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The characterization of the cut of funnel in a planar semidynamical system

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Abstract. We investigate the section of the funnel through a non-stationary point x in a planar semidynamical system $(\boldsymbol{R}^2, \boldsymbol{R}_+, \pi)$, i.e., the set $F(t, x) = \{y \in \boldsymbol{R}^2: \pi(t, y) = x\}$. The topological structure of $F(t, x)$ for $t \geqslant N(x)$, where by $N(x)$ we denote the negative escape time of x (according to McCann), is characterized.

Introduction

In the topological semidynamical system $(X, \boldsymbol{R}_+, \pi)$, we may consider $F(t, x)$ – "the past" of a given point x in time t (called a *cut of the funnel through* x). R. C. McCann ([13]) defined the negative escape time $N(x)$ of x and proved that in a locally compact metric space X every semidynamical system without start points is isomorphic to a semidynamical system with infinite negative escape time $N(x)$ for each x. Some results about upper semicontinuity of the function F: $(t, x) \mapsto F(t, x)$ are proved under the assumption that $t < N(x)$ ([5], [8]).

Many properties of different sets defined in semidynamical systems have been proved in a number of papers not cited here. Nevertheless, the set $F(t, x)$ was not investigated in much depth. Generally, this set does not have interesting topological properties (compare [2]). However, when the phase space X is equal to $\boldsymbol{R}^2$ and $t < N(x)$, then for a non-stationary point x the set $F(t, x)$ is a point or an arc ([7]). This need not hold for $t \geqslant N(x)$. When x is a stationary point these sets cannot be precisely characterized.

The purpose of this paper is to describe the set $F(t, x)$ for a non-stationary point x in a planar semidynamical system for every $t \geqslant 0$. We show that when x is not a point of negative unicity, $t \geqslant N(x)$ and $F(t, x)$ is not empty, then $F(t, x)$ is a one-dimensional topological manifold, possibly with boundary, not necessarily connected. At most two components of $F(t, x)$ are homeomorphic to $\boldsymbol{R}_+$, each of remaining components is homeomorphic to $\boldsymbol{R}$. Moreover, the "ends" of each component tend to infinity. If x is a point of negative unicity, then $F(t, x)$ is a point or the empty set.

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1. Preparatory definitions and theorems

By $\boldsymbol{R}$, N, $\boldsymbol{R}_+$ we denote the sets of real, natural and nonnegative real numbers, respectively. By S^n we denote an n-dimensional sphere in $\boldsymbol{R}^{n+1}$. By $B(p, \varepsilon)$ and $S(p, \varepsilon)$ we mean the subsets of $\boldsymbol{R}^2$: the open ball and the sphere of radius ε centred at p, respectively. Instead of p we may consider a set M. For a given set $A \subset \boldsymbol{R}^2$ we denote by $\operatorname{Int} A$, $\bar{A}$, ∂A and $\operatorname{Ext} A$ the sets: interior, closure, boundary and exterior (i.e. $\boldsymbol{R}^2 \setminus \bar{A}$) of A, respectively. Instead of $\bar{A}$ we may sometimes write $\operatorname{Cl} A$. For a given Jordan curve $\gamma \subset \boldsymbol{R}^2$, we denote by $\operatorname{Ins}\gamma$ and $\operatorname{Outs}\gamma$ the bounded and unbounded components of $\boldsymbol{R}^2 \setminus \gamma$, respectively. By an arc we mean the set contained in $\boldsymbol{R}^2$, homeomorphic to $[0, 1]$. If it is obvious which arc with ends a and b we mean ($a, b \in \boldsymbol{R}^2$) we denote it by ab. When c, $d \in ab$, we denote by cd the subarc of ab with the ends c and d. Throughout the paper, by a 1-dimensional manifold we mean a 1-dimensional manifold possibly with boundary, not necessarily connected.

Assume that there is given a region $D \subset \boldsymbol{R}^2$. We say that an arc ab is a *cross-cut* of D if $ab \setminus \{a, b\} \subset D$ and $a, b \in \partial D$.

If $a \in \boldsymbol{R}^2$ and $M \subset \boldsymbol{R}^2$, then by $\varrho(a, M)$ we mean the Euclidean distance between a point a and a set M.

If a function f is defined on a set containing an interval (x, y), we denote by $f(x, y)$ the set $f((x, y))$. In the same way we introduce the symbols $f[x, y)$, $f(x, y]$ and $f[x, y]$.

A (topological) semidynamical system on a topological space X is a triplet $(X, \boldsymbol{R}_+, \pi)$, where π is a continuous map from $\boldsymbol{R}_+ \times X \to X$ such that $\pi(0, x) = x$ for every $x \in X$ and $\pi(t, \pi(s, x)) = \pi(t+s, x)$ for every $x \in X$ and $t, s \in \boldsymbol{R}_+$. By π^s we denote $\pi|_{\{s\} \times X}$, by π_x we denote $\pi|_{\boldsymbol{R} \times \{x\}}$. A function σ: $(\alpha, 0] \to X$ is called a *left maximal solution through* x if $\sigma(0) = x$, $\pi(t, \sigma(s)) = \sigma(t+s)$ whenever t, $t+s \in (\alpha, 0]$ and $t \geqslant 0$ and it is maximal (with respect to inclusion) relative to the above properties. It is known ([2]) that every solution is continuous. By a (positive) trajectory through x we mean the set $\pi(\boldsymbol{R}_+ \times \{x\})$ and denote it by $\pi^+(x)$.

Assume that a semidynamical system $(\boldsymbol{R}^2, \boldsymbol{R}_+, \pi)$ is given. A point $x \in \boldsymbol{R}^2$ is said to be a *stationary point* if $\pi(t, x) = x$ for each $t \geqslant 0$. A point $x \in \boldsymbol{R}^2$ is said to be a *periodic point* if there exists a $T > 0$ such that $\pi(T, x) = x$ and x is not a stationary point. The smallest T with the above property is called the *period* of x. A point $x \in \boldsymbol{R}^2$ is said to be a *regular point* if it is neither stationary nor periodic.

A point x is said to be a *singular point* if there exist $y_1 \neq y_2 \in \boldsymbol{R}^2$ and $t > 0$ such that $\pi(t, y_1) = \pi(t, y_2) = x$ but $\pi(\theta, y_1) \neq \pi(\theta, y_2)$ for any $\theta \in [0, t)$. For a given $A \subset \boldsymbol{R}_+$ and an $M \subset \boldsymbol{R}^2$, we put $F(A, M) = \{y \in \boldsymbol{R}^2\colon\ \pi(t, y) \in M$ for some

$t \in A\}$. If A is a singleton, i.e. $A = \{t\}$, we write $F(t, M)$ instead of $F(\{t\}, M)$; in the similar way we denote $F(A, x)$ and $F(t, x)$. We call the set $F(t, x)$ a *cut* of the funnel through x. The set $F([s, t], x)$ is called a *section* of the funnel through x (the *funnel* $F(x)$ through x is defined to be the set $\bigcup\{F(t, x)\colon\ t \geqslant 0\}$). A point x is said to be a *point of negative unicity* if for any $t \geqslant 0$ the set $F(t, x)$ has at most one element. Generally we do consider also start point (a point x is said to be a *start point* if $F(t, x) = \emptyset$ for any $t > 0$), but it is known ([2]) that if $X = \boldsymbol{R}^2$, then a semidynamical system has no start points.

For the basic properties of semidynamical systems the reader is referred to [2], [13] and [14]. Now we recall some theorems. Recall that a semidynamical system $(\boldsymbol{R}^2, \boldsymbol{R}_+, \pi)$ is given; almost all the theorems mentioned below are true under more general assumptions.

1.1. Definition. By the *negative espace time* $N(x)$ of x we define $N(x) = \inf \{s \in (0, \infty]\colon (-s, 0]$ is the domain of a left maximal solution through $x\}$.

As one can easily verify, under our assumptions this definition is equivalent to the definition given by McCann in [13]. Note that it is not equivalent to the definition given in [1].

For $M \subset \boldsymbol{R}^2$ we put $N(M) = \inf\{N(x)\colon\ x \in M\}$.

1.2. Definition. The system $(\boldsymbol{R}^2, \boldsymbol{R}_+, \pi)$ is said to be *isomorphic* to a semidynamical system $(\boldsymbol{R}^2, \boldsymbol{R}_+, \varrho)$ if there is a continuous mapping $\varphi\colon \boldsymbol{R}_+ \times \boldsymbol{R}^2 \to \boldsymbol{R}_+$ such that: $\varphi(0, x) = 0$ and the mapping $\varphi(\cdot, x)\colon \boldsymbol{R}_+ \to \boldsymbol{R}_+$ is a homeomorphism for each $x \in \boldsymbol{R}^2$ and $\pi(t, x) = \varrho(\varphi(t, x), x)$ for each $(t, x) \in \boldsymbol{R}_+ \times \boldsymbol{R}^2$.

1.3. Theorem ([11]). *The semidynamical system $(\boldsymbol{R}^2, \boldsymbol{R}_+, \pi)$ is isomorphic to a semidynamical system $(\boldsymbol{R}^2, \boldsymbol{R}_+, \pi')$ which has infinite negative escape time for each $x \in \boldsymbol{R}^2$.*

1.4. Theorem ([3]). *For each neigbourhood U of a point $x \in \boldsymbol{R}^2$ there are an $s > 0$ and a neigbourhood V of x such that $F([0, s], V) \subset U$ and $F([0, \alpha], x)$ is compact for each $\alpha \in [0, s]$.*

1.5. Lemma ([13]). *For a point $x \in \boldsymbol{R}^2$ we have:*

$$N(x) = \sup\{s\colon\ F([0, s] \text{ is compact}\}.$$

1.6. Lemma ([11]). *If $K \subset \boldsymbol{R}^2$ and $J_1, J_2 \subset \boldsymbol{R}_+$, then $F(J_1, F(J_2, K)) = F(J_1 + J_2, K)$, where $J_1 + J_2 = \{\alpha + \beta\colon\ \alpha \in J_1,\ \beta \in J_2\}$.*

1.7. Lemma ([11]). *For every $a, b, t \in \boldsymbol{R}_+$ $(a \leqslant b)$, the sets $F(t, x)$ and $F([a, b], x)$ are closed.*

1.8. Lemma ([15]). *If the set $F(t, x)$ is compact, then $F([0, t], x)$ is also compact.*

1.9. Proposition ([15]). *If $F(t, x)$ is compact, then it is connected.*

1.10. Lemma ([11]). *If M is compact, then there exists an $\alpha > 0$ such that $F([0, s), M)$ is compact for each $s < \alpha$.*

1.11. Proposition ([5]). *Assume that M is compact and connected and W is a compact neigbourhood of M such that $F(t, W)$ is also compact. Then for every neighbourhood U of $F(t, M)$ there is a neighbourhood V of M such that $F(t, V) \subset U$.*

1.12. Proposition. *Assume that M is compact and connected, and that there exist an $s > 0$ and a compact neighbourhood W of M such that $F([0, s], W)$ is compact. Then for every $t \in [0, s]$ the set $F(t, M)$ is compact and connected.*

Proof. The compactness of $F(t, M)$ follows by Lemma 1.7. Suppose that $F(t, M)$ is not connected. We can present $F(t, M)$ as the union $A \cup B$, where A and B are closed non-empty disjoint sets. Notice that for every $y \in M$ we have $F(t, y) \subset A$ or $F(t, y) \subset B$ (because, by Proposition 1.9, $F(t, y)$ is connected). Define $A' = \{y \in M\colon F(t, y) \subset A\}$ and $B' = \{y \in M\colon F(t, y) \subset B\}$. The sets A' and B' are non-empty and disjoint; moreover, $F(t, A') = A$ and $F(t, B') = B$. We can find the open disjoint sets U_A and U_B such that $A \subset U_A$ and $B \subset U_B$. Take a $z \in A'$. The set $F(t, z)$ is compact. By Proposition 1.10 there exists an open neighbourhood V_z of z with $F(t, V_z) \subset U_A$, as $F(t, z) \subset U_A$. Thus $F(t, V_z) \cap B = \emptyset$ and $V_z \cap B' = \emptyset$. Put $V_A = \bigcup\{V_z\colon z \in A'\}$. The set V_A is the open neighbourhood of A' and $V_A \cap B' = \emptyset$. In the same way we can find an open neighbourhood V_B of B' with $V_B \cap A' = \emptyset$. This shows that the sets A' and B' are separated in M, which contradicts the connectedness of M.

