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On a certain class of functions whose derivatives have a positive real part in the unit disc

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1. Introduction. It is well known that a function $f(z)$ analytic in a convex domain D is univalent in D if $\operatorname{Re}(f'(z)) > 0$ for $z \in D$. Recently MacGregor [3] investigated the properties of functions $f(z)$ which are analytic in the unit disc, satisfying $\operatorname{Re} f'(z) > 0$ for $|z| < 1$. Subsequently the same author [4] studied the properties of the subclass of functions satisfying the condition $|f'(z)-1| < 1$ for $|z| < 1$. In this paper we investigate the class of functions $f(z)$ analytic in the unit disc, satisfying for $|z| < 1$ the condition

$$|f'(z)-1|/|f'(z)+1| < a, \quad 0 < a \leqslant 1,$$

with the normalization $f(0) = 0, f'(0) = 1$. Denoting by $R(a)$, the class defined above we observe that $R(1)$ coincides with the class of functions whose derivatives have a positive real part in the unit disc. We obtain sharp coefficient estimates, distortion theorems and determine the radius of convexity for the class $R(a)$. The results of MacGregor [3] are obtained as a particular case of our theorems for $a = 1$.

2. Coefficient estimates. We first derive a lemma which provides us with representation formula for functions in the class $R(a)$.

LEMMA 2.1. *Let $H(z)$ be analytic for $|z| < 1$ and satisfy*

$$\left|\big(1-H(z)\big)/\big(1+H(z)\big)\right| < a, \quad 0 < a \leqslant 1,$$

for $|z| < 1$. Let further $H(0) = 1$. Then we have

$$H(z) = \big(1-z\varphi(z)\big)/\big(1+z\varphi(z)\big), \tag{2.1}$$

where $\varphi(z)$ is analytic and satisfies $|\varphi(z)| \leqslant a$ for $|z| < 1$ and any function $H(z)$ given by the above formula is analytic and satisfies $\left|\big(1-H(z)\big)/\big(1+H(z)\big)\right| < a$ for $|z| < 1$.

Proof. Let $H(z) = 1+c_1z+c_2z^2+\ldots$ for $|z| < 1$. Then $h(z) = \big(1-H(z)\big)/\big(1+H(z)\big)$ is analytic and satisfies $|h(z)| < a$ for $|z| < 1$. Also

$h(0) = 0$. Hence we can write, using Schwarz's lemma, $h(z) = z\varphi(z)$ where $\varphi(z)$ is analytic and satisfies $|\varphi(z)| \leqslant a$ for $|z| < 1$.

Expressing $H(z)$ in terms of $\varphi(z)$ we obtain

$$H(z) = \frac{1-z\varphi(z)}{1+z\varphi(z)},$$

where $|\varphi(z)| \leqslant a$. Also, if $H(z)$ is given by (2.1), clearly $H(z)$ is regular for $|z| < 1$ since $|z\varphi(z)| \leqslant a|z| < 1$ for $|z| < 1$. The function $W = (1-az)/(1+az)$ maps $|z| < 1$ onto the disc $|(1-W)/(1+W)| < a$ in the W-plane and the converse statement in the lemma follows from the above observation.

We now proceed to prove the following

THEOREM (2.2). *Let $f(z) \epsilon R(a)$ and let $f(z)$ be given by*

$$f(z) = z + a_2 z^2 + \ldots + a_n z^n + \ldots \tag{2.2}$$

for $|z| < 1$. Then $|a_n| \leqslant (2a)/n$ for $n = 2, 3, \ldots$ and these bounds are sharp for the functions

$$f(z) = \int_0^z (1+az^{n-1})/(1-az^{n-1})\,dz \tag{2.3}$$

for $n \geqslant 2$ and $|z| < 1$.

