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DYNAMICS OF ENTIRE MAPS

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In this paper we summarize some of the recent work on the structure of the Julia sets of a large class of entire transcendental functions, namely the class of entire transcendental functions of finite type. As a dynamical system, a map in this class shares many of the properties of polynomial or rational maps. However, there are several significant differences. It is the aim of this paper to point out some of these distinctions.

The major difference between entire functions and polynomials or rational maps is, of course, the essential singularity at infinity. By Picard's Theorem, any neighborhood of ∞ is mapped over the entire plane infinitely often, missing at most one point. From a dynamical point of view, this result means that an entire map exhibits a tremendous amount of hyperbolicity near ∞. This is reflected in the Julia set of such a map, which assumes a form that is quite different from that of a polynomial or rational map. In particular, we will show below that the Julia set of an entire function of finite type contains Cantor bouquets – a topological structure that is quite different from the kinds of Julia sets that occur in the study of rational maps or polynomials, which extend to ∞.

§ 1. Basic facts

Let E be an entire transcendental function of finite type, i.e., for which there are only finitely many critical values and asymptotic values. Maps in this class include, among others, $\exp z$, $\sin z$, and $\cos z$. The *Julia set* of E, denoted by $J(E)$, is the set of points in C at which the family of iterates of E fails to be a normal family of functions in the sense of Montel. We denote by E^n the

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n-fold composition of E with itself. The following result, proved by Fatou [F] and I. N. Baker [Ba] shows that the Julia set has a dynamical definition as well.

THEOREM. *$J(E)$ is the closure of the set of repelling periodic points of E.*

The complement of the Julia set is called the *stable set*. Sullivan [Su] has recently classified the dynamics of rational maps on the components of the stable set. Basically, in the case of rational maps, the stable set consists of a finite number of periodic components, and all other components are eventually periodic. The periodic components fall into one of five basic categories: attractive basins, super-attractive basins, parabolic basins, Siegel disks, or Herman rings. See [Bl] for more details on this result.

The classification of the dynamics of entire functions in not yet complete, but there has been substantial recent progress. Goldberg and Keen [GK], Eremenko and Lubich [EL1], and Baker [Ba2] have extended the Sullivan result to the critically finite case, so there are no wandering domains for these maps. However, there are several examples of general entire maps which have components of the stable set which are not periodic. Herman [H] has a simple example, and Baker [Ba1] has found a quite different type of wandering domain which is not simply connected. Eremenko and Ljubic have examples of simply connected wandering domains on which all of the subsequent iterates of the map are univalent, and also examples for which the orbit of the domain has infinitely many limit points.

An example of a wandering domain similar to Herman's example is provided by the map $S_\lambda(z) = z + \lambda \sin z$, where $\lambda \in \boldsymbol{R}$ is chosen so that all critical points lie on one of two orbits which tend to ∞. See Fig. 1.

It will follow from our later remarks that the vertical lines $\operatorname{Re} z = k\pi$ for $k \in Z$ lie in the Julia set, so there is an open neighborhood of each critical point which lies in a separate component of the stable set and which tends to ∞.

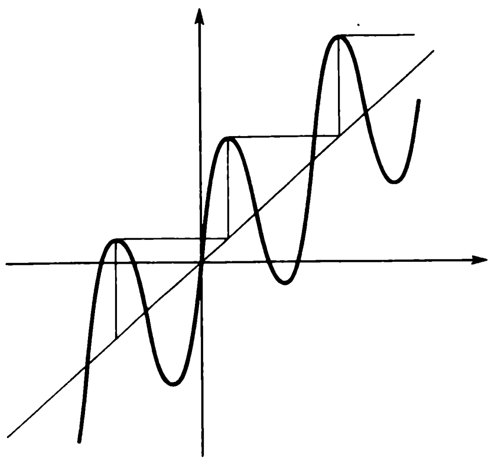

Fig. 1. The graph of s_λ with all critical points tending to ∞

For entire maps, the classification of the components of the stable set must admit the possibility of a new type of component, a *domain at infinity*. These are open sets which are invariant or periodic but which are not of the above types and which contain ∞ in their boundaries. For example, Fatou [F] has shown that the map $F(z) = z+1+\exp(-z)$ has a domain at ∞ which contains the half-plane $\operatorname{Re} z > 0$. Any point in this half-plane satisfies $\operatorname{Re} F(z) > \operatorname{Re} z$, so all such points tend to ∞. Note that F is not critically finite, however. Eremenko and Lyubich have recently announced [EL1] that such a domain cannot occur for a map which is critically finite.