1.13. Lemma. *If the set $F(t, x)$ is non-empty, then it is compact if and only if $t < N(x)$.*

Proof. By Theorem 1.4 and Lemma 1.5, $F(t, x)$ is compact for $t < N(x)$. Assume that $t > N(x)$ and suppose that $F(t, x)$ is compact. By Lemma 1.8 the set $F([0, t])$ is compact, which contradicts Lemma 1.5. If $F(N(x), x)$ is compact, then by Theorem 1.4 and Lemma 1.7 the set $F\big([0, \delta], F(N(x), x)\big)$ is compact for some $\delta > 0$ and by Lemmas 1.6 and 1.8 the set $F([N(x), N(x)+\delta], x)$ is compact, so $F([0, N(x)], x) \cup F([N(x), N(x)+\delta], x) = F([0, N(x)+\delta], x)$ which again contradicts Lemma 1.7.

1.14. Lemma ([7]). *Assume that $t, s_1, s_2 \geqslant 0$ and x is a non-stationary point. If $\pi^+(x)$ contains a periodic point of period T ($\pi^+(x)$ can contain a periodic point even in the case where x is regular) then we assume that $|s_1 - s_2| < T$. Then*:

(1.14.1) *if $a_i \in F(s_i, x)$ $(i = 1, 2)$ and $s_1 \neq s_2$, then $a_1 \neq a_2$,*

(1.14.2) *if $a_i \in F(t, x)$ $(i = 1, 2)$ and $\pi(s_1, a_1) = \pi(s_2, a_2)$ and $\pi(s_1, a_1)$ is not a stationary point, then $s_1 = s_2$.*

1.15. THEOREM ([7]). *If x is a non-stationary point and $t < N(x)$, then $F(t, x)$ is either a point or an arc, i.e. the set homeomorphic to the closed interval* $[0, 1]$.

1.16. THEOREM ([7]). *Assume that x is a non-stationary point, the arc $ab \subset F(t, x)$ $(a \neq b)$ and the positive number λ is such that $\pi([0, \lambda], a) \cap \pi([0, \lambda], b) = \emptyset$. If x is periodic, then also $\lambda < T$. Then*

(1.16.1) $\pi(\lambda, ab)$ *is an arc with the ends* $\pi(\lambda, a)$ *and* $\pi(\lambda, b)$,

(1.16.2) $\pi([0, \lambda], ab)$ *is homeomorphic to the square and* $\partial\pi([0, \lambda], ab) = \pi(\{0, \lambda\}, ab) \cup \pi([0, \lambda], \{a, b\})$.

1.17. PROPOSITION ([7]). *Assume that $t < N(x)$ and an arc $a_1 a_2$ $(a_1 \neq a_2)$ is contained in $F(t, x)$, where x is a non-stationary point. Let $\mu > 0$ and let $p \in \mathbf{R}^2$ be such that $\pi([0, \mu), a_1) \cap \pi([0, \mu), a_2) = \emptyset$, $\pi(\mu, a_1) = \pi(\mu, a_2) = p$. If x is periodic then we assume also that $t < T$.*

Then $\pi([0, \mu], a_1 a_2)$ is homeomorphic to a triangle and the edges of this triangle are the images of $a_1 a_2$, $\pi([0, \mu], a_1)$ and $\pi([0, \mu], a_2)$. Moreover, $\pi(\mu, a_1 a_2) = \{p\}$.

As the immediate consequence of 1.14–1.17 we get

1.18. COROLLARY. *Assume that $t < N(x)$, where x is a non-stationary point, $F(s, x)$ is not a point for some $s \in (0, t)$ and, if x is a periodic point, then $t-s$ is smaller than its period. Denote by a and b the ends of the arc $F(t, x)$. Then $\pi(t-s, a)$ and $\pi(t-s, b)$ are the ends of the arc $F(s, x)$ and $\pi([0, t-s], a) \cap \pi([0, t-s], b) = \emptyset$.*

By Lemma 1.14 and Proposition 1.17 we get

1.19. LEMMA. *Assume that x is a non-stationary point, $t < N(x)$ and $a, b \in F(t, x)$. Let a positive number λ be given; if x is a periodic point, then also λ is smaller than its period. Let $\pi(\xi, a) = \pi(\xi, b)$ for some $\xi \in (0, \lambda)$ and ab be an arc contained in $F(t, x)$. Then $\pi(\xi, p) = \pi(\xi, a)$ for every $p \in ab$.*

1.20. LEMMA ([7]). *Assume that x is a non-stationary point. Let us take a $t < N(x)$ and a $\lambda \in (0, t)$ (if x is periodic, then also $t < T$) and assume that $F(t, x)$ and $F(t-\lambda, x)$ are not singletons. Denote $F(t, x)$ by ab. Then for each $p \in ab \setminus \{a, b\}$ the set $\pi([0, \lambda], ab) \setminus \pi([0, \lambda], p)$ has two open components D_1 and D_2 which are bounded by Jordan curves. Moreover, for each $q \in ab$ we have:*

$$\pi([0, \lambda], q) \subset \overline{D}_1 \cup \pi([0, \lambda], p) \quad \textit{or} \quad \pi([0, \lambda], q) \subset \overline{D}_2 \cup \pi([0, \lambda], p).$$

1.21. LEMMA ([7]). *Assume that $t < N(x)$ and L is an arc (open or closed) contained in $F(t, x)$. Let us take a $\lambda \geqslant 0$ such that $t+\lambda < N(x)$. Then $F(\lambda, L)$ is an arc (open or closed, respectively) contained in $F(t+\lambda, x)$; if L is a point, then $F(\lambda, L)$ is a point or a closed arc.*

Now we recall some theorems from general topology.

1.22. Schönflies Theorem ([1], [10]). *Every homeomorphism from a Jordan curve γ contained in $\boldsymbol{R}^2$ onto S^1 may be extended to a homeomorphism from $\boldsymbol{R}^2$ onto $\boldsymbol{R}^2$. Every homeomorphism from an arc $L_1 \subset \boldsymbol{R}^2$ onto an arc $L_2 \subset \boldsymbol{R}^2$ may be extended to a homeomorphism from $\boldsymbol{R}^2$ to $\boldsymbol{R}^2$.*

1.23. Theorem ([3]). *Assume that X is a compact connected set, U is an open subset of X and C is a component of U. Then $(\bar{U}\setminus U) \cap \bar{C} \neq \emptyset$.*

1.24. Theorem ([10]). *If a metric continuum X contains two different points a and b such that for each $x \in X$ there exist two closed sets A and B with the properties: $X = A \cup B$, $a \in A$, $b \in B$, and $A \cap B = \{x\}$, then X is homeomorphic to the interval* $[0, 1]$.

1.25. Theorem ([1]). *Assume that X is a Hausdorff topological space and φ: $(0, 1) \to X$ is a homeomorphism onto its image. Let us define*

$$A = \bigcap \{\mathrm{Cl}\{\varphi(s)\colon 0 < s \leqslant \xi\},\ \xi \in (0, 1)\},$$

$$B = \bigcap \{\mathrm{Cl}\{\varphi(s)\colon \xi \leqslant s < 1\},\ \xi \in (0, 1)\}.$$

Then $A \cap \varphi(0, 1) = \emptyset$ and $B \cap \varphi(0, 1) = \emptyset$. Moreover, if X is compact, then A and B are nonempty, compact and connected.

1.26. Lemma. *Assume that γ is a Jordan curve contained in $\boldsymbol{R}^2$ and L is a cross-cut of* Ins γ, *where L_1 and L_2 are the components of $\gamma\setminus L$. Let us take an $a \in L_1\setminus L$ and a $b \in L_2\setminus L$. If a point p has the property "there exists an arc pb such that $pb\setminus\{b\} \in \mathrm{Ins}\gamma\setminus L$", then $p \in \mathrm{Ins}(L \cup L_2)$ and $a \in \mathrm{Outs}(L \cup L_2)$.*

This lemma follows immediately from the Jordan Curve Theorem, the θ-curve Theorem (see [1], [10]) and V.11.7 and V.11.8 in [12].

2. The characterization of $F(t, x)$

Throughout this section we assume as given a planar semidynamical system $(\boldsymbol{R}^2, \boldsymbol{R}_+, \pi)$ and a non-stationary point $x \in \boldsymbol{R}^2$.

2.1. Notation. By $(\boldsymbol{R}^2, \boldsymbol{R}_+, \pi^\infty)$ we denote the semidynamical system with infinite negative escape time for each point, isomorphic to $(\boldsymbol{R}^2, \boldsymbol{R}_+, \pi)$ (existing by Theorem 1.3). In this system we denote the cut of a funnel by $F^\infty(t, x)$ and the section of a funnel by $F^\infty([s, t], x)$.

By φ: $\boldsymbol{R}_+ \times \boldsymbol{R}^2 \to \boldsymbol{R}_+$ we denote the function which gives the isomorphism between $(\boldsymbol{R}^2, \boldsymbol{R}_+, \pi)$ and $(\boldsymbol{R}^2, \boldsymbol{R}_+, \pi^\infty)$, constructed by McCann in [11] (see Definition 1.2 and Theorem 1.3). We have:

(2.1.1) $\varphi(0, y) = 0$ for each $y \in \boldsymbol{R}^2$;

(2.1.2) φ_y: $\boldsymbol{R}_+ \to \boldsymbol{R}_+$ is a homeomorphism for each $y \in \boldsymbol{R}^2$ (we denote $\varphi(\cdot, y) = \varphi_y(\cdot)$);

(2.1.3) $\pi(t, y) = \pi^\infty(\varphi(t, y), y)$ for each $(t, y) \in \boldsymbol{R}_+ \times \boldsymbol{R}^2$.

