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On the structure of the set of solutions of a functional equation with applications to boundary value problems

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Abstract. An application of the Lasota–Opial covering theorem to the study of the structure of the set of solutions of a functional equation in a Banach space is given. In particular, it is proved that the set of solutions of the Nicoletti problem and of the Floquet boundary value problem for the first order differential equation is nonempty, compact and connected.

This note is strictly connected with Aronszajn's paper [1] whose main theorem states that the set of all fixed points of a completely continuous operator in a Banach space is an R_δ-set. The main idea of the proof is to find approximations and suitably associated vector fields so that the following conditions are satisfied: the vector fields are homeomorphisms, have a nonempty set of zeros, and give good frames (in the sense of the Hausdorff distance) of the set of all fixed points of the approximated operator. Approximations of a compact operator satisfying the above conditions have been studied by several authors by means of the topological degree arguments and have had many applications to integral or differential equations (compare, for example, [2], [3], [7], [8]).

In this note we prove two theorems concerning the topological structure of the set of all fixed points of a completely continuous (or compact) operator which has approximations with "multivalued regulators" of Lasota–Opial type ([6]). The Lasota–Opial conditions formulated with the help of a multivalued map allow using the domain invariance theorem to prove that the above approximations are ones of Aronszajn type. As a result we deduce that the set of all fixed points of the approximated operator is an R_δ-set. It is known that an R_δ-set is acyclic, in particular it is nonempty, compact, and connected. On the other hand, the Lasota–Opial conditions appear in a natural way in boundary value problems. As applications we prove that the sets of all solutions of the Cauchy problem, the Nicoletti problem and the Floquet boundary value problem for first order ordinary differential equations with Carathéodory right sides are R_δ-sets.

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1. The structure of the set of solutions of a functional equation. In this section we prove two topological theorems which describe the sets of fixed points of some maps in a normed linear space.

Recall that a subset of a metric space is said to be an *R_δ-set* if it is the intersection of a decreasing sequence of compact absolute retracts.

Let (X, ϱ) be a metric space. If A, B are two bounded closed nonempty subsets of X, $d(A, B)$ denotes the Hausdorff distance of A and B; that is,

$$d(A, B) = \max(\sup\{\varrho(x, A)\colon x \in B\}, \sup\{\varrho(x, B)\colon x \in A\}),$$

where $\varrho(x, A)$ denotes the distance of x from A. N. Aronszajn has proved the following theorem:

THEOREM A ([1]). *Suppose that there is a sequence $\{A_k\}$ of subsets of a metric space such that $A \subset A_k$ ($k \in \mathbf{N}$), where A_k are compact absolute retracts, and $d(A_k, A) \to 0$ as $k \to \infty$. Then A is an R_δ-set.*

Let E be a normed linear space. A map $h\colon E \to E$ is *compact* (*completely continuous*) if it is continuous and $\overline{h(E)}$ is compact ($\overline{h(A)}$ is compact for any bounded $A \subset E$). By $n(E)$ we denote the family of all nonempty subsets of E. For a set $A \subset E$ a multivalued map $H\colon A \to n(E)$ is called *upper semi-continuous* (u.s.c) if its graph $\{(x, y)\colon y \in H(x)\}$ is closed in $A \times E$. We say that $H\colon A \to n(E)$ is *completely continuous* if it is u.s.c. and for any bounded subset B of A, the set $\overline{\bigcup_{x \in B} H(x)}$ is compact. In what follows we use the following Lasota–Opial theorem.

THEOREM B ([6]). *Let U be a neighbourhood of 0 in a normed linear space E. Suppose that a continuous map $h\colon E \to E$ and a completely continuous map $H\colon U \to n(E)$ satisfy the following conditions:*

$$x \in H(x),\ x \in U \Rightarrow x = 0,$$

$$h(x) - h(y) \in H(x - y) \quad \textit{for } x - y \in U.$$

Then the equation $x = h(x)$ has exactly one solution and $I - h\colon E \to E$ (I denotes the identity map of E) is a homeomorphism. Furthermore, for every $\varepsilon > 0$ such that $B(0, 2\varepsilon) \subset U$ there exists $\delta > 0$, independent of x, such that for every $x \in E$

$$(I - h)(B(x, \varepsilon)) \supset B(x - h(x), \delta)$$

($B(x, \varepsilon)$ denotes the open ball about x of radius ε).