Proof. Since $f(z) \epsilon R(a), f'(z)$ satisfies the conditions of Lemma (2.1) and therefore has the representation

$$\begin{aligned} f'(z) &= \big(1-z\varphi(z)\big)/\big(1+z\varphi(z)\big) \\ &= \big(1-\omega(z)\big)/\big(1+\omega(z)\big), \end{aligned} \tag{2.4}$$

where $\omega(z) = z\varphi(z)$ is analytic for $|z| < 1$ and satisfies the conditions $|\omega(z)| < a$ for $|z| < 1$, $\omega(0) = 0$. Therefore

$$\omega(z)\big(1+f'(z)\big) = 1-f'(z)$$

and writing $\omega(z) = \sum \omega_k z^k$ for $|z| < 1$, we obtain

$$\Big(2 + \sum_{k=1}^{\infty}(k+1)a_{k+1}z^k\Big)\Big(\sum \omega_k z^k\Big) = -\sum_{k=1}^{\infty}(k+1)a_{k+1}z^k. \tag{2.5}$$

Equating corresponding coefficients on both sides of (2.5) we observe that the coefficient of z^n on the right-hand side of (2.5), depends only on $a_2, a_3, \ldots, a_n$ on the left of (2.5) and hence we write for $n \geqslant 1$

$$\Big(2 + \sum_{k=1}^{n-1}(k+1)a_{k+1}z^k\Big)\,\omega(z) = -\sum_{k=1}^{n}(k+1)a_{k+1}z^k + \sum_{k=n+1}^{\infty} b_k z^k, \tag{2.6}$$

say. Then, since $|\omega(z)| < a$, we have

$$a\left|2+\sum_{k=1}^{n-1}(k+1)a_{k+1}z^k\right| \geqslant \left|-\sum_{k=1}^{n}(k+1)a_{k+1}z^k+\sum_{k=n+1}^{\infty}b_kz^k\right|. \tag{2.7}$$

Squaring both sides of (2.7) and integrating round $|z| = r < 1$ we get

$$a^2\left\{4+\sum_{k=1}^{n-1}(k+1)^2|a_{k+1}|^2r^{2k}\right\} \geqslant \sum_{k=1}^{n}(k+1)^2|a_{k+1}|^2r^{2k}+\sum_{k=n+1}^{\infty}|b_k|^2r^{2k}$$

$$\geqslant \sum_{k=1}^{n}(k+1)^2|a_{k+1}|^2r^{2k}, \qquad n \geqslant 1.$$

Let $r \to 1$ and we find that

$$a^2\left(4+\sum_{k=1}^{n-1}(k+1)^2|a_{k+1}|^2\right) \geqslant \sum_{k=1}^{n}(k+1)^2|a_{k+1}|^2.$$

Hence

$$4a^2 \geqslant (1-a^2)\sum_{k=1}^{n-1}(k+1)^2|a_{k+1}|^2+(n+1)^2|a_{n+1}|^2$$

$$\geqslant (n+1)^2|a_{n+1}|^2.$$

Therefore, we obtain $|a_{n+1}| \leqslant (2a)/(n+1)$ for $n \geqslant 1$, whence follows $|a_n| \leqslant (2a)/n$ for $n \geqslant 2$.

Consider now the function $f(z) = \int_0^z (1+az^{n-1})/(1-az^{n-1})\,dz$, where $|z| < 1$. For this function we have

$$\frac{|f'(z)-1|}{|f'(z)+1|} = |az^{n-1}| < a \quad \text{for } |z| < 1.$$

Also, for $|z| < 1$, we have $f(z) = z+(2a/n)z^n+\ldots$ showing that the estimate is sharp.

3. Distortion theorems.

THEOREM (3.1). *If* $f(z) = z+a_2z^2+\ldots+a_nz^n+\ldots \in R(a)$, *then we have the following*

$$|f'(z)| \leqslant (1+a|z|)/(1-a|z|), \tag{3.1}$$

$$\mathrm{Re}f'(z) \geqslant (1-a|z|)/(1+a|z|), \tag{3.2}$$

$$|f(z)| \leqslant -|z|-(2/a)\log(1-a|z|), \tag{3.3}$$

$$|f(z)| \geqslant -|z|+(2/a)\log(1+a|z|). \tag{3.4}$$

All the above estimates are sharp.

