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FREE PRODUCTS OF LIE GROUPS

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1. Introduction. In [1] Graev proved that the algebraic free product of two Hausdorff topological groups can be equipped with a topology making it into a Hausdorff topological group and satisfying the appropriate conditions for a coproduct in the category of topological groups; see also [2] and [5].

In [3] and [4] we showed that a free product of connected locally compact groups is never locally compact. We prove here that a free product of connected locally compact groups G_1 and G_2 has no small subgroups if and only if G_1 and G_2 have no small subgroups (that is, G_1 and G_2 are Lie groups). A similar result is true if G_1 or G_2 is only assumed to be compact.

2. Preliminaries.

Definition 1. Let G_1 and G_2 be topological groups. Then the topological group F is said to be a *free product* of G_1 and G_2, denoted by $G_1 * G_2$, if it has the following properties:

(a) G_1 and G_2 are subgroups of F.

(b) F is generated algebraically by $G_1 \cup G_2$.

(c) If γ_1 and γ_2 are continuous homomorphisms of G_1 and G_2, respectively, into a topological group H, then there exists a continuous homomorphism Γ of F into H such that $\Gamma = \gamma_i$ on G_i for $i = 1, 2$.

Of course a free product is simply the coproduct in the category of all topological groups. Thus, if it exists, it is unique up to isomorphism. Free products of topological groups were first studied by Graev [1]. He showed that the free product of Hausdorff groups exists and is Hausdorff. His proof showed that the underlying group structure of $G_1 * G_2$ is the algebraic free product of G_1 and G_2.

Definition 2. Let G be a group and X a subset which generates G algebraically. Then $a \in G$ is said to be of *length n with respect to X* if n is the least integer N such that $a = x_1^{\varepsilon_1} \ldots x_N^{\varepsilon_N}$, where $\varepsilon_i = \pm 1$ and $x_i \in X$

for $i = 1, \ldots, N$. The set of all elements in G of length not greater than m will be denoted by $G_m(X)$.

Our first preliminary theorem is essentially proved in [4].

THEOREM A. *For $i = 1, 2$, let G^i be a Hausdorff group with a compact subspace X_i which generates G^i algebraically. Further, let the topology of G^i be the finest group topology on G^i which induces the same topology on X_i. If $G = G^1 * G^2$ is the free product of G^1 and G^2, then $X = X_1 \cup X_2$ is a compact set which generates G algebraically and has the property that a subset V of G is closed if and only if $V \cap G_n(X)$ is compact, for each n.*

THEOREM B (see [4]). *If G is a connected locally compact group, then there exists a compact subset X of G such that*

(i) *X generates G algebraically,*

(ii) *the topology of G is the finest group topology on G which induces the given topology on X.*

Notation. We will use e to denote the identity element of a group.

Whenever we write $x_1 x_2 \ldots x_n \in G_1 * G_2$ we will mean that, for any i, $x_i \neq e$ and x_i and x_{i+1} are not both in G_1 or G_2.

2. Results.

THEOREM 1. *Let G^1 and G^2 be connected locally compact groups. Then $G = G^1 * G^2$ has no small subgroups if and only if G^1 and G^2 have no small subgroups.*

Proof. If $G^1 * G^2$ has no small subgroups, then G^1 and G^2, being subgroups of $G^1 * G^2$, have no small subgroups.

Conversely, assume that G^1 and G^2 have no small subgroups; that is, they are Lie groups. Let O_1 and O_2 be open neighbourhoods of e in G^1 and G^2, respectively, which contain no non-trivial subgroups.

For $i = 1, 2$, by Theorem B, we can find a compact subset X_i of G^i which generates G^i algebraically and is such that the topology of G^i is the finest group topology on G^i which induces the same topology on X_i. Put $X = X_1 \cup X_2$. Then, by Theorem A, X generates G algebraically and a subset C of G is closed if and only if $C \cap G_n(X)$ is compact for every n.

Let d be a compatible metric on G^1 and define

$$A_n = G_n^1(X_1) \cap \left\{x\colon\ x \in G^1 \text{ and } d(x, e) \geqslant \frac{1}{n}\right\} \quad \text{for each } n \geqslant 1.$$

Then A_n is compact.