§ 2. Cantor bouquets

A characteristic topological feature of the Julia sets of critically finite entire functions is the presence of Cantor bouquets. These were first observed for the exponential map in [DK] and later shown to exist for a wide class of entire functions [DT].

A Cantor n-bouquet is defined as follows. Let Σ_n denote the Cantor set of one-sided sequences $(s_0, s_1, s_2, \ldots)$ where $s_j = 1, \ldots, n$. The shift automorphism σ: $\Sigma_n \to \Sigma_n$ is defined by $\sigma(s_0, s_1, s_2, \ldots) = (s_1, s_2, \ldots)$; the dynamics of σ are well understood [Sm]. A closed invariant subset B_n of $J(E)$ is called a *Cantor bouquet* if there is a homeomorphism h: $\Sigma_n \times [0, \infty) \to B_n$ satisfying:

1. $\pi \circ h^{-1} \circ E \circ h(s, t) = \sigma(s)$, where π: $\Sigma_n \times [0, \infty) \to \Sigma_n$ is projection.
2. $\lim_{t \to \infty} h(s, t) = \infty$.
3. $\lim_{n \to \infty} E^n \circ h(s, t) = \infty$ if $t \neq 0$.

A Cantor n-bouquet should be thought of as a "Cantor set" of curves, each of which extends to ∞. The map E preserves this set of curves and acts on them as the shift map. Each curve has a well-defined endpoint $h(s, 0)$ and the set of all endpoints is called the *crown* of the bouquet. The restriction of E to the crown is topologically conjugate to the shift map, and so periodic points are dense in the crown. By condition 3, any point in B_n not in the crown lies on a "tail" or "hair" of the form $h(s, t)$ with $t > 0$. Each such point tends to ∞ under iteration, so the Cantor bouquet separates into two distinct pieces, the crown, on which the map is modeled by the shift, and the attached hairs, on which all points simply tend to the essential singularity under iteration.

It is an easy exercise to construct Cantor n-bouquets for the map $E_\lambda(z) = \lambda \exp(z)$ where $0 < \lambda < 1/e$. The reason for the restriction on λ will become clear later. A 3-bouquet for this map is depicted in Figure 2. See [DK] for details.

We also note that the typical entire function which is critically finite usually admits an increasing sequence of Cantor n-bouquets $B_n \subset B_{n+1} \subset \ldots$

with natural inclusion maps. The union

$$B_{\infty} = \bigcup_{n=1}^{\infty} B_n$$

is the Cantor bouquet. See [DT] for examples of this construction.

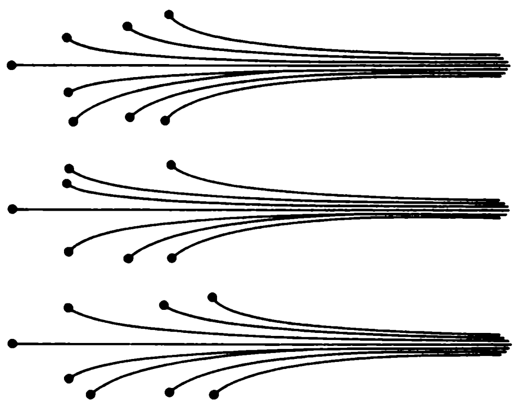

Fig. 2. A Cantor 3-bouquet for $z \mapsto \lambda \exp(z)$ when $\lambda < 1/e$

In analogy with Douady and Hubbard's use of the term "polynomial-like map" [DH1], critically finite entire maps may be considered "exponential-like". The rationale for this terminology comes from the following observation. Let D be a disk which contains all of the critical and asymptotic values of E. Let $\Gamma = C - D$, and let U be any component of $E^{-1}(\Gamma)$. We claim that $E|U$ is a universal covering of the punctured disk Γ. To see this, we first note that, since E has no critical points in U, E is locally one-to-one. Also, E is proper since there are no asymptotic values in Γ. Hence, E is a covering. It follows that U is either a punctured disk or a disk (in which case, E: $U \to \Gamma$ is a universal covering, as we claimed). Suppose U is a punctured disk. Then E: $U \to \Gamma$ is finite to one. If the puncture is at ∞, then E must be a polynomial, contradicting our assumption that E is transcendental. If, on the other hand, the puncture is at $a \neq \infty$, then E has a pole at a, contradicting our assumption that E is entire. Therefore it follows that $E|U$ is a universal covering on U and so is "exponential-like".