Proof. From the representation (2.4) for $f'(z)$ we have

$$|f'(z)| = \left|\frac{1-z\varphi(z)}{1+z\varphi(z)}\right| \leqslant \frac{1+|z\varphi(z)|}{1-|z\varphi(z)|} \leqslant \frac{1+a|z|}{1-a|z|}$$

proving (3.1).

Again

$$\begin{aligned}\operatorname{Re} f'(z) &= \operatorname{Re}\{(1-z\varphi(z))/(1+z\varphi(z))\}\\ &= \operatorname{Re}\{(1-z\varphi(z))(1+\overline{z\varphi(z)})\}/(|1+z\varphi(z)|)^2\\ &= (1-|z\varphi(z)|^2)/(|1+z\varphi(z)|)^2\\ &\geqslant \frac{(1-|z\varphi(z)|)(1+|z\varphi(z)|)}{(1+|z\varphi(z)|)^2}\\ &= \frac{1-|z\varphi(z)|}{1+|z\varphi(z)|}\\ &\geqslant \frac{1-a|z|}{1+a'|z|}.\end{aligned}$$

This proves result (3.2).

We shall use (3.1) and (3.2) to derive (3.3) and (3.4) by integration as follows:

$$f(z) = \int_0^z f'(s)\,ds = \int_0^r f'(te^{i\varphi})e^{i\varphi}\,dt, \qquad \text{where } r = |z|.$$

Therefore

$$\begin{aligned}|f(z)| &\leqslant \int_0^r |f'(te^{i\varphi})|\,dt\\ &\leqslant \int_0^r (1+at)/(1-at)\,dt\\ &= -r-(2/a)\log(1-ar).\end{aligned}$$

Again

$$\begin{aligned}|f(z)| &\geqslant \int_0^r \operatorname{Re} f'(te^{i\varphi})\,dt\\ &\geqslant \int_0^r (1-at)/(1+at)\,dt\\ &= -r+(2/a)\log(1+ar).\end{aligned}$$

In all the preceding steps $r = |z| < 1$.

For the function $f(z) = -z-(2/a)\log(1-az)$ which satisfies $|(1-f'(z))/(1+f'(z))| = a|z| < a$ for $|z| < 1$, equality is attained in (3.3).

For the function $f(z) = -z+(2/a)\log(1+az)$ which also belongs to $R(a)$, equality is attained in (3.4).

If w is any complex number such that $|w| < -r+(2/a)\log(1+ar)$, it follows from Rouche's theorem that $f(z)$ and $f(z)-w$ have the same number of zeros in $|z| < r$, that is, precisely one.

Letting $r \to 1$, we deduce that $f(z) = w$ has exactly one root in $|z| < 1$ if $|w| < -1+(2/a)\log(1+a)$. Hence we have.

COROLLARY. *Every function $f(z) \in R(a)$ maps the unit disc onto a domain which contains the disc $|w| < -1+(2/a)\log(1+a)$.*

4. Radius of convexity for the class $R(a)$.

THEOREM (4.1). *Let $f(z) = z+a_2z^2+\ldots+a_nz^n+\ldots \in R(a)$, $0 < a \leqslant 1$. Then we have*

(a) *Each function $f \in R(a)$ maps $|z| < (\sqrt{2}-1)/a$ onto a convex domain or $(2-\sqrt{2})(1+\sqrt{3})/2 \leqslant a \leqslant 1$.*

(b) *Each function $f \in R(a)$ maps $|z| < \left\{\dfrac{a^2-1+\sqrt{(a^2-1)(a^2-4a-1)}}{2a(1+a)}\right\}^{1/2}$ onto a convex domain if $0 < a \leqslant (2-\sqrt{2})(1+\sqrt{3})/2$.*

The bounds for $|z|$ in (a) *and* (b) *are sharp.*

Proof. From the representation (2.4) for $f'(z)$ we get on differentiation

$$\frac{zf''(z)}{f'(z)} = -\frac{2\big(z\varphi(z)+z^2\varphi'(z)\big)}{1-\big(z\varphi(z)\big)^2}. \tag{4.1}$$