We define a function ψ: $\boldsymbol{R}_+ \times \boldsymbol{R}^2 \to \boldsymbol{R}^2$ as follows:

(2.1.4) $$\psi(s, y) = \varphi_y^{-1}(s).$$

Write $\psi_y(\cdot) = \psi(\cdot, y)$. Then we have

(2.1.5) $\varphi(s, y) = \psi_y^{-1}(s)$, as ψ_y is a homeomorphism.

If x is a periodic point, then we denote its period in the system $(\boldsymbol{R}^2, \boldsymbol{R}_+, \pi)$ by T and in the system $(\boldsymbol{R}^2, \boldsymbol{R}_+, \pi^\infty)$ by T^∞.

2.2 PROPOSITION. *The following properties hold*:

(2.2.1) $\pi^\infty(s, y) = \pi(\psi(s, y), y)$ *for every* $(s, y) \in \boldsymbol{R}_+ \times \boldsymbol{R}^2$;

(2.2.2) $$\varphi(t, y) \geqslant t;$$

(2.2.3) $$\psi(s, y) \leqslant s;$$

(2.2.4) *the function* ψ *is continuous.*

Proof. The first property follows immediately by the definition of ψ. The second was proved by McCann in the proof of the existence of a suitable isomorphism (Theorem 2.2 in [11]). The third follows immediately by (2.2.2) and (2.1.4). The last one follows by the results of Carlson (see Lemma 1 in [4]).

2.3. PROPOSITION. *If x is a periodic point, then $T^\infty \geqslant T$.*

Proof. We know that $\pi(T, x) = x$ and $\pi(s, x) \neq x$ for each $s \in (0, t)$. So $\pi^\infty(\varphi(T, x), x) = x$ and $\pi^\infty(\varphi(s, x), x) \neq x$ for each $s \in (0, T)$. This means that $T^\infty = \varphi(T, x)$ and by (2.2.2) $\varphi(T, x) \geqslant T$.

2.4. DEFINITION. Assume that x is not a point of negative unicity and $N(x) = \infty$. If x is a regular point, then by T we denote an arbitrary fixed positive number (if x is a periodic point, then by T we denote its period, as was assumed). By λ we denote $\sup\{t$: $F(t, x)$ has exactly one element$\}$.

Assume that $F(\lambda + \frac{1}{2}T, x) = a_1 b_1$. If $F(\lambda + \frac{1}{2}nT, x) = a_n b_n$, then by $a_{n+1} b_{n+1}$ we mean $F(\lambda + \frac{1}{2}(n+1)T, x)$; we choose a_{n+1} and b_{n+1} in such a way that $\pi(\frac{1}{2}T, a_{n+1}) = a_n$ and $\pi(\frac{1}{2}T, b_{n+1}) = b_n$. By Corollary 1.18 there is only one way to choose such a_{n+1} and b_{n+1}. Now we may define the boundary solutions. Under the above assumptions we define σ_a: $(-\infty, 0] \to \boldsymbol{R}^2$ by:

$$\sigma_a(-t) = \begin{cases} \pi(\lambda + \frac{1}{2}T - t, a_1) & \text{for } t \in [0, \lambda + \frac{1}{2}T], \\ \pi(\frac{1}{2}(n+1)T + \lambda - t, a_{n+1}) & \text{for } t \in [\frac{1}{2}nT + \lambda, \frac{1}{2}(n+1)T + \lambda]. \end{cases}$$

The mapping σ_b is defined in an analogous way. It is very simple to verify that σ_a and σ_b are solutions. The ranges of these solutions will be called the *boundary trajectories*.

2.5. Remark. From Definition 2.4 it follows that $\sigma_a(-t)$ and $\sigma_b(-t)$ are the end-points of the arc $F(t, x)$.

2.6. Lemma. *Assume that $p \in F(t, x)$ $(t > 0)$. Let $\varphi(t, p) = s$. Assume also that p is not the end-point of the arc $F^{\infty}(s, x)$ (by Theorem 1.15 $F^{\infty}(s, x)$ is an arc). Then:*

(2.6.1) *There exists a homeomorphism on its image h: $(0, 1) \to F(t, x)$ such that $h(\frac{1}{2}) = p$.*

(2.6.2) *There exists a neighbourhood U of p such that $U \cap F(t, x) = h(0, 1)$ and $U = \operatorname{Int} \pi^{\infty}([0, \varepsilon], cd)$, where $cd \subset F^{\infty}(s_1, x)$, $0 < s_1 - s < \varepsilon < \frac{1}{2}T$ and $\pi^{\infty}([0, \varepsilon], c) \cap \pi^{\infty}([0, \varepsilon], d) = \emptyset$.*

(2.6.3) *For each point $q \in cd$ there exists exactly one point belonging to $\pi^{\infty}((0, \varepsilon), q) \cap F(t, x)$.*

Proof. At first we show that there exist s_2, $\bar{s}$ with:

(2.6.4) $0 < \bar{s} - s_2 < \frac{1}{2}T$, $\pi^{\infty}([0, \bar{s} - s_2], \bar{a}) \cap \pi^{\infty}([0, \bar{s} - s_2], \bar{b}) = \emptyset$, where $\bar{a}$ and $\bar{b}$ are the ends of the arc $F^{\infty}(\bar{s}, x)$ and $\varphi(t, p) = s \in (\bar{s}, s_2)$ (Fig. 1).

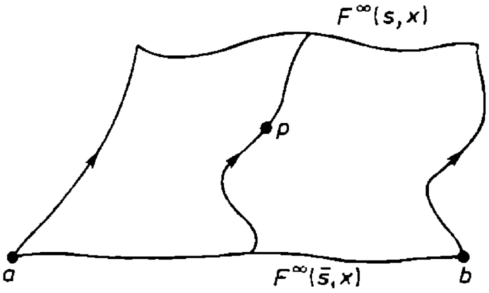

Fig. 1

We have $\pi^{\infty}(s, p) = \pi(t, p) = x$, as $s = \varphi(t, p)$. Thus $p \in F^{\infty}(s, x)$, which is an arc, not a point. Let $F^{\infty}(s, x) = ab$ $(p \neq a, p \neq b)$. Let us take the neighbourhoods U_a of a and U_b of b with $U_a \cap U_b = \emptyset$ (Fig. 2). There is a $\lambda \in (0, \frac{1}{4}T) \subset (0, \frac{1}{4}T^{\infty})$ such that $\lambda < s$ and $\pi^{\infty}([0, \lambda], a) \subset U_a$, $\pi^{\infty}([0, \lambda], b) \subset U_b$, $F^{\infty}([0, \lambda], a) \subset U_a$ and $F^{\infty}([0, \lambda], b) \subset U_b$ (from Theorem 1.4). Consider $F^{\infty}(s + \lambda, x)$. This is the arc with the end-points $\bar{a}$ and $\bar{b}$. We may assume that $\pi^{\infty}(\lambda, \bar{a}) = a$ and $\pi^{\infty}(\lambda, \bar{b}) = b$. It is enough to put $s_2 = s - \lambda$ and $\bar{s} = s + \lambda$.

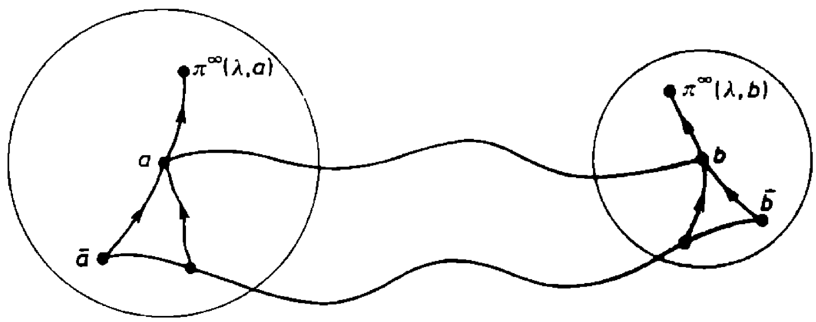

Fig. 2

By the continuity of ψ and the property $\psi(s, p) = t$ (see (2.1.4)) there is an $s_1 \in (s, \bar{s})$ such that

(2.6.5) $$\psi(s_1, F^\infty(s_1-s, p)) \subset (t-\tfrac{1}{4}T, t+\tfrac{1}{4}T).$$

In order to show this denote $F^\infty(\bar{s}-s, p)$ by J. There are an $\omega > 0$ and a neighbourhood V of p with $\psi((s-\omega, s+\omega), V) \subset (t-\tfrac{1}{4}T, t-\tfrac{1}{4}T)$. By Theorem 1.4 there is an $\eta \in (0, \omega)$ such that $\pi^\infty([\bar{s}-s-\eta, \bar{s}-s], J) = F^\infty([0, \eta], p) \subset V$. If we take $s_1 = s+\eta$ then $F^\infty(s_1-s, p) = \pi^\infty(\bar{s}-s_1, J) \subset V$ and $\psi(s_1, F^\infty(s_1-s, p)) \subset (t-\tfrac{1}{4}T, t+\tfrac{1}{4}T)$.

By (2.6.4) we have also (as $s_1 \in (s, \bar{s})$)

(2.6.6) $\pi^\infty([0, s_1-s_2], \tilde{a}) \cap \pi^\infty([0, s_1-s_2], \tilde{b}) = \emptyset$, $0 < s_1-s_2 < \tfrac{1}{2}T$, where $\tilde{a}$, $\tilde{b}$ are the ends of the arc $F^\infty(s_1, x)$.

By Theorem 1.16 and (2.6.6) the set $\pi^\infty([0, s_1-s_2], F^\infty(s_1, x))$ is homeomorphic to a rectangle and contains p in its interior. Write $L = F^\infty(s_1, x)$, $\Gamma = F^\infty(s_1-s, p)$ (Fig. 3). Of course $\Gamma \subset L$ and Γ is an arc or a point. Moreover, $\tilde{a}, \tilde{b} \notin \Gamma$, as $p \in \operatorname{Int} \pi^\infty([0, s_1-s_2], L)$.

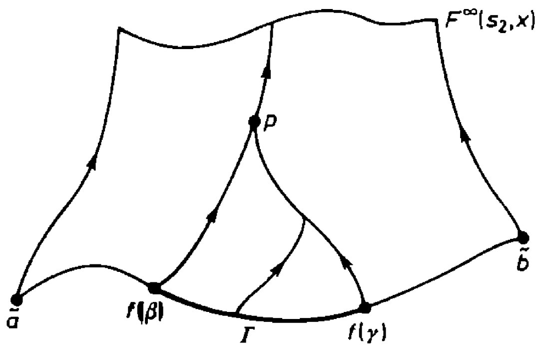

Fig. 3

Let us define a parametrization:

(2.6.7) $$f\colon [0, 1] \to L, \quad 0 < \beta \leqslant \gamma < 1, \quad f(\beta)f(\gamma) = \Gamma.$$

Now we show that

(2.6.8) $$\psi(s_1, q) = t+\psi(s_1-s, q) \quad \text{for each } q \in \Gamma.$$

We have

$$\pi(\psi(s_1, q), q) = \pi^\infty(s_1, q) = x,$$
$$\pi(t-\psi(s_1-s, q), q) = \pi\big(t, \pi(\psi(s_1-s, q), q)\big) = \pi(t, \pi^\infty(s_1-s, q)) = \pi(t, p) = x.$$

Thus $\psi(s_1, q) = t+\psi(s_1-s, q)$ for a regular point x and $\psi(s_1, q) = t+\psi(s_1-s, q)+nT$ for a periodic point x. But from (2.6.5) it follows that $\psi(s_1, q) \in (t-\tfrac{1}{4}T, t+\tfrac{1}{4}T)$ and from (2.2.3) it follows that $0 \leqslant \psi(s_1-s, q) \leqslant s_1-s \leqslant s_1-s_2 \leqslant \tfrac{1}{2}T$, so $nT = \psi(s_1, q)-t-\psi(s_1-s, q) \in (-\tfrac{3}{4}T, \tfrac{1}{4}T)$. According to this $n = 0$ and (2.6.8) holds also for a periodic point x.