THEOREM 1. *Let U be a neighbourhood of 0 in a normed linear space $(E, \|\cdot\|)$. Suppose that a completely continuous map $h\colon E \to E$, continuous maps $h_k\colon E \to E$ ($k \in \mathbf{N}$), and completely continuous maps $H_k\colon U \to n(E)$ ($k \in \mathbf{N}$) satisfy* *the following conditions*:

$$\text{(1.1)} \qquad h_k \text{ tends to } h \text{ uniformly on } E \text{ as } k \to \infty,$$

$$\text{(1.2)} \qquad \forall k \in \mathbf{N}\ \forall x \in U\ (x \in H_k(x) \Rightarrow x = 0),$$

$$\text{(1.3)} \qquad \forall k \in \mathbf{N}\ \forall x, y \in E\ (x - y \in U \Rightarrow h_k(x) - h_k(y) \in H_k(x-y)),$$

$$\text{(1.4)} \qquad \bigcup_{k \in \mathbf{N}} \mathrm{Fix}(h_k) \text{ is bounded}.$$

Then Fix(h), *the set of all fixed points of* h, *is an* R_δ-*set*.

Proof. First, we show that Fix(h) is nonempty. By (1.1), without loss of generality we may assume that

$$\text{(1.5)} \qquad \|h_k(x) - h(x)\| \leqslant 1/k, \qquad x \in E,\ k \in \mathbf{N}.$$

By Theorem B for any $k \in \mathbf{N}$ there exists a fixed point x_k of h_k. By (1.4) the sequence $\{x_k\}$ is bounded. Since h is completely continuous, we may assume (passing to a subsequence if necessary) that $\{h(x_k)\}$ is convergent to some x. By (1.5) we have

$$\|x_k - h(x_k)\| = \|h_k(x_k) - h(x_k)\| \leqslant 1/k.$$

Therefore, $x_k \to x$, $h(x_k) \to h(x)$, and $x = h(x)$.

Define T_k: $E \to E$ ($k \in \mathbf{N}$) by $T_k(x) = x - h_k(x)$. Also, define sets A_k and B_k ($k \in \mathbf{N}$) by

$$\text{(1.6)} \qquad A_k = T_k^{-1}(\{x \in E\colon \|x\| \leqslant 1/k\}),$$

$$\text{(1.7)} \qquad B_k = T_k^{-1}\big(\overline{\mathrm{co}}\big(T_k(\mathrm{Fix}(h))\big)\big)$$

(if $X \subset E$, then $\overline{\mathrm{co}}(X)$ denotes the closed convex hull of X). By (1.5) we have

$$\text{(1.8)} \qquad \mathrm{Fix}(h) \subset B_k \subset A_k, \qquad k \in \mathbf{N}.$$

From Theorem B it follows that T_1: $E \to E$ is a homeomorphism and if $B(0, 2\varepsilon) \subset U$ ($\varepsilon > 0$), then there exists $\delta > 0$ such that $T_1(B(x, \varepsilon)) \supset B(T_1(x), \delta)$ for every $x \in E$. It easily follows that for every $m \in \mathbf{N}$ and $x \in E$