§ 3. Entire functions vs. polynomials

There are a number of additional features of the dynamics of critically finite entire functions which distinguish them from polynomials. For a polynomial, ∞ is always an attracting fixed point and its basin of attraction never lies in the Julia set. This should be contrasted with the following result.

THEOREM. *Suppose a critically finite entire transcendental function E satisfies certain growth conditions. Then any point which tends to ∞ under iteration of E lies in $J(E)$. Moreover, $J(E)$ is precisely the closure of the set of points which escape to ∞ under iteration.*

The precise growth conditions necessary for this result are specified in [DT]. We note that this class of maps therefore has two quite different but equivalent definitions of its Julia set. On the one hand, the Julia set is the closure of the set of repelling periodic points. All of these orbits are bounded. On the other hand, the Julia set is also the closure of the set of points which tend to ∞. This set consists of only unbounded orbits and so is quite distinct from the set of periodic points.

Recently, in the lectures during the Semester on Dynamical Systems and Ergodic Theory at the Banach Center, 1986, Eremenko has announced the result that the Julia set is the frontier of the escaping set for any entire transcendental function, not necessarily of finite type.[1]

Unlike polynomials, the Julia set of E may be the entire complex plane. Indeed, Sullivan's Theorem gives an easy criterion for this event (modulo domains at ∞).

THEOREM. *Suppose that a critically finite entire transcendental function E has no domains at ∞. If all critical and asymptotic values tend to ∞ under iteration of E, then $J(E) = C$.*

For example, the only asymptotic value for $E_\lambda(z) = \lambda\exp(z)$ is 0 and there are no critical values. It is easy to show that this map has no domains at infinity. Hence the exponential map $z \mapsto \exp(z)$ has Julia set the entire plane, since 0 tends to ∞ under iteration. This fact was first proved by Misiurewicz [Mi], answering a sixty year old conjecture of Fatou [F].

If all critical and asymptotic values are preperiodic (but not periodic), then again the Julia set must be the entire plane. So it follows that the maps $z \mapsto 2\pi i\exp(z)$ and $z \mapsto \pi i\exp(z)$ each have Julia set the entire plane.

These ideas lead to a rather spectacular occurrence in the dynamics of entire maps: the Julia sets may explode as a parameter is varied. For example, consider the one parameter family of maps $E_\lambda(z) = \lambda\exp(z)$ with $\lambda > 0$. The reader may check that the graphs of E_λ assume two different forms depending upon whether $\lambda < 1/e$ or $\lambda > 1/e$. See Figure 3.

If $\lambda < 1/e$, then E_λ admits two fixed points in R, an attracting fixed point at q_λ and a repelling fixed point at p_λ. Consider any vertical line which meets the real axis at a point z_0 which lies between q_λ and p_λ. Then E_λ maps this vertical line onto the circle centered at the origin with radius $|\lambda|\exp(z_0)$. More importantly, E_λ maps the half plane $\operatorname{Re} z < z_0$ inside this circle and,

[1] See the paper "On the iteration of entire function" by A. E. Eremenko, this volume, pp. 339–346.

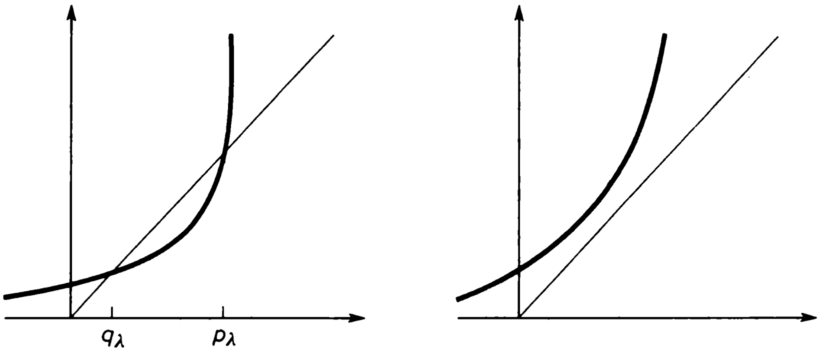

Fig. 3. The graphs of $\lambda\exp(x)$ when a. $\lambda < 1/e$, b. $\lambda > 1/e$