Now we put

(2.6.9) $$\lambda_y = \varphi(\psi(s_1, y)-t, y).$$

For $y \in \Gamma$ we have $\psi(s_1, y) > t$, so (by the compactness of Γ) $\psi(s_1, \Gamma) \geqslant t+\chi$ for some $\chi > 0$. This shows that for each point y from some neighbourhood of Γ we have $\psi(s_1, y) > t$, which means that λ_y is well defined for these y. Thus the function λ of the variable y is defined in some neighbourhood of Γ. Moreover, in this neighbourhood $\lambda\colon y \mapsto \lambda_y$ is a continuous function, as the composition of continuous functions.

Now we show that:

(2.6.10) $$\pi^\infty(\lambda_y, q) = p \quad \text{for every } q \in \Gamma.$$

Using (2.6.8), (2.1.3) and (2.2.1), we get

$$\pi^\infty\big(\varphi(\psi(s_1, q)-t, q), q\big) = \pi^\infty\big(\varphi(t+\psi(s_1-s, q)-t, q), q\big)$$
$$= \pi^\infty\big(\varphi(\psi(s_1-s, q), q), q\big) = \pi\big(\psi(s_1-s, q), q\big) = \pi^\infty(s_1-s, q) = p.$$

Now we show that

(2.6.11) $$\pi(t, \pi^\infty(\lambda_y, y)) = x \quad \text{if } y \in L \text{ is such that } \lambda_y \text{ is defined.}$$

Using (2.1.3) and (2.2.1), we get:

$$\pi\Big(t, \pi^\infty\big(\varphi(\psi(s_1, y)-t, y), y\big)\Big) = \pi\Big(t, \pi(\psi(s_1, y)-t, y)\Big)$$
$$= \pi(\psi(s_1, y), y) = \pi^\infty(s_1, y) = x, \quad \text{as } y \in L.$$

Next we prove that there are neighbourhoods U_β and U_γ of $f(\beta)$ and $f(\gamma)$ such that λ_y is defined for each $y \in (U_\beta \cup U_\gamma) \cap L$ and

(2.6.12) $$\lambda_y \in (0, s_1-s_2) \quad \text{for } y \in (U_\beta \cup U_\gamma) \cap L.$$

In order to prove this it is enough to show that $\lambda_q \in (0, s_1-s_2)$ for every $q \in \Gamma$. We have $\lambda_q > 0$, as $\psi(s_1, q) > t$. Let us notice that $s_1-s_2 \leqslant s_1-s$ and $\psi(s_1-s_2, q) \leqslant \psi(s_1-s, q)$, so $\varphi_q^{-1}(s_1-s_2) \leqslant \psi(s_1-s, q)$, and by (2.6.8) $s_1-s_2 \leqslant \varphi(\psi(s_1-s, q), q) = \lambda_q$ which finishes the proof of (2.6.12).

From (2.6.12) it follows that $\pi^\infty(\lambda_y, y) \in \pi^\infty((0, s_1-s_2), y)$ for $y \in (U_\beta \cup U_\gamma) \cap L$, so (by (2.6.11)) $F(t, x) \cap \pi^\infty((0, s_1-s_2), y) \neq \emptyset$. Thus there exist α and δ such that $0 \leqslant \alpha < \beta \leqslant \gamma < \delta \leqslant 1$ and $\pi^\infty((0, s_1-s_2), y) \cap F(t, x) \neq \emptyset$ for every $y \in f[\alpha, \beta]$ (we use the definitions of β, γ and Γ).

Now we show that

(2.6.13) For each $y \in f[\alpha, \delta]$ the set $\pi^\infty((0, s_1-s_2), y) \cap F(t, x)$ has exactly one element.

Assume that $0 < \mu \leqslant \nu < s_1-s_2$ and $\pi(t, \pi^\infty(\mu, y)) = \pi(t, \pi^\infty(\nu, y)) = x$. Thus

$$x = \pi\Big(t, \pi(\psi(\mu, y), y)\Big) = \pi\Big(t, \pi(\psi(\nu, y), y)\Big) \quad \text{and}$$
$$x = \pi(t+\psi(\mu, y), y) = \pi(t+\psi(\nu, y), y).$$

But $\psi(\mu, y) \leqslant \psi(\nu, y)$, so

$$
\begin{aligned}
x &= \pi(t+\psi(v, y)-\psi(\mu, y)+\psi(\mu, y), y) \\
&= \pi\big(\psi(v, y)-\psi(\mu, y), \pi(t+\psi(\mu, y), y)\big) \\
&= \pi(\psi(v, y)-\psi(\mu, y), x).
\end{aligned}
$$

This means that $\psi(v, y) = \psi(\mu, y)$ for a regular point x and $\psi(v, y)-\psi(\mu, y) = nT$ for a periodic point x. However, in the second case $\psi(v, y) \leqslant v < s_1-s_2 < \frac{1}{2}T$, $0 < \psi(\mu, y) \leqslant \psi(v, y)$ and $\psi(v, y)-\psi(\mu, y) \in [0, \frac{1}{2}T)$. Thus in every case $\psi(v, y) = \psi(\mu, y)$ and $v = \mu$.

Write

(2.6.14) $$U = \operatorname{Int} \pi^{\infty}([0, s_1-s_2], f[\alpha, \delta]).$$

Then

(2.6.15) $$\bar{U} = \pi^{\infty}([0, s_1-s_2], f[\alpha, \delta]).$$

We show that $F(t, x) \cap \bar{U}$ is compact and connected. By (2.6.11) and (2.6.13) we have:

$$
\begin{aligned}
F(t, x) \cap \bar{U} &= \bigcup\{\pi^{\infty}((0, s_1-s_2), y) \cap F(t, x)\colon y \in f[\alpha, \delta]\} \\
&= \bigcup\{\pi^{\infty}(\lambda_y, y)\colon y \in f[\alpha, \delta]\} = \Phi(f[\alpha, \delta]), \quad \text{where } \Phi = \pi^{\infty} \circ (\lambda, \mathrm{id})
\end{aligned}
$$

(by id we denote the identity function from $\boldsymbol{R}^2$ onto itself). Thus $\bar{U} \cap F(t, x)$ is the continuous image of a compact and connected set.

We show also that:

(2.6.16) for every $q \in F(t, x) \cap \bar{U}$ there exist closed sets A and B such that $A \cap B = \{q\}$, $\pi^{\infty}(\lambda_{f(\alpha)}, f(\alpha)) \in A$, $\pi^{\infty}(\lambda_{f(\delta)}, f(\delta)) \in B$ and $A \cup B = F(t, x) \cap \bar{U}$.

Let $q = \pi^{\infty}(\lambda_y, y)$ (Fig. 4), $\xi \in [\alpha, \delta]$, $y = f(\xi)$. Put $A = \Phi(f[\alpha, \xi])$, $B = \Phi(f[\xi, \gamma])$. Of course A and B are compact, $A \cup B = F(t, x) \cap \bar{U}$.

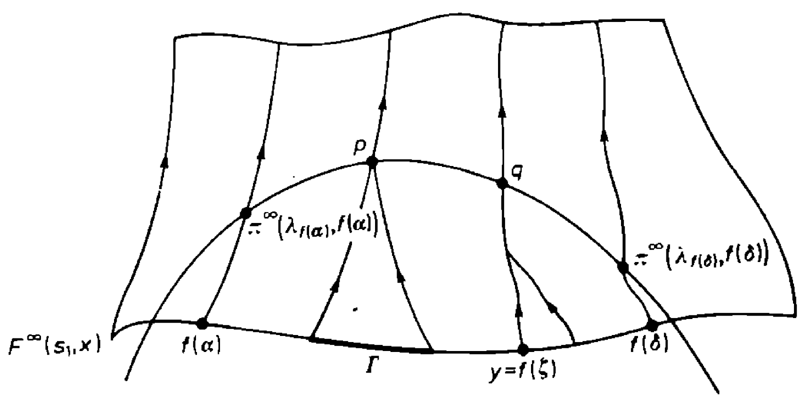

Fig. 4

Let $q_1 \in A \cap B$. Then $q_1 = \pi^{\infty}(\lambda_{f(\zeta)}, f(\zeta)) = \pi^{\infty}(\lambda_{f(\tau)}, f(\tau))$ $(\zeta \leqslant \xi, \tau \geqslant \xi)$ and (by (2.6.12)) $\lambda_{f(\zeta)}, \lambda_{f(\tau)} \in (0, s_1-s_2)$. By Lemma 1.14, $\lambda_{f(\zeta)} = \lambda_{f(\tau)}$ and by

Lemma 1.19 we get $q_1 = \pi^\infty(\lambda_{f(\zeta)}, (\zeta)) = \pi^\infty(\lambda_{f(\tau)}, f(\tau)) = \pi^\infty(\lambda_{f(\xi)}, f(\xi)) = q$. We have proved (2.6.16).

Using Theorem 1.24 we conclude that $\bar{U} \cap F(t, x)$ is homeomorphic to the interval $[0, 1]$, let $\bar{U} \cap F(t, x) = h[0, 1]$, where h is a suitable homeomorphism. Then U is the required neighbourhood of p, $\varepsilon = s_1 - s_2$, $c = f(\alpha)$, $d = f(\delta)$, $cd = f[\alpha, \delta]$. The property (2.6.3) follows from (2.6.13).