$$B(x, m\varepsilon) \supset T_1^{-1}\big(B(T_1(x), m\delta)\big).$$

Therefore, A_1 (and so Fix(h)) is bounded. Since h is completely continuous, we conclude that Fix(h) is compact. Thus, the sets $\overline{\mathrm{co}}\big(T_k(\mathrm{Fix}(h))\big)$ are nonempty, compact, and convex, and so they are compact absolute retracts. Since by Theorem B, the T_k are homeomorphisms, by virtue of (1.7) the sets B_k are also compact absolute retracts. From Theorem A it follows that the proof will be completed if we show that $d(B_k, \mathrm{Fix}(h)) \to 0$ as $k \to \infty$. By (1.8), it suffices to show that $d(A_k, \mathrm{Fix}(h)) \to 0$ as $k \to \infty$. Suppose, on the contrary, that there exists a sequence $\{a_k\}$ such that $a_k \in A_k$ and $d(a_k, \mathrm{Fix}(h)) \geqslant c$ for $k \in \mathbf{N}$, where $c > 0$. We may choose $\{a_k\}$ in such a way that $d(a_k, \mathrm{Fix}(h)) = c$ $(k \in \mathbf{N})$ because $\mathrm{Fix}(h) \subset A_k$ and A_k is connected. Since $\mathrm{Fix}(h)$ is bounded, we conclude that $\{a_k\}$ is bounded. As h is completely continuous, we may assume (passing to a subsequence if necessary) that $\{h(a_k)\}$ is convergent to some a. From (1.5) and (1.6) it follows that

$$\|h(a_k)-a_k\| \leqslant \|h(a_k)-h_k(a_k)\| + \|h_k(a_k)-a_k\| \leqslant 2/k. \tag{1.9}$$

Thus, $a_k \to a$, $h(a_k) \to h(a)$, $a = h(a)$, $a \in \mathrm{Fix}(h)$, and $d(a, \mathrm{Fix}(h)) = c$. This contradiction finishes the proof.

THEOREM 2. *Let U be a neighbourhood of* 0 *in a normed linear space* $(E, \|\cdot\|)$. *Suppose that a compact map h: $E \to E$, continuous maps h_k: $E \to E$ $(k \in \mathbf{N})$, and completely continuous maps H_k: $U \to n(E)$ $(k \in \mathbf{N})$ satisfy* (1.1)–(1.3). *Then* $\mathrm{Fix}(h)$ *is an R_δ-set.*

Proof. This is a consequence of Theorem 1. As before,we may assume that $\|h(x)-h_k(x)\| \leqslant 1/k$ for $x \in E$ and $k \in \mathbf{N}$. We must only show that (1.4) holds. To this end, suppose that $h_k(x) = x$ for some $x \in E$ and $k \in \mathbf{N}$. Then we have

$$\|x-h(x)\| = \|h_k(x)-h(x)\| \leqslant 1/k.$$

This ends the proof because $h(E)$ is bounded.

2. Applications. We say that f: $[a, b] \times \mathbf{R}^n \to \mathbf{R}^n$ is a *Carathéodory map* if $f(\cdot, x)$: $[a, b] \to \mathbf{R}^n$ is measurable for every $x \in \mathbf{R}^n$ and $f(t, \cdot)$: $\mathbf{R}^n \to \mathbf{R}^n$ is continuous for every $t \in [a, b]$. By E we denote the space $C([a, b], \mathbf{R}^n)$ of all continuous maps from $[a, b]$ to $\mathbf{R}^n$ with the norm $\|x\| = \sup\{|x(t)|: t \in [a, b]\}$, where $|\cdot|$ denotes the Euclidean norm in $\mathbf{R}^n$.

Consider the Cauchy problem

$$\begin{cases} x'(t) = f(t, x(t)) & \text{a.e. on } [a, b], \\ x(a) = x_0 & (x_0 \in \mathbf{R}^n), \end{cases} \tag{2.1}$$

where f: $[a, b] \times \mathbf{R}^n \to \mathbf{R}^n$ is a Carathéodory map. By a *solution* of (2.1) we mean any absolutely continuous map x: $[a, b] \to \mathbf{R}^n$ satisfying (2.1).