Also, since $|\varphi(z)| \leqslant a$ for $|z| < 1$, we have ([1], p. 18),

$$\frac{|\varphi'(z)|}{a} \leqslant \frac{1-|\varphi(z)/a|^2}{1-|z|^2}. \tag{4.2}$$

Using (4.1) and (4.2) we get

$$\begin{aligned}\frac{|zf''(z)}{f'(z)} &= \left\{\frac{2\,|z\varphi(z)+z^2\varphi'(z)|}{\big|1-\big(z\varphi(z)\big)^2\big|}\right\}\\ &\leqslant \left\{\frac{2\,\big(|z\varphi(z)|+|z^2\varphi'(z)|\big)}{1-|z\varphi(z)|^2}\right\}\\ &\leqslant \left\{\frac{\big(2\,|z\varphi(z)|(1-|z|^2)+|z|^2\big(a-|\varphi(z)|^2/a\big)\big)}{(1-|z|^2)\big(1-|z\varphi(z)|^2\big)}\right\}.\end{aligned}$$

Setting $a = |z|$, $t = |z\varphi(z)|$, the above inequality becomes on simplification

$$\left|\frac{zf''(z)}{f'(z)}\right| \leqslant \frac{2(a^2+t/a)(a-t)}{(1-a^2)(1-t^2)} \tag{4.3}$$

(here we observe that $0 \leqslant a < 1, 0 \leqslant t \leqslant \alpha a$ since $|\varphi(z)| \leqslant a$).

Therefore, $|zf''(z)/f'(z)| < 1$ provided

$$2(a^2 + t/\alpha)(a - t) < (1 - a^2)(1 - t^2),$$

that is, provided

$$t^2(1 - 2/\alpha - a^2) + 2t(1 - a^2) + a^2(1 + 2\alpha) - 1 < 0. \tag{4.4}$$

Denoting by $E(t)$ the left-hand side of (4.4), we observe that the derivative $E'(t)$ vanishes for $t = t_1 = \alpha(1 - a^2)/\big(2 + \alpha(a^2 - 1)\big)$. Now $E''(t) < 0$. Also $t_1 < \alpha a$ provided

$$\alpha a^3 + a^2 + a(2 - \alpha) - 1 > 0. \tag{4.5}$$

Let a_0 denote the only positive root in $(0, 1)$ of the equation

$$P(a) \equiv \alpha a^3 + a^2 + a(2 - \alpha) - 1 = 0. \tag{4.6}$$

Clearly, $t_1 > \alpha a$ if $a < a_0$ and hence $E'(t) < 0$ for $0 < t < \alpha a$. Therefore $E(\alpha a) < 0$ implies that $E(t) < 0$ for $0 \leqslant t \leqslant \alpha a$. The condition (4.4) is therefore satisfied if

$$a^2\alpha^2(1 - a^2 - 2/\alpha) + 2\alpha a(1 - a^2) + a^2(1 + 2\alpha) - 1 < 0.$$

The above inequality reduces to

$$(1 - a^2)(a^2\alpha^2 + 2\alpha a - 1) < 0$$

which is true for $a < (\sqrt{2} - 1)/\alpha$.

Again $(\sqrt{2} - 1)/\alpha < a_0$ if

$$P\big(((2)^{1/2} - 1)/\alpha\big) < 0. \tag{4.7}$$

Condition (4.7) will be satisfied provided

$$\big\{\alpha^2 - \alpha(2 - \sqrt{2}) + 2\sqrt{2} - 3\big\} > 0,$$

which, in turn, holds for

$$(2 - \sqrt{2})(1 + \sqrt{3})/2 < \alpha < 1,$$

and for

$$\alpha = (2 - \sqrt{2})(1 + \sqrt{3})/2, \quad (\sqrt{2} - 1)/\alpha = a_0 = (\sqrt{6} - \sqrt{2})/2.$$