2.7. PROPOSITION. *If we change the assumptions of Lemma 2.6 in that p is now the end-point of the arc $F^\infty(s, x)$ we get (repeating the above proof in the simpler case):*

(2.7.1) *there exists a homeomorphism onto its image h: $[0, 1) \to F(t, x)$ such that $h(0) = p$,*

(2.7.2) *there exist $\tilde{s}$, ε (with $0 < \tilde{s} - s < \varepsilon < \frac{1}{2}T$), c, d and an arc $cd \subset F^\infty(\tilde{s}, x)$ such that $\pi^\infty([0, \varepsilon], c) \cap \pi^\infty([0, \varepsilon], d) = \emptyset$, $\pi^\infty(\tilde{s} - s, c) = p$, $\pi^\infty((0, \varepsilon), cd) \cap F(t, x) = h[0, 1]$, $h(1) \in \pi^\infty((0, \varepsilon), d)$, $h(0, 1) \subset \operatorname{Int} \pi^\infty([0, \varepsilon], cd)$.*

2.8. LEMMA. *For every left maximal solution σ^∞: $(-\infty, 0] \to \boldsymbol{R}^2$ through x in the system $(\boldsymbol{R}^2, \boldsymbol{R}_+, \pi^\infty)$ and $t > 0$ there exists at most one number $s > 0$ with the property $\varphi(t, \sigma^\infty(-s)) = s$. Then $\pi(t, \sigma^\infty(-s)) = x$ and $\sigma^\infty(-s) \in F(t, x)$.*

Proof. Let us define the function $\tilde{\psi}$: $\boldsymbol{R}_+ \to \boldsymbol{R}_+$ as follows:

(2.8.1) $$\tilde{\psi}(s) = \psi(s, \sigma^\infty(-s)).$$

Let $u < s$. Then $\pi^\infty\big(\psi(s-u, \sigma^\infty(-s)), \sigma^\infty(-s)\big) = \pi^\infty(s-u, \sigma^\infty(-s)) = \sigma^\infty(-u)$. Moreover,

$$\begin{aligned}\pi^\infty\big(\psi(s, \sigma^\infty(-s)), \sigma^\infty(-s)\big) &= x = \pi^\infty\big(\psi(u, \sigma^\infty(-u)), \sigma^\infty(-u)\big)\\ &= \pi^\infty\Big(\psi(u, \sigma^\infty(-u)), \pi^\infty\big(\psi(s-u, \sigma^\infty(-s)), \sigma^\infty(-s)\big)\Big)\\ &= \pi^\infty\big(\psi(u, \sigma^\infty(-u)) + \psi(s-u, \sigma^\infty(-s)), \sigma^\infty(-s)\big).\end{aligned}$$

The point x is non-stationary, so

$$\psi(s, \sigma^\infty(-s)) = \psi(u, \sigma^\infty(-u)) + \psi(s-u, \sigma^\infty(-s))$$

for a regular point x and

$$\psi(s, \sigma^\infty(-s)) = \psi(u, \sigma^\infty(-u)) + \psi(s-u, \sigma^\infty(-s)) + kT^\infty$$

for a periodic point x.

Let us fix s. By the continuity of σ^∞ it follows that there exists a $\delta > 0$ such that if $0 < s-u < \delta$, then $|\psi(s-u, \sigma^\infty(-s))| < \frac{1}{4}T^\infty$ (because $\psi(0, \sigma^\infty(-s)) = 0$) and $|\psi(s, \sigma^\infty(-s)) - \psi(u, \sigma^\infty(-u))| < \frac{1}{4}T^\infty$. This means that $k = 0$. Thus in every case $\psi(u, \sigma^\infty(-u)) < \psi(s, \sigma^\infty(-s))$ for $0 < s-u < \delta$ (as $\psi(s-u, \sigma^\infty(-s)) > 0$).

Now fix u. In the same way as above we show that there exists a $\delta > 0$ such that if $0 < s-u < \delta$, then $|\psi(s, \sigma^\infty(-s)) - \psi(u, \sigma^\infty(-u))| < \frac{1}{4}T^\infty$ and

$|\psi(s-u, \sigma^\infty(-s))| < \frac{1}{4}T^\infty$ (because $\psi(0, \sigma^\infty(-u)) = 0$ and σ^∞ is continuous). Again $k = 0$ and $\psi(u, \sigma^\infty(-u)) < \psi(s, \sigma^\infty(-s))$.

From the above investigation it follows that for each $w \in \boldsymbol{R}_+$ there is a neighbourhood W of w such that $\tilde{\psi}$ is strictly increasing in W. This means that $\tilde{\psi}$ is strictly increasing in $\boldsymbol{R}_+$.

Let s_1, s_2 be such that $\varphi(t, \sigma^\infty(-s_i)) = s_i$ $(i = 1, 2)$. By (2.1.4) we conclude that $\psi(s_i, \sigma^\infty(-s_i)) = t$, so $\tilde{\psi}(s_1) = \tilde{\psi}(s_2) = t$ and $s_1 = s_2$.

In order to finish the proof it is enough to notice that

$$\begin{aligned}\pi(t, \sigma^\infty(-s)) &= \pi^\infty(\varphi(t, \sigma^\infty(-s)), \sigma^\infty(-s)) \\ &= \pi^\infty(s, \sigma^\infty(-s)) = \sigma^\infty(s-s) = \sigma^\infty(0) = x\end{aligned}$$

if $\varphi(t, \sigma^\infty(-s)) = s$.

2.9. Remark. In the case when x is a periodic point it is possible that there are more than one point belonging to $F(t, x)$ (for fixed t) on one negative trajectory being the image of the left maximal solution σ through x. However, in such case only one of these points is given on $F(t, x)$ by the solution σ. The others are given by the other solutions which have the same negative trajectory as the image.

2.10. Remark. It is possible to describe quite precisely the behaviour of the left maximal solution through the given point x and introduce the natural order and topology in the set of solutions. This can be found in [6].

2.11. Lemma. *There are at most two points in $F(t, x)$ which do not fulfil the conclusion of Lemma 2.6. If a point p does not fulfil this lemma, then it fulfils the conclusion of Proposition 2.7 or it is the isolated point in $F(t, x)$; in the last case $F(t, x) = \{p\}$.*

Proof. Notice that if $p \in F(t, x)$ does not fulfil the conclusion of Lemma 2.6 and $\varphi(t, p) = s$, then p is the end-point of the arc $F^\infty(s, x)$ or $\{p\} = F^\infty(s, x)$. In both cases $p = \sigma_0^\infty(-s)$, where σ_0^∞ is a boundary solution through x in the system $(\boldsymbol{R}^2, \boldsymbol{R}_+, \pi^\infty)$. By Lemma 2.8 if follows that there are at most two such points. If p is the end-point of the arc $F^\infty(s, x)$, then p satisfies the conclusion of Proposition 2.7. If $\{p\} = F^\infty(s, x)$, then $\sigma^\infty(-s) = p$ for every left maximal solution σ^∞ through x in the system $(\boldsymbol{R}^2, \boldsymbol{R}_+, \pi^\infty)$. Suppose that there exists a $q \in F(t, x)$, $q \neq p$. Then for some left maximal solution σ_1^∞ through x in $(\boldsymbol{R}^2, \boldsymbol{R}_+, \pi^\infty)$ we have $\sigma_1^\infty(-s_1) = q$, $\sigma_1^\infty(-s) = p$ and $\varphi(t, q) = s_1$, where s_1 is a number different from s. This contradicts Lemma 2.8.

In order to characterize precisely $F(t, x)$ we present some lemmas.

2.12. Lemma. *If $p \in y_1y_2 \subset F(t, x)$, $y_1 \neq p$, $y_2 \neq p$ and $\varphi(t, p) = s$, then p is not the end-point of the arc $F^\infty(s, x)$.*

Proof. As in the proof of Lemma 2.10 notice that $F^\infty(s, x)$ is not a point (in the other case $F(t, x) = \{p\}$). Let $F^\infty(s, x) = ab$.

Let us fix a $\lambda < s$ (if x is periodic, then also $\lambda < \frac{1}{2}T \leqslant \frac{1}{2}T^{\infty}$) such that $\pi^{\infty}([0, \lambda], a) \cap \pi^{\infty}([0, \lambda], b) = \emptyset$. Write $F^{\infty}(s+\lambda, x) = a_1 b_1$, where $\pi^{\infty}(\lambda, a_1) = a$, $\pi^{\infty}(\lambda, b_1) = b$. The set $\pi^{\infty}([0, 2\lambda], a_1 b_1)$ is homeomorphic to the rectangle.

Suppose to the contrary that $p = a$. We can find a neighbourhood U of a such that $\varphi(t, U) \subset (s-\lambda, s+\lambda)$ and $U \cap \pi^{\infty}([0, 2\lambda], b_1) = \emptyset$. Thus there exist points c and d such that $a \in cd \subset y_1 y_2 \cap U$ and $c \neq a$, $d \neq a$ (Fig. 5). For each $y \in cd$ we have $x = \pi(t, y) = \pi^{\infty}(\varphi(t, y), y) \in \pi^{\infty}((s-\lambda, s+\lambda), y)$, so $y \in F^{\infty}((s-\lambda, s+\lambda), x) = \pi^{\infty}((0, 2\lambda), a_1 b_1)$. By Lemma 1.14 it follows that $c, d \notin \pi([0, 2\lambda], a_1)$. We show that

(2.12.1) there exist $q \in a_1 b_1$ and $z, \bar{z} \in cd \cap \pi^{\infty}((0, 2\lambda), q)$ $(z \neq \bar{z})$.

Denote $\partial\pi([0, 2\lambda], a_1 b_1)$ by Γ.

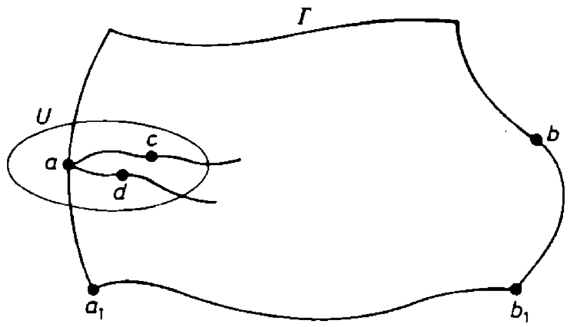

Fig. 5

By Theorem 1.16 it follows that there are $c_1, d_1 \in a_1 b_1$ (different from a_1 and b_1) such that $c \in \pi^{\infty}([0, 2\lambda], c_1)$ and $d \in \pi^{\infty}([0, 2\lambda], d_1)$. If $c_1 = d_1$ then we put $z = c$, $\bar{z} = d$ and $q = c_1 = d_1$. If $c_1 \neq d_1$ then we may assume that $c_1 \in a_1 d_1 \subset a_1 b_1$ (Fig. 6).

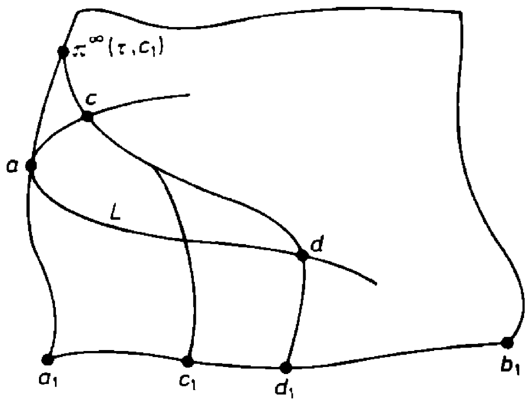

Fig. 6

By Lemma 1.20 the arc $L = \pi^{\infty}([0, 2\lambda], c_1)$ is a cross-cut of $\operatorname{Ins}\Gamma$. If $d \in L$, then it is enough to take $q = c_1$, $z = c$, $\bar{z} = d$. If $d \notin L$, then by Lemma 1.20 d belongs to the component of $\operatorname{Ins}\Gamma \setminus L$ which contains b_1 and d_1 in its boundary (as $\pi^{\infty}([0, \alpha], d_1)$ is contained in this component for some α).