LEMMA. *Let g: $[a, b] \times \mathbf{R}^n \to \mathbf{R}^n$ be a Carathéodory map such that*

$$|g(t, x)| \leqslant \gamma(t), \qquad t \in [a, b],\ x \in \mathbf{R}^n,$$

where γ: $[a, b] \to [0, \infty)$ is an integrable function. Then for every $\varepsilon > 0$ there exists a locally Lipschitzian map $\bar{g}$: $[a, b] \times \mathbf{R}^n \to \mathbf{R}^n$ such that

$$\int_a^b \sup_{z \in \mathbf{R}^n} |g(t, z)-\bar{g}(t, z)|\, dt < \varepsilon.$$

This can be proved in the same way as Lemma 2 in [3].

THEOREM 3. *Let $f\colon [a, b] \times \mathbf{R}^n \to \mathbf{R}^n$ be a Carathéodory map satisfying*

$$|f(t, x)| \leqslant \alpha(t) + \beta(t)|x|, \qquad t \in [a, b],\ x \in \mathbf{R}^n, \tag{2.2}$$

where $\alpha, \beta\colon [a, b] \to [0, \infty)$ are integrable functions. Then the set of all solutions of (2.1) *is an R_δ-set in E.*

Proof. A map x is a solution of (2.1) if and only if it is a continuous solution of the integral equation

$$x(t) = x_0 + \int_a^t f(s, x(s))\,ds, \qquad t \in [a, b]. \tag{2.3}$$

From the Gronwall inequality and (2.2) it follows that there exists $M > 0$ such that for every solution x of (2.3), $\|x\| \leqslant M$. Let $r\colon \mathbf{R}^n \to \overline{B(0, M)} \subset \mathbf{R}^n$ be a retraction such that $|r(x) - r(y)| \leqslant |x - y|$ for $x, y \in \mathbf{R}^n$. Note that $g\colon [a, b] \times \mathbf{R}^n \to \mathbf{R}^n$ defined by $g(t, x) = f(t, r(x))$ is Carathéodory. It is easy to check that (2.3) is equivalent to

$$x(t) = x_0 + \int_a^t g(s, x(s))\,ds, \qquad t \in [a, b]. \tag{2.4}$$

Notice that by the definition of r we have

$$|g(t, x)| = |f(t, r(x))| \leqslant \alpha(t) + \beta(t)|r(x)| \leqslant \alpha(t) + \beta(t)M.$$

Therefore, by the Lemma for each $k \in \mathbf{N}$ there exists a continuous map $g_k\colon [a, b] \times \mathbf{R}^n \to \mathbf{R}^n$ such that

$$\int_a^b \sup_{w \in \mathbf{R}^n} |g_k(t, w) - g(t, w)|\,dt < 1/k,$$

$$|g_k(t, x) - g_k(t, y)| \leqslant C_k|x - y|, \qquad |x| \leqslant M,\ |y| \leqslant M,\ t \in [a, b],$$

where C_k is a constant. Define $h\colon E \to E$, $h_k\colon E \to E$, and $H_k\colon E \to n(E)$ ($k \in \mathbf{N}$) by

$$h(x)(t) = x_0 + \int_a^t g(s, x(s))\,ds,$$

$$h_k(x)(t) = x_0 + \int_a^t g_k(s, r(x(s)))\,ds,$$

$$H_k(x) = \Big\{z \in E\colon z(a) = 0,\ \forall s, t \in [a, b]\ \Big(s < t \Rightarrow |z(s) - z(t)| \leqslant C_k \int_s^t |x(u)|\,du\Big)\Big\}.$$

It is easy to show that all the hypotheses of Theorem 2 are satisfied, which finishes the proof.

Consider the Floquet problem

$$\text{(2.5)} \qquad \begin{cases} x'(t) = f(t, x(t)) & \text{a.e. on } [a, b], \\ x(a)+\lambda x(b) = \xi & (\lambda > 0,\ \xi \in \mathbf{R}^n), \end{cases}$$

where $f\colon [a, b] \times \mathbf{R}^n \to \mathbf{R}^n$ is a Carathéodory map. Any absolutely continuous map $x\colon [a, b] \to \mathbf{R}^n$ which satisfies the above equations is called a *solution* of (2.5).

