Vector-valued analytic functions

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Abstract. The problems arising in the theory of vector-valued analytic functions when the topological vector spaces are not locally convex were discussed in this talk. In this summary, I have chosen to stress a class of "good analytic functions" which behave as analytic functions do, rather than the counter-examples which one can meet when the analytic functions are not assumed to be good. Sufficient conditions, that ensure that a vector-valued function is good are also given.

1. Good analytic functions.

DEFINITION 1. Let U be a complex manifold and E a complete topological vector space. A mapping $f\colon U \to E$ is a *good analytic function* if we can associate to each $z \epsilon U$ a neighbourhood V_z of z, a Banach space A_z, a holomorphic mapping $f_z\colon V_z \to A_z$ and a continuous linear mapping $\varphi_z\colon A_z \to E$ in such a way that $f = \varphi_z \circ f_z$ on V_z.

It is no loss in generality to assume that $A_z \subset E$, and that $\varphi_z\colon A_z \to E$ is the identity. Just replace A_z by $A_z/\mathrm{Ker}\,\varphi_z$, and identify this with the range of φ_z.

If K is compact in U, we find a finite number of $z_1, \ldots, z_r \epsilon K$ such that the neighbourhoods $V_{z_1}, \ldots, V_{z_r}$ cover K, and define $A = \sum_{1}^{r} A_{z_i}$, with the norm

$$\|a\| = \inf\left\{\sum \|a_i^r\|_i \mid a_i \epsilon A_{z_i}, \sum a_i = a\right\},$$

A is a Banach space, it is normed because its unit ball is a bounded subset of E, and a quotient of the Banach space $A_{z_1} \oplus \ldots \otimes A_{z_r}$ by a closed subspace. This proves

PROPOSITION 1. *Let* $f\colon U \to E$ *be a good analytic function. We can associate to each compact subset* $K \subseteq U$ *a neighbourhood* V *of* K *and a Banach space* $A \subseteq E$ *with continuous inclusion, such that* $fV \subseteq A$, *the mapping* $f\colon V \to A$ *being holomorphic.*

Good analytic functions have all the properties we can hope for since they are, at least locally, and on neighbourhoods of compact sets, ana-

lytic mappings into Banach spaces. In a way, the introduction of these analytic functions is a lazy generalization of analytic function theory.

It turns out that I do not know of a generalization of analytic function theory, where the functions have the properties expected of holomorphic functions, but where the analytic functions are not good. This is not a theorem. It is not even a conjecture, I have not said what a reasonable class of analytic functions would be, nor what part of standard theory should generalize. It is an observation.

2. The galb of a topological vector space.

DEFINITION 2. Let E be a topological vector space. A sequence $\{\lambda_k\}_{k\in N}$ belongs to the galb $G(E)$ of E if we can associate to each neighbourhood U of the origin in E a neighbourhood V of the origin in such a way that

$$(1) \qquad U \supseteq \bigcup_n \sum_0^n \lambda_k V.$$

A filter $\mathfrak{F}$ *tends to zero* in $G(E)$ if for each neighbourhood U, we can find a neighbourhood V and an $A \in \mathfrak{F}$ in such a way that inclusion (1) holds for all $\{\lambda_n\} \in A$.

The galb of a topological vector space has been defined by P. Turpin ([1], Chapter II). Turpin defines also the galb of a linear mapping $E \to F$, the galb defined above being the galb of the identity $E \to E$, but we shall not need this more general class of galbs here. He gives many examples of spaces with a non trivial galb, of spaces whose galb can be computed, among others some countable intersections of non-locally convex Orlicz spaces.

Except when E is a vector space with the gross topology (or simply the null space) we always have the continuous inclusion $G(E) \subseteq l_1$. For the topology to be locally convex, it is necessary and sufficient that $G(E) = l_1$ bicontinuously. E is locally p-convex if $l_p \subseteq G(E)$ continuously and locally pseudo-convex if $\bigcap_{p>0} l_p \subseteq G(E)$ continuously.