If $a \in L$, then we put $q = c_1$, $z = a$, $\bar{z} = c$. If $a \notin L$, define $\tau = \inf\{\xi: \pi^\infty(\xi, c_1) \in \Gamma\}$; $\lambda < \tau \leqslant 2\lambda$. Denote by $a_1 \pi^\infty(\tau, c_1)$ the arc contained in Γ with the ends a_1 and $\pi^\infty(\tau, c_1)$, which contains a. Then $L \cup \big(\Gamma \backslash (c_1 a_1 \cup a_1 \pi^\infty(\tau, c_1))\big)$ is a Jordan curve obtained after cutting the region $\operatorname{Ins} \Gamma$ by the arc L. The assumption of Lemma 1.26 (with the points a and d) are fulfilled, a belongs to the exterior of this curve and d belongs to the interior. Thus this curve has a common point with ad. By the properties of cd it follows that $ad \cap L \neq \emptyset$ and $c \notin ad \cap L$, so it is enough to take as z any point belonging to $ad \cap L$ and put $\bar{z} = c$ and $q = c_1$. We have proved (2.12.1).

We may assume that $z = \pi^\infty(\mu, q)$, $\bar{z} = \pi^\infty(\nu, q)$, $0 < \mu < \nu < 2\lambda$. Then $z = \pi^\infty(\mu, q) = \pi(\psi(\mu, q), q)$ and $x = \pi(t, z) = \pi(t + \psi(\mu, q), q)$; in the same way $x = \pi(t + \psi(\nu, q), q)$. The function ψ is strictly increasing, so $0 < \psi(\mu, q) < \psi(\nu, q) < \psi(2\lambda, q) \leqslant 2\lambda < T$; thus $\pi(t + \psi(\mu, q), q) = \pi(t + \psi(\nu, q), q)$ and $0 < \psi(\nu, q) - \psi(\mu, q) < T$, which contradicts Lemma 1.14.

2.13. LEMMA. *If $F(t, x)$ is not a point, then each component of $F(t, x)$ is a 1-dimensional manifold. Moreover, either at most two components of $F(t, x)$ are homeomorphic to $\boldsymbol{R}_+$ and none of them to $[0, 1]$, or exactly one component of $F(t, x)$ is homeomorphic to $[0, 1]$ and none of them to $\boldsymbol{R}_+$.*

If $f: \boldsymbol{R}_+ \to S_i$ is the parametrization of the component S_i, then $\lim |f(w)| = \infty$, when $w \to \infty$. If $f: \boldsymbol{R} \to S_i$ is the parametrization, then $\lim |f(w)| = \infty$, when $w \to \infty$ and when $w \to -\infty$.

Proof. Consider the component H of $F(t, x)$. By Lemma 1.7, H is closed. By Lemma 2.11 at most two points in $F(t, x)$ do not fulfil the conclusion of Lemma 2.6. Denote them (if they exist) by a and b. If $a, b \notin H$, then, by Lemma 2.6, H is a 1-dimensional manifold without boundary.

Assume that $H = \{a\}$. Then by Lemma 2.11 if follows that a does not fulfil the conclusion of Lemma 2.6, so $F(t, x) = \{a\}$.

Let $a \in H$, $\{a\} \neq H$ and $b \notin H$. Let us present $H \backslash \{a\}$ as the union of its components: $H \backslash \{a\} = \bigcup \{S_i: i \in I\}$. As above, every S_i is the manifold without boundary, so is homeomorphic to $\boldsymbol{R}$ or S^1.

Suppose that H is a bounded set. Then H is a continuum, $H \backslash \{a\}$ is an open subset of H and by Theorem 1.23 we get that $a \in \bar{S}_i$ for each $i \in I$, so $\bar{S}_i = S_i \cup \{a\}$. Thus S_i is homeomorphic to $\boldsymbol{R}$ and $S_i = f(0, 1)$. Let us put

$$A_i = \bigcap \{\mathrm{Cl}\{f(w): 0 < w \leqslant \xi\}: \xi \in (0, 1)\},$$

$$B_i = \bigcap \{\mathrm{Cl}\{f(w): \xi \leqslant w < 1\}: \xi \in (0, 1)\}.$$

By Theorem 1.25, $A_i = B_i = \{a\}$. Thus we can construct the arc $f[\frac{2}{3}, 1) \cup \{a\} \cup f(0, \frac{1}{3}]$ containing a in its interior. Let $\varphi(t, a) = s$. Then by Lemma 2.12 it follows that a is not the end-point of the arc $F^\infty(s, x)$ and fulfils the conclusion of Lemma 2.6. We have a contradiction.

Now assume that H is not bounded. Let $\hat{H} = H \cup \{\infty\}$, so $\hat{H}$ is a continuum contained in S^2 and $\hat{H} \setminus \{a, \infty\}$ is its open subset. As above we denote by S_i the components of $H \setminus \{a\}$ and conclude that $\{\infty, a\}$ is not disjoint from the closure of S_i in S^2. Define in the same way as above $\hat{A}_i$ and $\hat{B}_i$. Using Theorem 1.25 we show that the three cases are possible:

(2.13.1) $$\hat{A}_i = \hat{B}_i = \{a\},$$

(2.13.2) $$\hat{A}_i = \{a\}, \quad \hat{B}_i = \{b\} \quad \text{(or vice versa)},$$

(2.13.3) $$\hat{A}_i = \hat{B}_i = \{\infty\}.$$

As above we show that (2.13.1) cannot hold. Suppose that (2.13.3) holds. Then S_i is a closed subset of $\boldsymbol{R}^2$, so it is a closed subset of H. Recall that $a \notin S_i$. But S_i is also an open subset of H, because by Lemma 2.6 it follows that for each point belonging to S_i there is a neighbourhood U of this point such that $U \cap F(t, x) \subset S_i \subset H$. Thus, using connectedness of H we get that $S_i = H$. This is impossible, because $a \in H$ and $a \notin S_i$.

We have shown that for each component S_i condition (2.13.2) holds. This means that there is only one such component. If there existed two such components, then we could find an arc contained in $F(t, x)$, containing a in its interior, so a would fulfil the conclusion of Lemma 2.6 which would contradict the assumption.

We have proved that if $a \in H$ and $b \notin H$ then H is homeomorphic to $\boldsymbol{R}_+$. Thus we get (using Lemma 2.11) that at most two components of $F(t, x)$ are homeomorphic to $\boldsymbol{R}_+$.

Consider the last case: $a, b \in H$, $a \neq b$. At first suppose that H is unbounded. Then $\hat{H} = H \cap \{\infty\}$ is a continuum contained in S^2. Considering the components S_i of the set $H \setminus \{a, b\}$ and using Theorem 1.25 (as in the previous case) we get (changing the parametrization, if necessary) that the only possible cases are:

(2.13.4) $$\hat{A}_i = \hat{B}_i = \{a\} \quad \text{or} \quad \hat{A}_i = \hat{B}_i = \{b\},$$

(2.13.5) $$\hat{A}_i = \hat{B}_i = \{\infty\},$$

(2.13.6) $$\hat{A}_i = \{a\}, \quad \hat{B}_i = \{\infty\} \quad \text{or} \quad \hat{A}_i = \{b\}, \quad \hat{B}_i = \{\infty\},$$

(2.13.7) $$\hat{A}_i = \{a\}, \quad \hat{B}_i = \{b\},$$

where $\hat{A}_i$, $\hat{B}_i$ are defined in the same way as previously.

In the same way as above we exclude the cases (2.13.4) and (2.13.5). Suppose that for some component S_0 (2.13.6) holds. In this case S_0 in the only one component of $H \setminus \{a, b\}$. If there is another component, it fulfils (2.13.6) or (2.13.7) and we can construct an arc contained in $F(t, x)$, containing a (or b) in its interior, which contradicts the assumption (according to Lemma 2.6).

This means that either $H = S_0 \cup \{a\} \cup \{b\}$ or $H = \{a\} \cup \{b\} \cup \bigcup \{S_i\colon i \in I\}$, where each from S_i $(i \in I)$ fulfils (2.13.6). But if for one of the components (say S_j) we have $\hat{A}_j = \{a\}$, $\hat{B}_j = \{\infty\}$, then (as above) for any component S_k different from S_j we have $\hat{A}_k \neq \{a\}$ and $\hat{B}_k \neq \{a\}$. The point b has the same properties. Thus we get $H \backslash \{a, b\} = S_{j_0}$, where j_0 fulfils (2.13.6) (for instance $\hat{A}_{j_0} = \{a\}$) or $H \backslash \{a, b\} = S_{j_1} \cup S_{j_2}$, where j_1 and j_2 satisfy (2.13.6) with $\hat{A}_{j_1} = \{a\}$ and $\hat{A}_{j_2} = \{b\}$. However, if $H \backslash \{a, b\} = S_{j_0}$, then $H = \bar{S}_{j_0} \cup \{b\}$, where $b \notin \bar{S}_{j_0}$; this contradicts the connectedness of H. If $H \backslash \{a, b\} = S_{j_1} \cup S_{j_2}$, then $H = \bar{S}_{j_1} \cup \bar{S}_{j_2}$; $a \in \bar{S}_{j_1}$ and $a \notin \bar{S}_{j_2}$, $b \in \bar{S}_{j_2}$ and $b \notin \bar{S}_{j_1}$. We have presented H as the union of two non-empty, disjoint (as $S_{j_1} \cap S_{j_2} = \emptyset$) closed subsets which is a contradiction.

We have proved that if H is unbounded then $H = S_0 \cup \{a\} \cup \{b\}$ and ∞ does not belong to the closure of H in S^2. This shows that H is bounded.

If H is bounded, then in the same way as above we get that the only possible cases are:

$$A_i = B_i = \{a\} \quad \text{or} \quad A_i = B_i = \{b\}, \tag{2.13.8}$$

$$A_i = \{a\}, \quad B_i = \{b\}. \tag{2.13.9}$$

In the same way as in the case when H is unbounded we get that $H = S_0 \cup \{a\} \cup \{b\}$, where $S_0 = f(0, 1)$ and $\lim f(t) = a$ $(t \to 0)$, $\lim f(t) = b$ $(t \to 0)$.

Now the last conclusions of the lemma follow quickly by using the fact that each component S_i is closed and the result that if S_i is homeomorphic to $\boldsymbol{R}$ then it cannot have the cluster point in S_i when $w \to \infty$ (we use Theorem 1.25).

2.14. LEMMA. *The set $F(t, x)$ does not contain any subset homeomorphic to S^1.*

Proof. Suppose the contrary. Denote by Γ the subset of $F(t, x)$, homeomorphic to S^1. Of course $t \geqslant N(x)$ (by Theorem 1.15). Considering the suitable isomorphism between $(\boldsymbol{R}^2, \boldsymbol{R}_+, \pi)$ and $(\boldsymbol{R}^2, \boldsymbol{R}_+, \pi^\infty)$ we have that for each $p \in \Gamma$ there is an s_p such that $\varphi(t, p) = s_p$. By Lemma 2.12 p is not the end-point of the arc $F^\infty(s, x)$. Thus we may use Lemma 2.6. For a given p denote by h_p, ε_p, c_p, d_p the suitable homeomorphism, number and points existing by Lemma 2.6, respectively. Let us put $U_p = \operatorname{Int} \pi^\infty([0, \varepsilon_p], c_p d_p)$, where $c_p d_p \subset F^\infty(s_p, x)$. The family $\{U_p\colon p \in \Gamma\}$ is a cover of Γ; moreover, by Lemma 2.6 $\bigcup \{U_p\colon p \in \Gamma\} \cap F(t, x) = \Gamma$. By the compactness of Γ we may choose a finite subcover, say $\{U_i\colon i = 1, \ldots, n\}$. Let us note that $c_i d_i \subset F^\infty(s_i, x)$ for $i = 1, \ldots, n$. Define $s = \max\{s_1, \ldots, s_n\}$ and $c_i' d_i' = F^\infty(s - s_i, c_i d_i)$, which is a compact arc by Lemma 1.21. The set $S = \bigcup \{c_i d_i\colon i = 1, \ldots, n\}$ is a compact subset of $F^\infty(s, x)$. Considering a homeomorphism $f\colon [0, 1] \to F^\infty(s, x)$ we may assume that:

(2.14.1) there is a $\lambda > 0$ with $f(\lambda) = c_1$ and $f[0, \lambda) \cap S = \emptyset$.