THEOREM 4. *Suppose that $f\colon [a, b] \to \mathbf{R}^n$ is a bounded Carathéodory map which satisfies*

$$\text{(2.6)} \qquad |f(t, x)-f(t, y)| \leqslant p(t)|x-y| \quad \textit{for } t \in [a, b] \textit{ and } x, y \in \mathbf{R}^n,$$

where $p\colon [a, b] \to [0, \infty)$ is an integrable function such that

$$\text{(2.7)} \qquad \int_a^b p(s)\,ds \leqslant \sqrt{\pi^2+\ln^2\lambda}.$$

Then the set of all solutions of (2.5) is an R_δ-set in E.

Proof. Problem (2.5) is equivalent to the equation

$$\text{(2.8)} \qquad x(t) = x(a)+Lx-\xi+\int_a^t f(s, x(s))\,ds, \qquad t \in [a, b],$$

where $L\colon E \to \mathbf{R}^n$ is given by $Lx = x(a)+\lambda x(b)$. Define $h\colon E \to E$, $h_k\colon E \to E$, and $H_k\colon E \to n(E)$ $(k \in \mathbf{N})$ by

$$\text{(2.9)} \qquad h(x)(t) = x(a)+Lx-\xi+\int_a^t f(s, x(s))\,ds,$$

$$\text{(2.10)} \qquad h_k(x)(t) = x(a)+Lx-\xi+\int_a^t (1-(1/k))f(s, x(s))\,ds,$$

$$\text{(2.11)} \qquad H_k(x) = \Big\{x(a)+Lx+z\colon z \in E,\ z(a) = 0,$$

$$\forall s, t \in [a, b]\ \Big(s < t \Rightarrow |z(s)-z(t)| \leqslant \int_s^t (1-(1/k))p(u)|x(u)|\,du\Big)\Big\}.$$

First, we prove that if $x \in H_k(x)$ for some $x \in E$ and $k \in \mathbf{N}$, then $x = 0$.

If $x \in H_k(x)$, we have

$$\text{(2.12)} \qquad x(t) = x(a)+Lx+z(t), \qquad t \in [a, b],$$

where $z\colon [a, b] \to \mathbf{R}^n$ is absolutely continuous and

$$\text{(2.13)} \qquad |x'(t)| = |z'(t)| \leqslant (1-(1/k))p(t)|x(t)| \quad \text{a.e. on } [a, b].$$

From (2.12) and (2.11) it follows that $Lx = 0$. By Theorem 3 of [4], only the trivial solution of (2.13) satisfies $Lx = 0$ and so $x = 0$.

Let K be a constant such that $|f(t, x)| \leqslant K$ for $t \in [a, b]$ and $x \in \mathbf{R}^n$. Now suppose that $x = h_k(x)$ for some $x \in E$ and $k \in \mathbf{N}$. By (2.10) we have

$$|x(b)-x(a)| \leqslant \int_a^b (1-(1/k))|f(s, x(s))|\, ds \leqslant K(b-a).$$

Notice that $|x(b)| \leqslant |\xi|+K|b-a|$; if not, we would have

$$|x(a)+\lambda x(b)| = |x(a)-x(b)+(1+\lambda)x(b)| \geqslant (1+\lambda)|x(b)|-|x(a)-x(b)| > |\xi|,$$

which is a contradiction. Thus, for $t \in [a, b]$ we have

$$|x(t)-x(b)| \leqslant K(b-a), \qquad |x(t)| \leqslant |\xi|+2K(b-a).$$

It is easy to check that all the remaining hypotheses of Theorem 1 are satisfied, which finishes the proof.

