^{n-1} |p_{s+j}|(t_{s+j} - t_{s+j-1}) \geqslant \sum_{j=1}^{m-1} |p'_{r+j}|(t'_{r+j} - t'_{r+j-1}) \tag{3.12}$$

holds true.

In order to prove Lemma 3 we start with the following

Remark 3.1. If a vector v is a solution of difference equation (3.7) and if there is an index $i \in \{s+1, \ldots, s+n-1\}$ such that $v_i = 0$, then $\nabla_\tau \Delta_\tau v_i = 0$; this means that the points (t_{i-1}, v_{i-1}), (t_i, v_i), (t_{i+1}, v_{i+1}) of the plane $\boldsymbol{R}^2$ lie on the same straight line.

Proof of Lemma 3. The definition of coefficients p_i' and the remark just made immediately imply the first part of the theorem. Thus, to complete the proof, it remains, owing to induction argument, to show that inequality (3.12) holds true if $r = s$ and $m = n+1$; i.e. if to the points $t_s, \ldots, t_{s+n}$ of the τ-net we add only one point t such that $t \in (t_s, t_{s+n})$.

Without loss of generality we can assume that $t_s < t < t_{s+1} < \ldots < t_{s+n}$. In this case the extended net τ' is of the form

$$t_i' = \begin{cases} t_i & \text{for } i \leqslant s, \\ t & \text{for } i = s+1, \\ t_{i-1} & \text{for } i \geqslant s+2, \end{cases}$$

and the coordinates of the vector v' are given by the formula

$$v_i' = \begin{cases} v_i & \text{for } i \leqslant s, \\ \varphi(t) & \text{for } i = s+1, \\ v_{i-1} & \text{for } i \geqslant s+2. \end{cases}$$

Let us put $I = \{i \in \{s+1, \ldots, s+n-1\} \colon v_i \neq 0\}$, and notice that I, for non-trivial solutions of difference equation (3.7), is a non-empty set (the proof in the case of $v = 0$ is trivial). Now, by the assumption that v is a solution of difference equation (3.7) we have

$$\sum_{j \in I} |p_j| (t_j - t_{j-1}) = \sum_{j \in I} \left| \frac{\Delta_\tau v_j - \Delta_\tau v_{j-1}}{v_j} \right|,$$

and by the definition of the set I we can write the inequality

$$\text{(3.13)} \qquad \sum_{j=1}^{n} |p_{s+j}| (t_{s+j} - t_{s+j-1}) \geqslant \sum_{j \in I} \left| \frac{\Delta_\tau v_j - \Delta_\tau v_{j-1}}{v_j} \right|.$$

Similarly, owing to the first part of our theorem, we obtain

$$\text{(3.14)} \quad \sum_{j=1}^{n} |p_{s+j}'| (t_{s+j}' - t_{s+j-1}') = |p_{s+1}'| (t_{s+1}' - t_s') + \sum_{j \in I} \left| \frac{\Delta_{\tau'} v_{j+1}' - \Delta_{\tau'} v_j'}{v_{j+1}'} \right|.$$

Notice that for $j \in I$ we have

$$\frac{\Delta_{\tau'} v_{j+1}' - \Delta_{\tau'} v_j'}{v_{j+1}'} = \frac{\Delta_\tau v_j - \Delta_\tau v_{j-1}}{v_j},$$

and that p_{s+1}' vanishes also if $v_{s+1}' = \varphi(t)$ is different from zero.

Therefore, equality (3.14) takes the form

$$\sum_{j=1}^{n} |p'_{s+j}|(t'_{s+j}-t'_{s+j-1}) = \sum_{j\in I} \frac{\Delta_\tau v_j - \Delta_\tau v_{j-1}}{v_j}, \tag{3.15}$$

and together with inequality (3.13) immediately gives the required inequality (3.12) in the considered case. Thus the proof is completed.

4. The lemmae stated in the preceding section allow us to prove the main theorem of this paper, namely the following

THEOREM 1. *If for a fixed τ-net such that*

$$t_{i+1}-t_i = t_{i+n+1}-t_{i+n}, \quad i \in \boldsymbol{Z} \tag{4.1}$$

(n denotes an integer >1) *the vector v of $\boldsymbol{R}^{\boldsymbol{Z}}$ is a non-trivial solution of the difference equation*

$$\nabla_\tau \Delta_\tau v_i + p_i v_i = 0, \quad i \in \boldsymbol{Z}, \tag{4.2}$$

satisfying the periodicity condition

$$v_i = v_{i+n}, \quad i \in \boldsymbol{Z}, \tag{4.3}$$

and if the coefficients p_i in equation (4.2) *fulfil the inequality*

$$\sum_{j=1}^{n} p_j(t_j-t_{j-1}) \geqslant 0, \tag{4.4}$$

then the inequality

$$\sum_{j=1}^{n} |p_j|(t_j-t_{j-1}) \geqslant \frac{16}{t_n-t_0} \tag{4.5}$$

holds true.