Recall that for each point $q \in c_i d_i$ the set $\pi^\infty((0, \varepsilon_i), q) \cap F(t, x)$ has exactly one element. Denote the only element of $\pi^\infty([0, \varepsilon_1), c_1) \cap F(t, x)$ by c^*. The family $\{U_i\colon i = 1, \ldots, n\}$ is the cover of Γ, so $c^* \in \pi^\infty((0, \varepsilon_k), c_k d_k \setminus \{c_k, d_k\})$ for some $k \in \{2, \ldots, n\}$. Notice that $c^* \in F^\infty(s^*, x)$, where $0 < s_1 - s^*$ and $0 < s_k - s^*$ (by Lemma 2.6); also $s_1 - s^* < \frac{1}{2}T$ and $s_k - s^* < \frac{1}{2}T$ if x is periodic of period T.

Denote by L the set $((\pi^\infty)^{s_k - s^*})^{-1}(\{c^*\})$; by Lemma 1.21, L is an arc. Moreover, $L \subset c_k d_k$ as $c_k d_k \cap L \neq \emptyset$, $c_k d_k \subset F^\infty(s_k, x)$ and $c_k, d_k \notin L$.

Let us take three left maximal solutions $\sigma_c, \sigma_d, \sigma^*$ through x in the system $(\boldsymbol{R}^2, \boldsymbol{R}_+, \pi^\infty)$ such that: $\sigma_c(-s_k) = c_k$, $\sigma_d(-s_k) = d_k$, $\sigma^*(-s^*) = c^*$ and $\sigma^*(-s) = c'_1$. Then $\sigma^*(-s_1) = c_1$ and $\sigma^*(-s_k)$ must lie on the arc $F^\infty(s_k, x)$ between $\sigma_c(-s_k)$ and $\sigma_d(-s_k)$, as $\sigma^*(-s_k) \in L$. If so, then $\sigma^*(-s)$ must lie on the arc $F^\infty(s, x)$ between $\sigma_c(-s)$ and $\sigma_d(-s)$ — the proof of this fact is simple but technical, so it will be omitted here (in case of periodic point x this has to be proved by induction). But this contradicts (2.14.1) as $\sigma_c(-s) \in S$ and $\sigma_d(-s) \in S$.

2.15. LEMMA. *The set $F(t, x)$ has at most a countable number of components.*

Proof. We may assume that $F(t, x)$ has more than one element. Consider $F(t, x) \setminus \{a, b\}$, where a and b are the only points (if they exist) not fulfilling the conclusion of Lemma 2.6. By Lemma 2.13 the number of components of the set $F(t, x) \setminus \{a, b\}$ is the same as the number of components of $F(t, x)$.

Consider a component S_i of $F(t, x) \setminus \{a, b\}$. For each $p \in S_i$ we can find an open neighbourhood U_p of p such that $U_p \cap F(t, x) \subset S_i$ (we use Lemma 2.6). Each set $U_i = \bigcup \{U_p\colon p \in S_i\}$ is open, connected and contains S_i. Moreover, $U_i \cap F(t, x) = S_i$.

We have constructed the family of open and pairwise disjoint sets; each one of them contains some S_i. This family is at most countable, so the set $\{S_i\colon i \in I\}$ is at most countable as well.

Now we state the first main theorem of this paper.

2.16. THEOREM. *If $F(t, x)$ has more than one point, then it is a 1-dimensional manifold.*

Proof. Denote by a and b the only points which do not fulfil the conclusion of Lemma 2.6 (if they exist). For the other points the neighbourhood required in the definition of manifold exists by Lemma 2.6. Consider a point a. Denote by S_a the component of $F(t, x)$ containing a and by S_b the component of $F(t, x)$ containing b. Let f be a parametrization of S_a (if $b \notin S_a$, then $f\colon \boldsymbol{R}_+ \to S_a, f(0) = a$; if $b \in S_a$, then $f\colon [0, 1] \to S_a, f(0) = a$, $f(1) = b$). Notice that for each component S of $F(t, x)$ different from S_a the distance $\varrho(a, S)$ is greater than 0 (in the other case $a \in \bar{S} = S$).

Map the plane homeomorphically onto itself to get $[0, 1] \times \{0\}$ as the image of $f[0, 1]$ and $(0, 0)$ as the image of a, which is possible by the Schönflies Theorem. This homeomorphism gives the new dynamical system and preserves the investigated topological properties. We will not change the notation of the sets and points of the plane and by a, $F(t, x)$ etc. we will mean the images of a, $F(t, x)$ etc. The component S_a is a manifold, so we can find an $\varepsilon_0 \in (0, 1)$ such that $S_a \cap B(a, \varepsilon_0) \subset f[0, 1]$ and if $S_b \neq S_a$ then also $S_b \cap B(a, \varepsilon_0) = \emptyset$.

In order to finish the proof we show that there exists and $\varepsilon > 0$ with $B(a, \varepsilon) \cap F(t, x) \subset f[0, 1]$. Suppose not. Then there is a sequence $\{z_n\} \subset F(t, x)$ such that $z_n \to a$, $z_n \notin [0, 1]$. We may assume that each z_n belongs to a different component of $F(t, x)$ (say: $z_n \in S_n$) as $\varrho(a, S) > 0$ for every component S different from S_a. By Lemma 2.13 it follows that $S_n \cap S(a, \frac{1}{2}\varepsilon_0) \neq \emptyset$ for each n greater than some n_0 (as $|f_n(w)| \to \infty$ if $w \to \infty$, f_n is the parametrization of S_n, which – by Lemma 2.14 – is homeomorphic to $\boldsymbol{R}$). Let $y_n \in S_n \cap S(a, \frac{1}{2}\varepsilon_0)$ (Fig. 7).

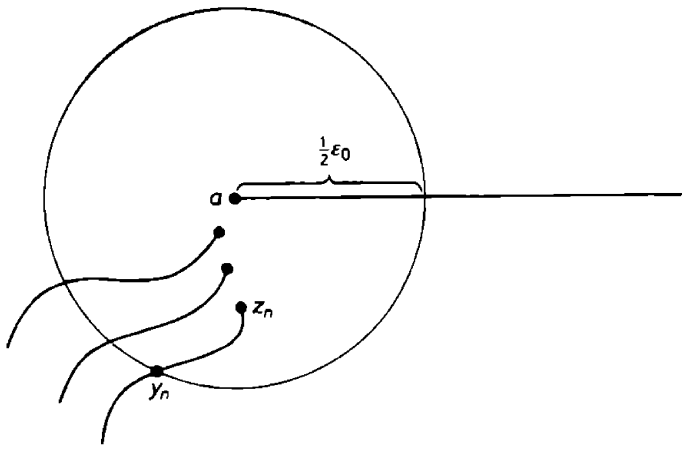

Fig. 7

The sequence $\{y_n\}$ has elements belonging to $S(a, \frac{1}{2}\varepsilon_0)$, so we can choose a subsequence tending to y_0. We get $y_0 \in F(t, x)$, as $\{y_n\} \subset F(t, x)$. Thus in every neighbourhood of y_0 there are points from an infinite number of components of $F(t, x)$. This means that y_0 does not fulfil the conclusion of Lemma 2.6, so $y_0 = a$ or $y_0 = b$. But $y_0 \in S(a, \frac{1}{2}\varepsilon_0)$, so $y_0 \neq a$. Moreover, $S_b \cap S(a, \frac{1}{2}\varepsilon_0) = \emptyset$, so $y_0 \neq b$. This contradiction finishes the proof.

Before stating our two main theorems which characterize precisely $F(t, x)$ we present the following

2.17. LEMMA. *If the set* $F(t, x)$ *contains a component* S_0 *homeomorphic to* $[0, 1]$, *then* $F(t, x) = S_0$.

Proof. At first we show that

(2.17.1) if for a given $s > 0$ the set $F(s, x)$ is not connected and contains a compact component C, then there exists a $\lambda > 0$ such that for each $\mu \in [s-\lambda, s]$ the set $F(\mu, x)$ is not connected and contains a compact component, which is equal to $\pi(s-\mu, C)$.

Assume that $C \neq F(s, x)$. By Lemmas 2.13 and 2.14 C is an arc and every component of $F(s, x)$ different from C is unbounded. Denote $F(s, x) \setminus C$ by B. Map the plane homeomorphically onto itself to get the segment $[-1, 1] \times \{0\}$

as the image of C. As in the proof of Theorem 2.16 we do not change the notation. By U_τ we denote the set $(-1-\tau, 1+\tau)\times(-\tau, \tau)$ (for any $\tau > 0$). By Theorem 2.16 and the compactness of C it follows that we can find an ε with the property $U_\varepsilon \cap F(s, x) = C$. Then $B = F(s, x)\backslash U_\varepsilon$ is the closed set. We will show that $\pi([0, \delta], B) \cap \bar{U}_{\varepsilon/3} = \emptyset$ for some $\delta > 0$.

Suppose not. Then there exist a sequence $\{\delta_n\}$ tending to 0 and a sequence $\{y_n\} \subset B$ such that $\pi(\delta_n, y_n) \in \bar{U}_{\varepsilon/3}$. If there is a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ with $y_{n_k} \to y_0$, then the sequence $\{\pi(\delta_{n_k}, y_{n_k})\}$ tends to $\pi(0, y_0) \in \bar{U}_{\varepsilon/3}$. However, $\bar{U}_{\varepsilon/3} \cap B = \emptyset$ and $y_0 \in \bar{B} = B$, which means that $|y_n| \to \infty$. Thus for sufficiently large n we have $y_n \notin \bar{U}_{\varepsilon/3}$ and $\pi(\delta_n, y_n) \in \bar{U}_{\varepsilon/3}$. We can find a $v_n \in (0, \delta_n)$ such that $\pi(v_n, y_n) \in \partial U_{\varepsilon/2}$ and from the sequence $\{\pi(v_n, y_n)\}$ we can choose a subsequence tending to $z_0 \in \partial U_{\varepsilon/2}$ (Fig. 8). We get $\pi(\delta_n, y_n) = \pi(\delta_n - v_n + v_n, y_n) = \pi(\delta_n - v_n, \pi(v_n, y_n))$, but $\delta_n - v_n \to 0$ and $\pi(v_n, y_n) \to z_0$, which means that $\pi(\delta_n, y_n) \to z_0 \in \partial U_{\varepsilon/2}$. This is impossible, as $\{\pi(\delta_n, y_n)\} \subset \bar{U}_{\varepsilon/3}$.