We recall that a p-semi-norm is a functional $\nu_p \colon E \to \boldsymbol{R}_+$ on a vector space, such that $\nu_p(x+y) \leqslant \nu_p(x) + \nu_p(y)$, $\nu_p(\lambda x) = |\lambda|^p \nu_p(x)$. A vector space topology is locally p-convex when it can be defined by a family of p-semi-norms with constant p. It is locally pseudo-convex (or locally 0_+ convex) if it can be determined by a family of p-semi-norms, with $p > 0$ for each semi-norm, but p depending on the semi-norm. In our characterization of locally p-convex and locally pseudo-convex spaces, we topologize l_p by the p-norm $\| \ \|_p$, and $\bigcap_{p>0} l_p$ by the upper bound of the topologies induced by the spaces l_p.

Another galb is important in the theory of analytic functions. After P. Turpin, we shall call it the exponential galb. It is the smallest galb

of a topological vector space that contains the sequence $\{a^k\}_{k \in N}$ for some $a > 0$. This galb contains the sequence $\{\lambda_k\}$ if $M < \infty$, $r \epsilon N$, $a < 1$ can be found such that $|\lambda_k^+| < Ma^{k^{1/r}}$, where $|\lambda_k^+|$ is the decreasing rearrangement of the sequence $|\lambda_k|$. A subset B of the analytic galb is bounded if $M < \infty$, $a < 1$, $r \epsilon N$ can be found such that $|\lambda_k^+| < Ma^{k^{1/r}}$ for all $\lambda \epsilon A$. A filter $\mathfrak{F}$ tends to zero if $\mathfrak{F}$ is finer than the filter $\{\epsilon B \mid \epsilon > 0\}$ for some bounded, balanced B.

3. Analytic functions and the exponential galb.

PROPOSITION 2. *Let* $(E, \mathscr{T})$ *be a topological vector space with the exponential galb,* $\{e_{n_1 \ldots n_k}\}$ *a bounded system of elements in* E. *The series*

$$\sum e_{n_1 \ldots n_k} z_1^{n_1} \ldots z_k^{n_k}$$

converges then on the unit polydisc. Its sum is a good analytic function there.

This is easy. The crucial step is the observation that the decreasing rearrangement of the countable set $a^{r_1 + \cdots + r_k} (\forall i: r_i \epsilon N)$ is less than $a^{r^{1/k}}$, and this is in the galb. We consider the set of series

$$\sum u_{r_1 \ldots r_k} e_{r_1 \ldots r_k}, \qquad |u_{r_1 \ldots r_k}| \leqslant a_1^{r_1 + \cdots + r_k}$$

for some $a_1 < a$. These series all converge. The set B of their sums is bounded, absolutely convex, and $\sum e_{r_1 \ldots r_k} z_1^{r_1} \ldots z_k^{r_k}$ converges in E_B, the vector space absorbed by B normed by the Minkowski functional of B. From there, everything is standard.

4. Holomorphy and pseudo-convexity.

A differentiable function can be approached faster by functions of finite rank than a continuous one. A function of class C_r or C_∞ can be approached still faster. If f is approached fast enough by functions of finite rank, and if the galb of E is large, f is continuous E_B-valued for some bounded absolutely convex B. When the circumstances are even more favorable, this happens when the derivatives of f are approximated fast enough, f is even an E_B-differentiable function. If now f is a complex-differentiable function, and is E_B-differentiable, we see that f is an E_B-valued holomorphic function, i.e. f is a good holomorphic function.

DEFINITION 3. Let U be an open set in R^n, E a topological vector space, and $r \epsilon R_+$. A mapping f: $U \to E$ is said to be a *class* C_r, to belong to $C_r(U, E)$ when functions $f_k = f_{k_1 \cdots k_n}$ exist, for $k \epsilon N^n$, $|k| = k_1 + \ldots + k_n \leqslant r$ such that $f_0 = f$, and

$$\omega_k(x, y) = \frac{1}{|x - y|^{r - |k|}} \left[f_k(y) - \sum_{|l| \leqslant r - |k|} f_{k+l}(x) \frac{(y - x)^l}{l!} \right]$$

is a continuous function of (x, y) which vanishes on the diagonal.