Similarly define compact subsets B_n of G^2. We note that

(1) $$A_n \supseteq A_{n-1} \text{ and } B_n \supseteq B_{n-1} \quad \text{for every } n \geqslant 2$$

and

(2) $$\bigcup_{n=1}^{\infty} A_n = G^1 - \{e\} \quad \text{and} \quad \bigcup_{n=1}^{\infty} B_n = G^2 - \{e\}.$$

Define

$$V_1 = A_1 \cup B_1, \quad V_2 = A_2 B_2 \cup B_2 A_2, \quad V_3 = A_3 B_3 A_3 \cup B_3 A_3 B_3$$

and so on. Clearly, each V_n is compact.

For $n = 0, 1, 2, \ldots$ and $m = 1, 2, \ldots$, define

$$W_n^m = \{(x_1 \ldots x_n)^{-1} y x_1 \ldots x_n \colon \text{each } x_i \in G_m(X) \text{ and } y \in [A_m \cap (G^1 - O_1)] \cup [B_m \cap (G^2 - O_2)]\}.$$

We note that each W_n^m is compact; every element in W_n^m has length, with respect to X, greater than n.

Without loss of generality assume that there is an integer N such that

$$O_i \subseteq \left\{x \colon x \in G^i \text{ and } d(x, e) \leqslant \frac{1}{N}\right\}, \quad i = 1, 2.$$

Then, for $m > k > N$,

$$A_m \cap (G^1 - O_1) \cap G_k(X) = A_k \cap (G^1 - O_1) \cap G_k(X).$$

Thus, for $m > k > N$ and any $n > 0$,

$$W_n^m \cap G_k(X) = W_n^k \cap G_k(X).$$

Our comments on the sets W_n^m show that if

$$V_0 = \bigcup_{n=0}^{\infty} \bigcup_{m=1}^{\infty} W_n^m,$$

then

$$V_0 \cap G_k(X) = \Big[\bigcup_{n=0}^{k} \bigcup_{m=1}^{k} W_n^m\Big] \cap G_k(X) \quad \text{for any } k > N.$$

Thus $V_0 \cap G_k(X)$ is compact for every $k \geqslant 1$. By Theorem A, this implies that V_0 is closed in G.

We now define

$$V = \bigcup_{n=0}^{\infty} V_n.$$

For any k we have

$$V \cap G_k(X) = [V_0 \cap G_k(X)] \cup \Big[\bigcup_{n=1}^{\infty} V_n \cap G_k(X)\Big].$$

Since each element in V_n, $n \geqslant 1$, has the length greater than or equal to n, we see that

$$V \cap G_k(X) = [V_0 \cap G_k(X)] \cup \Big[\bigcup_{n=1}^{k} V_n \cap G_k(X)\Big].$$

Since V_0 is closed in G and each V_n, $n \geqslant 1$, is compact, we infer that $V \cap G_k(X)$ is compact for all k. Thus, by Theorem A, V is closed in G. Further, $e \notin V$. Thus $G - V$ is an open neighbourhood of e.

To complete the proof we only have to show that

(*) *For any $a \in G$ with $a \neq e$ we have $a^k \in V$ for some k.*

Suppose $a \in G$ and $a \neq e$. Then $a = x_1 \ldots x_n$. It follows from (1) and (2) that there exists an m such that $x_i \in A_m \cup B_m$ for $i = 1, \ldots, n$.

Case 1. The length of a^2 is greater than the length of a. Then there exists a k such that a^k has the length greater than m. Clearly, $a^k \in V_r$, where r denotes the length of a^k, since a^k is in the group generated by $\{x_1, \ldots, x_n\}$ and each $x_i \in A_r \cup B_r$. So $a^k \in V$, as required.

Case 2. The length of a^2 is less than or equal to the length of a. This can happen only if $a = (z_1 \ldots z_t)^{-1} y z_1 \ldots z_t$ for some $y \in G^1 \cup G^2$ and $t \geqslant 0$. Without loss of generality assume $y \in G^1$. Since O_1 contains no non-trivial subgroup, there exists a $k \geqslant 1$ such that $y^k \notin O_1$. By (2), there exists an m such that $y^k \in A_m$. Thus

$$a^k = (z_1 \ldots z_t)^{-1} y^k z_1 \ldots z_t \in V_0.$$

Hence $a^k \in V$ and the proof of (*) is complete.

In a similar manner we can prove

THEOREM 2. *Let G^1 and G^2 be locally compact groups which are either connected or compact. Then $G = G^1 * G^2$ has no small subgroups if and only if G^1 and G^2 have no small subgroups.*

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