Remark 1. In [4] it is proved that if in Theorem 4 we replace the weak inequality in (2.7) by the strong inequality, then (2.5) has exactly one solution. On the other hand, if in (2.7) equality holds, then (2.5) may have more than one solution. To demonstrate this consider the following problem, which is a slight modification of an example in [4]:

$$(2.14) \qquad \begin{cases} x_1' = r(-(\pi^{-1}\ln\lambda)x_1+x_2), \\ x_2' = r(-x_1-(\pi^{-1}\ln\lambda)x_2), & \text{a.e. on } [0, \pi], \\ x_i(0)+x_i(\pi) = 0, \quad i = 1, 2, \end{cases}$$

where $r\colon \mathbf{R} \to [-1, 1]$ is a retraction. Notice that for sufficiently small $|C_1|$ and $|C_2|$ ($C_1, C_2 \in \mathbf{R}$) we have the following solution $x = (x_1, x_2)$ of (2.14):

$$x_1(t) = \exp(-(t/\pi)\ln\lambda)(C_1\cos t+C_2\sin t),$$

$$x_2(t) = \exp(-(t/\pi)\ln\lambda)(-C_1\sin t+C_2\cos t).$$

Consider the Nicoletti problem

$$(2.15) \quad \begin{cases} x_i'(t) = f_i(t, x_1, \ldots, x_n), \quad i = 1, \ldots, n, \\ x_i(t_i) = \xi_i \quad (\xi_i \in \mathbf{R},\ t_i \in [a, b],\ i = 1, \ldots, n,\ \xi = (\xi_1, \ldots, \xi_n)), \end{cases}$$

where $f = (f_1, \ldots, f_n)\colon [a, b] \times \mathbf{R}^n \to \mathbf{R}^n$ is a Carathéodory map. Any absolutely continuous map $x\colon [a, b] \to \mathbf{R}^n$ which satisfies the above equations is called a *solution* of (2.15).

THEOREM 5. *Suppose that $f\colon [a, b] \to \mathbf{R}^n$ is a bounded Carathéodory map which satisfies*

$$(2.16) \qquad |f(t, x)-f(t, y)| \leqslant p(t)|x-y| \quad \textit{for } t \in [a, b] \textit{ and } x, y \in \mathbf{R}^n,$$

where $p\colon [a, b] \to (0, \infty)$ is an integrable function such that

$$\int_a^b p(s)\,ds \leqslant \pi/2. \tag{2.17}$$

Then the set of all solutions of (2.15) *is an* R_δ*-set in* E.

Proof. Problem (2.15) is equivalent to the equation

$$x(t) = x(a) + Lx - \xi + \int_a^t f(s, x(s))\,ds, \qquad t \in [a, b],$$

where $L\colon E \to \mathbf{R}^n$ is given by

$$Lx = (x_1(t_1), \ldots, x_n(t_n)).$$

Define $h\colon E \to E$, $h_k\colon E \to E$, and $H_k\colon E \to n(E)$ $(k \in \mathbf{N})$ by (2.9), (2.10), and (2.11) respectively. Using Theorem 3 of [5] one can show as before that if $x \in H_k(x)$ for some $x \in E$ and $k \in \mathbf{N}$, then $x = 0$.

It is easy to check that all the remaining hypotheses of Theorem 1 hold, which finishes the proof.

Remark 2. In [5] it is proved that if in Theorem 5 we replace the weak inequality in (2.17) by the strong inequality, then (2.15) has exactly one solution. On the other hand, if in (2.17) equality holds, then (2.15) may have more than one solution. To demonstrate this consider the following slight modification of an example given in [5]:

$$\begin{cases} x_1' = -(\pi/2)r(x_2), \quad x_2' = (\pi/2)r(x_1) \quad \text{a.e. on } [0, 1], \\ x_1(0) = x_2(0) = 0, \end{cases} \tag{2.18}$$

where $r\colon \mathbf{R} \to [-1, 1]$ is a retraction. Notice that for every $C \in [-1, 1]$ we have the following solution $x = (x_1, x_2)$ of (2.18):

$$x_1(t) = -C\sin(\pi t/2), \qquad x_2(t) = C\cos(\pi t/2).$$

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