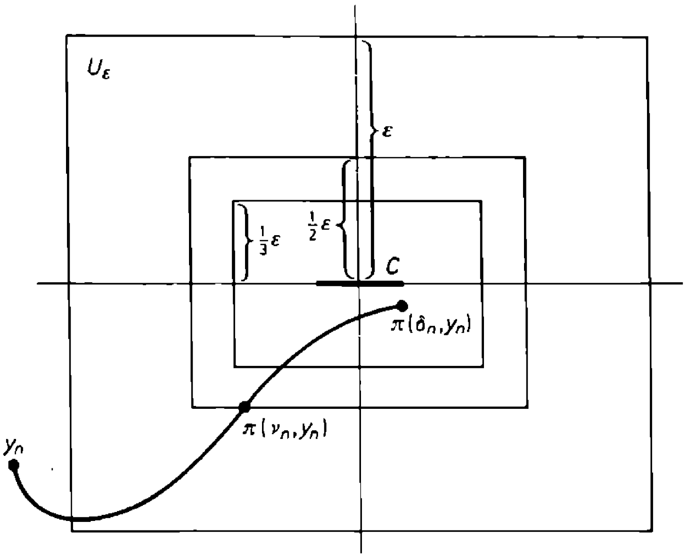

Fig. 8

By the continuity of π it follows that there is a δ_1 such that $\pi([0, \delta_1], c) \subset U_{\varepsilon/3}$. By the compactness of $\bar{U}_{\varepsilon/3}$, Lemma 1.7 and Theorem 1.4 we get that there is a δ_2 with the properties: $F([0, \delta_2], \bar{U}_{\varepsilon/3}) \subset U_\varepsilon$ and $F(\xi, \bar{U}_{\varepsilon/3})$ is non-empty and compact for each $\xi \in [0, \delta_2]$. Notice that for $\lambda = \min\{\delta, \delta_1, \delta_2\}$ and for every $\mu \in [s-\lambda, s]$ we have: $(F(\mu, x)\backslash\pi(s-\mu, C)) \cap \bar{U}_{\varepsilon/3} = \emptyset$. To show this take a $y \in F(\mu, x) \cap \bar{U}_{\varepsilon/3}$; then $\emptyset \neq F(s-\mu, y) \subset U_\varepsilon$, but $F(s-\mu, y) \subset F(s, x)$ and $\emptyset \neq F(s-\mu, y) \subset C$, because $U_\varepsilon \cap F(s, x) = \emptyset$. This means that $y \in \pi(s-\mu, C)$. Thus $\lambda = \min\{\delta, \delta_1, \delta_2\}$ satisfies (2.17.1), because for every $\mu \in [s-\lambda, s]$ we have that $\pi(s-\mu, C)$ is a compact and connected subset of $U_{\varepsilon/3}$ and

$$\emptyset \neq \pi(s-\mu, B) \subset F(\mu, x)\backslash\pi(s-\mu, C) \subset \mathbf{R}^2\backslash\bar{U}_{\varepsilon/3}.$$

Now we turn to the proof of the theorem. Suppose the contrary: the set $F(t, x)$ is not compact and contains a compact component S_0. Let us define $N = \{\beta \in [0, t]\colon \pi(t-\beta, S_0)$ is a compact component of $F(\beta, x)\}$. By the hypothesis we get that $t \in N$. The set $\pi(t-\beta, S_0)$ is compact, so if $\beta \in N$, then also $[\gamma, \beta] \subset N$ for some $\gamma < \beta$ (by (2.17.1) and Lemma 1.13).

Write $K = \bigcup \{\gamma \in [0, t]: (\gamma, t] \subset N\}$ and $\alpha = \inf K$.

By the hypothesis and Lemma 1.13 we have $t \geqslant N(x)$, and by (2.17.1) we get $t > N(x)$. Notice that $F(N(x), x)$ does not contain any compact component. In order to show this suppose that the non-compact set $F(N(x), x)$ contains a compact component. Then by (2.17.1) for some $\gamma > 0$ the set $F(N(x)-\gamma, x)$ is also non-compact and contains a compact component, which contradicts Lemma 1.13. Thus $t \notin N$ and $\alpha < t$.

Now notice that $\alpha \notin N$. This is because $(\alpha, t] \subset N$ and, as above, if $\alpha \in N$ then there is a γ with $(\gamma, \alpha] \subset N$, so $(\gamma, t] \subset N$ which contradicts the definition of α.

The above properties are shown in Fig. 9.

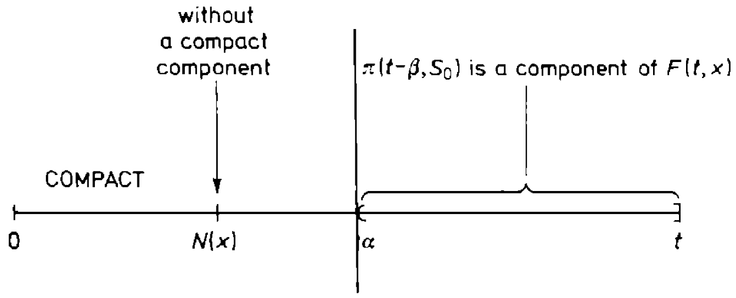

Fig. 9

After this preparation we can turn to the main part of the proof. Consider the set $\pi(t-\alpha, S_0)$. This set is compact and connected, so it is contained in the component S_1 of $F(\alpha, x)$. Moreover, $\pi(t-\alpha, S_0) \subset S_1$, because $\alpha \notin N$. By Lemmas 2.13 and 2.15 it follows that S_1 is a closed set homeomorphic to $\boldsymbol{R}$, $\boldsymbol{R}_+$ or $[0, 1]$, so $\pi(t-\alpha, S_0)$ is a compact arc contained in S_1 and different from S_1, or the point contained in S_1. Thus there exists an arc $L \subset S_1$ such that the set $L \cap \pi(t-\alpha, S_0)$ has exactly one element. Let a be the only element of $L \cap \pi(t-\alpha, S_0)$ and $b \in L \setminus \pi(t-\alpha, S_0)$ (Fig. 10). We can find a $u < t-\alpha$ and a closed neighbourhood W of the arc L such that $F([0, u], W)$ is compact (see Lemma 1.10) and $F(u, b) \neq \emptyset$ $(u < N(a),\ u < N(b))$. By Proposition 1.12 the set $F(u, L)$ is connected. Moreover, $\emptyset \neq F(u, a) \subset F(u, L)$.

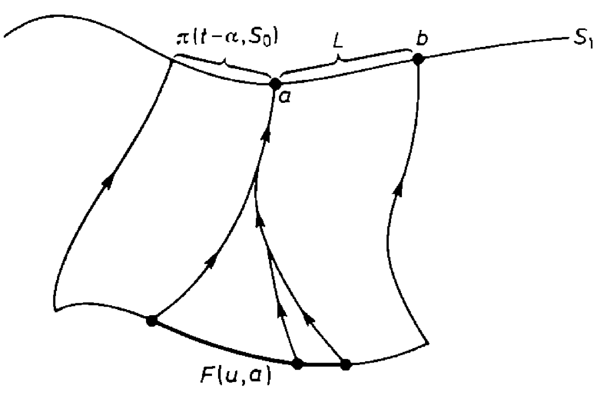

Fig. 10

We have that $a = \pi(t-\alpha, p)$ for some $p \in S_0$, as $a \in \pi(t-\alpha, S_0)$. Thus $a = \pi(u, \pi(t-\alpha-u, p))$. By the definition of α we have that $\pi(t-\alpha-u, p)$ belongs to the compact component of $F(\alpha+u, x)$, but also $\pi(t-\alpha-u, p) \in F(u, a) \subset F(u, L)$.

The set $\pi(t-\alpha-u, S_0) \cup F(u, L)$ is the connected set contained in $F(\alpha+u, x)$. By the properties of u there is a $q \in F(u, b) \subset F(u, L)$. However, $q \notin \pi(t-\alpha-u, S_0)$, because if $q \in \pi(t-\alpha-u, S_0)$, then $b = \pi(u, q) \in \pi(t-\alpha, S_0)$, which does not hold.

This shows that $\pi(t-\alpha-u, S_0)$ is not a component of $F(\alpha+u, x)$, which contradicts the definitions of α and N and finishes the proof of the lemma.

Now as the simple corollaries we get two main theorems of the paper, describing the set $F(t, x)$.

2.18. THEOREM. *Assume that $F(t, x)$ is a non-empty set which contains more than one element. Then $F(t, x)$ is a 1-dimensional manifold and exactly one of the following two conditions holds:*

(a) *$F(t, x)$ is homeomorphic to $[0, 1]$,*

(b) *$F(t, x)$ is at most countable union of pairwise disjoint closed connected 1-dimensional manifolds. At most two components of $F(t, x)$ are homeomorphic to $\boldsymbol{R}_+$, the rest of them are homeomorphic to $\boldsymbol{R}$. Denote by f the parametrization of the arbitrarily chosen component of $F(t, x)$. Then in the first case* $\lim|f(w)| = \infty$ *when $w \to \infty$ and in the second case* $\lim|f(w)| = \infty$ *when $w \to \infty$ and when $w \to -\infty$.*

If $t < N(x)$, then the set $F(t, x)$ is a point or condition (a) *holds. If $t \geqslant N(x)$, then the set $F(t, x)$ is empty or condition* (b) *holds.*

Proof. The theorem is an immediate corollary of Theorem 2.16 and Lemmas 1.13, 2.13, 2.14, 2.15 and 2.17.

2.19. THEOREM. *Exactly one of the following two conditions holds:*

(a) *the set $F(t, x)$ is a point for $t < N(x)$ and the empty set for $t \geqslant N(x)$,*

(b) *there exist α, β such that $0 \leqslant \alpha < N(x) \leqslant \beta \leqslant \infty$ and $F(t, x)$ is a point for $t \in [0, \alpha]$, the arc for $t \in (\alpha, (N(x))$, the set described in point* (b) *of Theorem* 2.18 *for $t \in [N(x), \beta]$ and the empty set for $t \in [\beta, \infty)$. Note that the sets $[N(x), \beta)$ and $[\beta, \infty)$ may be empty.*

Condition (a) *holds if and only if x is a point of negative unicity.*

Proof. The theorem is an immediate corollary of Theorem 2.18 and the following obvious facts:

(2.19.1) if $F(s, x)$ is a singleton, then for every $\alpha \in [0, s]$ the set $F(\alpha, x)$ is also a singleton,

(2.19.2) if $F(t, x)$ is not a singleton and is not empty, then there exists an $\varepsilon > 0$ such that $F(t-\alpha, x)$ is not a singleton for every $\alpha \in [0, \varepsilon)$, (2.19.3) if $F(t, x) = \emptyset$, then $F(s, x) = \emptyset$ for each $s \geqslant t$.

The last property follows by Lemma 1.6, as $F(s, x) = F(s-t, F(t, x)) = F(s-t, \emptyset) = \emptyset$.

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