This shows that $|zf''(z)/f'(z)| < 1$ and hence $\operatorname{Re}\{1 + zf''(z)/f'(z)\}$ is positive for $|z| < (\sqrt{2} - 1)/\alpha$ if α lies in the interval mentioned in part (a) of the theorem. It follows immediately that $f(z)$ maps $|z| < (\sqrt{2} - 1)/\alpha$ onto a convex domain. To see that the result is sharp, we choose

$$f(z) = -z+(2/a)\log(1+az)$$

for which $1+zf''(z)/f'(z) = 0$ with $z = (\sqrt{2}-1)/a$. Hence the function $f(z)$ of our choice is not convex in any disc $|z| < R$ if R exceeds $(\sqrt{2}-1)/a$. This proves the first part of the theorem.

To prove the second part, we observe that the value $t = t_1$ which makes $E'(t)$ vanish lies in the interval $(0, \alpha a)$ if a exceeds a_0. The maximum of $E(t)$ for $0 \leqslant t \leqslant \alpha a$ occurs at $t = t_1$ and condition (4.4) is satisfied if $E(t_1) < 0$. This reduces to the following inequality on simplification, i.e.,

$$Q(a) \equiv \alpha(1+\alpha)a^4+(1-\alpha^2)a^2+(\alpha-1) < 0. \tag{4.8}$$

Evidently condition (4.8) is satisfied for

$$a < a_1 = \left\{\frac{\sqrt{(1-\alpha^2)(1+4\alpha-\alpha^2)}-(1-\alpha^2)}{2\alpha(1+\alpha)}\right\}^{1/2}.$$

We observe that a_1 (< 1) is the smallest positive root of the equation $Q(a) = 0$. We shall now show that $a_1 > a_0$, if

$$\alpha < \frac{(2-\sqrt{2})(1+\sqrt{3})}{2}.$$

For

$$\alpha = \frac{(2-\sqrt{2})(1+\sqrt{3})}{2},$$

we see by computation that $a_1 = (\sqrt{6}-\sqrt{2})/2$ and $a_0 = (\sqrt{2}-1)/\alpha = (\sqrt{6}-\sqrt{2})/2$ and so $a_1 = a_0$ in this case. Also for $0 < \alpha \leqslant 1$ and a fixed a the expression $Q(a)$ increases (as α increases). Thus, if condition (4.8) holds for a certain interval of values of a with $\alpha = \alpha_1$, then (4.8) also holds with any $\alpha < \alpha_1$ and the same interval of values of a. Hence a_1 increases with decreasing α. And $\alpha = (2-\sqrt{2})(\sqrt{3}+1)/2$ corresponds to $a_1 = a_0$.

Thus for $\alpha < (2-\sqrt{2})(\sqrt{3}+1)/2$, we have $a_1 > a_0$. Therefore, it follows that if $\alpha < (2-\sqrt{2})(\sqrt{3}+1)/2$, $f(z)$ is convex for $|z| < a_1$. To see that the bound a_1 is sharp, we construct a function $f(z)$ as follows. We first define a number b by means of

$$\frac{a_1(a_1-b)}{1-ba_1} = \frac{(1-a_1^2)}{2+\alpha(a_1^2-1)}. \tag{4.9}$$

For $\alpha < (2-\sqrt{2})(\sqrt{3}+1)/2$, $a_1 > a_0$, the only positive root of the equation $P(a) = 0$. Therefore, we have $\alpha a_1^3+a_1^2+a_1(2-\alpha)-1 > 0$ which implies that

$$\frac{(1-a_1^2)}{2+a(a_1^2-1)} < a_1.$$

Therefore, we conclude from (4.9) that $0 < (a_1-b)/(1-ba_1) < 1$. This leads to $(a_1-b)^2 < (1-ba_1)^2$, that is, $(a^2-1)(1-b^2) < 0$. Since $a_1 < 1$, it follows that $b^2 < 1$. The bilinear transformation $(z-b)/(1-bz)$ maps $|z| < 1$ onto itself since $|b| < 1$. We define $\varphi(z)$ by