The idea of the proof of Theorem 1 was suggested to me by A. Lasota. We start with the following remark, which is a simple consequence of Remark 3.1.

Remark 4.1. If a vector v of $\boldsymbol{R}^{\boldsymbol{Z}}$ is a solution of difference equation (3.7) (but with $i \in \boldsymbol{Z}$) and if there is an index $i \in \boldsymbol{Z}$ such that

$$v_i = v_{i+1} = 0,$$

then the vector v is a trivial solution (i.e. $v_i = 0$ for $t \in \boldsymbol{Z}$).

Proof of Theorem 1. Suppose that all coordinates of the vector v have the same sign. We can assume that $v_i > 0$ for $i \in \boldsymbol{Z}$ (the case of $v_i < 0$ for $i \in \boldsymbol{Z}$ is quite analogous). Thus, from equation (4.2) we obtain

$$\frac{\nabla_\tau \Delta_\tau v_i}{v_i} = -p_i, \quad i = 1, \ldots, n.$$

Multiplying both sides of this equality by $(t_i - t_{i-1})$ and summing with respect to "i", we get

$$(4.6) \qquad \sum_{i=1}^{n} \frac{\nabla_\tau \Delta_\tau v_i}{v_i}(t_i - t_{j-1}) = -\sum_{i=1}^{n} p_i(t_i - t_{i-1}).$$

The left-hand side of this equality can be transformed by the summation-by-parts formula (2.7),

$$\sum_{i=1}^{n} \frac{1}{v_i} \nabla_\tau(\Delta_\tau v_i)(t_i - t_{i-1}) = \sum_{i=0}^{n-1} \Delta_\tau\left(\frac{\Delta_\tau v_i}{v_i}\right)(t_{i+1} - t_i) + \sum_{i=0}^{n-1} \frac{(\Delta_\tau v_i)^2}{v_i \cdot v_{i+1}}.$$

Hence, by assumption (4.3) and by (4.6) we obtain

$$(4.7) \qquad \sum_{i=0}^{n-1} \frac{(\Delta_\tau v_i)^2}{v_i \cdot v_{i+1}} = -\sum_{i=1}^{n} p_i(t_i - t_{i-1}),$$

which, by (4.4), is impossible.

This contradiction excludes the case considered above.

Thus, suppose that there exists a pair of indexes, k and l ($k, l \in \boldsymbol{Z}$), such that $v_k \cdot v_l \leqslant 0$.

By Remark 4.1 and the non-triviality and periodicity of v we can assume without loss of generality that

$$v_0 \leqslant 0, \quad v_1 > 0.$$

Let $k \in \{2, \ldots, n\}$ be the smallest index such that $v_k < 0$. The existence of such an index easily follows from assumption (4.3) (in the case of $v_0 = 0$ it follows from Remark 3.1 that $v_{-1} < 0$).

We thus have the following inequalities:

$$v_0 \leqslant 0, \quad v_1 > 0, \quad v_k < 0, \quad v_n \leqslant 0, \quad v_{n+1} > 0.$$

Now, we introduce the function φ defined in the preceding section and we extend the τ-net by adding all zeros of φ. Let the points t'_s, t'_r, t'_m of the extended net be equal, respectively, to the points t_0, t_k, t_n of the τ-net.

Introducing the vector v' and the coefficients p'_i ($i \in \boldsymbol{Z}$) in the same way as in the proof of Lemma 3, we have

1° $v'_{s+1} = v'_{r-1} = v'_{m+1} = 0$ in the case $v_0 < 0$ and

2° $v'_s = v'_{r-1} = v'_m = 0$ in the case $v_0 = 0$.