hence, strictly inside itself. One may check easily that q_λ lies inside this circle, and so it follows that the basin of attraction of q_λ contains the entire half plane $\operatorname{Re} z < z_0$, so the Julia set lies entirely in the right half plane for these λ-values. In fact, as we discussed in the previous section, the Julia set of E_λ is a Cantor bouquet in this half plane. On the other hand, as soon as $\lambda > 1/e$, we have $E_\lambda^n(0) \to \infty$ and Sullivan's Theorem guarantees that $J(E_\lambda) = C$ for these λ-values. So the Julia set explodes as λ passes through $1/e$.

§ 4. Bifurcation diagrams

As is well known in the case of quadratic polynomials, there is also an apparent similarity between the bifurcation diagrams for families of critically finite entire maps and their corresponding Julia sets. We will restrict our attention here to the exponential family $E_\lambda(z) = \lambda\exp(z)$, although similar results hold for such families as $z \mapsto \lambda\sin(z)$ and $z \mapsto \lambda\cos(z)$.

Since there is only one asymptotic value for E_λ, this family is a natural one parameter family of maps. Indeed, any entire map which has a single omitted value and no critical values is affinely conjugate to a member of this family.

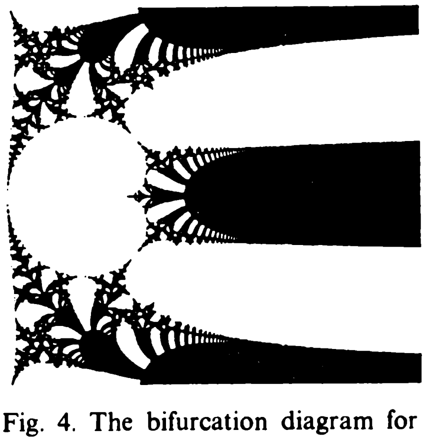

Fig. 4. The bifurcation diagram for E_λ

Since there is a unique singular value, E_λ can admit at most one attracting periodic orbit or at most two Siegel disks. Hence the dynamics of this family may be suitably represented by a plot in the λ-plane which indicates which λ-values admit components in their stable sets of the above types. This set is displayed in Figure 4.

In Fig. 4, the white regions represented λ-values for which E_λ admits an attracting periodic point of a given period. Black regions indicate λ-values for which the orbit of 0 under E_λ grows "too large". This picture suggests that there are curves in the bifurcation diagram which consist of λ-values for which $E_\lambda^n(0) \to \infty$. This is indeed the case, as was shown in [DGH]. In this paper the set of "escaping" λ-values is shown to have a structure similar to the Cantor bouquets discussed in § 2. In this sense, the Julia sets and bifurcation set of the exponential family are similar.

We remark that the curves in the λ-plane above are intimately related to the external arguments of Douady and Hubbard [DH], who showed that the exterior of the Mandelbrot set [Ma] is a disk filled with such curves or "external rays". The difference in the quadratic map case is that parameter values outside the Mandelbrot set have Julia sets which are Cantor sets. In the case of E_λ, these curves consist of λ-values for which $J(E_\lambda) = C$.

There is an interesting relationship between the Mandelbrot set and the bifurcations set for E_λ. Let

$$P_{d,\lambda}(z) = \lambda\left(1+\frac{z}{d}\right)^d.$$

Each $P_{d,\lambda}$ has a unique critical point at $-d$ and critical value 0. Of course, $P_{d,\lambda}(z) \to E_\lambda(z)$. But this convergence also occurs in a dynamical sense. Consider the bifurcation set B_d for the $P_{d,\lambda}$. When $\lambda = 2$, B_2 is essentially the same as the Mandelbrot set. As $d \to \infty$, the corresponding bifurcation sets grow to resemble more and more the bifurcation set of E_λ. Indeed, the paper [DGH] makes this convergence precise.

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