$$\varphi(z) = a(z-b)/(1-bz). \tag{4.10}$$

Then $|\varphi(z)| < a$ for $|z| < 1$. Direct computation shows that

$$\varphi'(z) = \frac{a^2-(\varphi(z))^2}{a(1-z^2)}. \tag{4.11}$$

Using (4.11) we find that

$$1-\frac{2(z\varphi(z)+z^2\varphi'(z))}{1-(z\varphi(z))^2} = 0,$$

when

$$1-(z\varphi(z))^2-2z\varphi(z)-2z^2(a^2-(\varphi(z))^2)/(a-az^2) = 0, \tag{4.12}$$

that is, when

$$(z\varphi(z))^2(2-a+az^2)-2az\varphi(z)(1-z^2)+a-z^2(a+2a^2) = 0. \tag{4.13}$$

Now using (4.9) and remembering that a_1 satisfies equation (4.8), it is easy to verify that equation (4.13) holds with $z = a_1$. A function $f(z)$ can now be constructed for which $f'(z) = (1-z\varphi(z))/(1+z\varphi(z))$, with $\varphi(z)$ given by (4.10). Then $f(z)\in R(a)$ since $|\varphi(z)| \leqslant a$, for $|z| < 1$. For this function $f(z)$, we have

$$1+\frac{zf''(z)}{f'(z)} = 1-\frac{2(z\varphi(z)+z^2\varphi'(z))}{1-(z\varphi(z))^2}.$$

The right-hand side of the above equation vanishes for $z = a_1$. Hence the function $f(z)$ of our choice is not convex in any disc $|z| < r$ if r exceeds a_1. The proof of the theorem is complete.

5. Some extremal properties of $f(z) = -z-(2/a)\log(1-az)$.

THEOREM (5.1). *The area of the image of $|z| < r$ $(0 < r < 1)$, for functions in $R(a)$ is maximal for $f(z) = -z-(2/a)\log(1-az)$. The length of the image of $|z| = r$ by functions in $R(a)$ is also maximised by the same function.*

Proof. If $f(z)\in R(a)$, $|(f'(z)-1)/(f'(z)+1)| < a$ and so $f'(z)$ assume values which lie in the disc on the segment joining the points $(1-a)/(1+a)$, $(1+a)/(1-a)$ as diameter.

Setting $F(z) = (1+az)/(1-az)$, it is clear that $F(z)$ is analytic and univalent for $|z| < 1$ and maps $|z| < 1$. Onto the open disc on the line-segment joining the points $(1-a)/(1+a)$, $(1+a)/(1-a)$ as diameter. Also $f'(0) = F(0) = 1$. Thus $f'(z)$ is subordinate to $F(z)$ for $|z| < 1$. $F(z)$ is the derivative of $f_0(z) = -z-(2/a)\log(1-az)$. By a theorem of Littlewood ([2], p. 484, Theorem 2) if $g(z)$ is subordinate to $G(z)$ for $|z| < 1$, then for any $k > 0$, we have

$$\int_0^{2\pi} |g(re^{i\varphi})|^k d\varphi \leqslant \int_0^{2\pi} |G(re^{i\varphi})|^k d\varphi.$$

Let $A_r(f)$ denote the area of the image of $|z| < r$ under $f(z)$. Application of Littlewood's inequality to the derivatives of functions in $R(a)$ with $k = 2$ yields

$$\begin{aligned} A_r(f) &= \int_0^r \Big[\int_0^{2\pi} |f'(te^{i\varphi})|^2 d\varphi \Big] t\,dt \\ &\leqslant \int_0^r \Big[\int_0^{2\pi} |F(te^{i\varphi})|^2 d\varphi \Big] t\,dt \\ &= A_r(f_0). \end{aligned}$$

If $L_r(f)$ denotes the length of the image of $|z| = r$ under $f(z)$ application of the inequality with $k = 1$ gives

$$\begin{aligned} L_r(f) &= \int_0^{2\pi} |f'(re^{i\varphi})| r\,d\varphi \\ &\leqslant \int_0^{2\pi} |F(re^{i\varphi})| r\,d\varphi \\ &= L_r(f_0). \end{aligned}$$

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