Now, applying successively Lemma 3 and Lemma 2 to the vectors $(\ldots, 0, v'_{s+1}, \ldots, v'_{r-1}, 0, \ldots)$ and $(\ldots, 0, v'_{r-1}, \ldots, v'_{m+1}, 0, \ldots)$ in case 1°, and using the evident equality $p_k(t_k - t_{k-1}) = p'_r(t'_r - t'_{r-1})$, we obtain the following inequalities:

$$\sum_{i=1}^{k-1} |p_i|(t_i - t_{i-1}) \geqslant \sum_{i=s+2}^{r-2} |p'_i|(t'_i - t'_{i-1}) \geqslant \frac{4}{t'_{r-1} - t'_{s+1}}$$

and

$$\sum_{i=k}^{n-1}|p_i|(t_i-t_{i-1}) \geqslant \sum_{i=r}^{m}|p_i'|(t_i'-t_{i-1}') \geqslant \frac{4}{t_{m+1}'-t_{r-1}'}.$$

By a similar reasoning on vectors $(\ldots, 0, v_s', \ldots, v_{r-1}', 0, \ldots)$ and $(\ldots, 0, v_{r-1}', \ldots, v_m', 0, \ldots)$ in case 2^0 we can get the following two inequalities:

$$\sum_{i=1}^{k-1}|p_i|(t_i-t_{i-1}) \geqslant \sum_{s+1}^{r-2}|p_i'|(t_i'-t_{i-1}') \geqslant \frac{4}{t_{r-1}'-t_s'},$$

$$\sum_{i=k}^{n-1}|p_i|(t_i-t_{i-1}) \geqslant \sum_{i=r}^{m-1}|p_i'|(t_i'-t_{i-1}') \geqslant \frac{4}{t_m'-t_{r-1}'}.$$

From these inequalities and from the evident equality

$$t_{m+1}'-t_{s+1}' = t_m'-t_s' = t_n-t_0,$$

it easily follows that

$$\sum_{i=1}^{n-1}|p_i|(t_i-t_{i-1}) \geqslant \frac{16}{t_n-t_0},$$

which completes the proof of our theorem.

From this theorem we obtain by contraposition the following

COROLLARY. *If the vector v of $\boldsymbol{R}^{\boldsymbol{Z}}$ is a periodic solution of difference equation* (4.2), *where the τ-net is such that* (4.1) *holds true and the coefficients p_i satisfy inequality* (4.4) *and are such that*

$$\sum_{j=1}^{n}|p_j|(t_j-t_{j-1}) < \frac{16}{t_n-t_0}, \tag{4.8}$$

then v is a trivial solution (i.e. $v_i = O$ for $i \in \boldsymbol{Z}$).

The corollary just stated is a discrete analogue of the well-known criterion of uniqueness for periodic solutions of linear differential equations (see [3], p. 86). Such a criterion in the discrete case, as a simple conclusion from the above corollary, may be stated as follows:

THEOREM 2. *If the coefficients p_i, q_i ($i \in \boldsymbol{Z}$) of the linear difference equation*

$$\nabla_\tau \Delta_\tau v_i + p_i v_i = q_i, \quad i \in \boldsymbol{Z}, \tag{4.9}$$

satisfy the condition

$$p_{i+n} = p_i, \quad q_{i+n} = q_i \tag{4.10}$$

and if inequalities (4.4), (4.8) *are fulfilled, then equation* (4.9) *has at most one periodic solution.*

For the proof it suffices to notice that the difference of two periodic solutions of equation (4.9) is a periodic solution of equation (4.2) and, by the corollary, it is equal to zero.

5. The example presented below proves that inequality (4.8) is the best possible in the sense that, if we replace the number 16 by any greater number, then the corollary fails (moreover, it is impossible to replace the sign "$<$" by "$\leqslant$").

Consider, namely, the difference equation

$$\nabla_\tau \Delta_\tau v_i + p_i v_i = 0, \quad i \in \mathbf{Z},$$

where

$$p_{2i} = 0, \quad p_{2i+1} = -2 \quad (i \in \mathbf{Z}), \quad \tau = \{i\}_{i \in \mathbf{Z}}.$$

For this equation we have

$$\sum_{i=1}^{4} |p_i|(t_i - t_{i-1}) = 4 = \frac{16}{t_4 - t_0},$$

but there exists a non-trivial periodic solution $v \in \mathbf{R}^{\mathbf{Z}}$ of the form

$$v_{2i} = 0, \quad v_{2i+1} = (-1)^i.$$

In a similar way one can show that inequality (3.8) (a discrete analogue of Beurling's inequality) is the best possible.

References

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[2] Z. Denkowski, *On the existence and uniqueness of periodic solutions for difference equations of second order*, Ann. Polon. Math. (to appear).

[3] A. Lasota et Z. Opial, *Sur les solutions périodic des équations différentielles ordinaires*, ibidem 16 (1964), p. 69 - 94.

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