A function is thus of class C_r when it has derivatives to the order $[r]$ (the largest interger in r) and when these derivatives have the properties expected from the limited Taylor expansion theorem. Counter-examples show that these properties do not follow from the fact that f has derivatives, which are themselves differentiable, to the order $[r]$, the last derivatives satisfying a o-Holder condition with exponent $r-[r]$, when f is E-valued, E not locally convex.

Of course, a function f is of class C_∞ when it is of class C_r for all r. The following approximation theorem has been proved by P. Turpin and the author [3].

THEORES 3. *Let $r_1 < r_2$ and let V be open relatively compact in $U \subseteq R^n$. It is possible to find a bounded sequence of elements $u_1, ..., u_k, ...$ of $C_{r_1}(U)$ and an equicontinuous sequence of linear mappings $t_1, ..., t_k, ...$ of $C_{r_2}(U,E)$ into E in such a way that*

$$f(x) = \sum k^{(r_1-r_2)/n} t_k(f) u_n(x)$$

for all $f \in C_{r_2}(U, E)$, all $x \in V$, the convergence of the rigtht-hand side to f being valid in the vector space topology of $C_{r_1}(U,E)$.

THEORES 4. *Let now $U \subseteq C$ be open, $V \subseteq U$ relatively compact, let E be locally p-convex. and $f \in C_r(U, E)$ with $pr+p > 2$. A bounded, absolutely convex subset B of E exists, such that $f \in C_1(V, E_B)$.*

We note that $r > (2/p)-1$, and $(2/p)-1 > 1$, so $1 < r$. Apply Theorem 3 with $r_1 = 1$, $r_2 = r$. We have $f = \sum k^{(1-r)/2} t_k(f) u_k$. The set B is the closed, absolutely convex hull of the sequence $k^{(1-r)/2} t_k(f)$.

COROLLARY. *An E-valued function of class C_r on $U \subseteq C$ is a good analytic function if it is complex differentiable, if E is locally p-convex and $pr+p > 2$. If E is locally pseudo-convex, if f is an E-valued function of class C_∞, and is complex differentiable, then E is a good analytic function.*

5. Some counter-examples.

Counter-example show that some assumptions about a topological vector space E are necessary if we want E-valued functions on a complex domain to have the properties we hope for when they satisfy one, or another definition of an analytic function

The mapping $t \mapsto u_t$, where $u_t(z) = \dfrac{1}{t-z}$, $C^* \to L_p(I+iI)$ is a vector-valued function of class C_r on the complex sphere, as soon as $pr+p < 2$, and it is a solution of the Cauchy Riemann equation yet it is not a good analytic function (it is not of class C_∞ on the unit square $I+iI$). We know that an E-valued function u on a complex domain is a good analytic function if E is p-normed, if u is of class C_r, and a solution of the Cauchy-Riemann equation, and if $pr+p > 2$. (See [5], p. 146–150 for a proof that u_t is of class C_r when $pr+p < 2$.)

Turpin shows ([1], proposition 9.2.3) that the boundedness of E is the analytic galb if $\sum \lambda^n e_n$ converges on the open unit disc as soon as $\sum e_n$ converges. In other words, the boundedness of E has the analytic galb if we can associate to every power series $\sum e_n z^n$ an open disc D such that $D \subseteq X \subseteq \bar{D}$, where X is the domain of convergence of the series.

If E is metrizable and does not have the analytic galb, and if X is an infinite compact space, it can be shown (unpublished) that $C(X, E)$ contains a sequence f_n, which tends to zero, but is such that $\sum z^n f_n$ diverges for all $z \neq 0$. W. Żelazko [6] has shown that a sequence g_n of elements of $C(X, E)$ can then be associated to every finite set of points, in such a way that $\sum z^n g_n$ converges exactly on the union of this finite set of points and the origin.

Finally, Turpin ([1], paragraph 9, and [2]) has given an example of a nonconstant vector-valued function on the complex sphere, which is locally the sum of a power series, in the neighbourhood of each point of its domain and at infinity.

References

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