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VARIETIES OF GENERALIZED LIE GROUPS

BY

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0. Introduction. In [2] and [7], varieties of topological groups generated by (finite dimensional) Lie groups are investigated. The reason to study varieties of topological groups generated by Lie groups instead of varieties of Lie groups is partially that the category of Lie groups is not closed under the operation of taking Cartesian products. This difficulty has been overcome by introducing the category Γ of generalized Lie groups in [3]. In this paper, some theorems of [2] and [7] are extended to the category of generalized Lie groups. Many well-known categories are full subcategories of the category of generalized Lie groups: the category of compact groups, the category of locally compact groups and the category of finite-dimensional Lie groups. We also consider the category of Hausdorff topological groups and the variety of that category generated by a class of generalized Lie groups.

1. Generalized Lie groups and locally convex topological Lie algebras.

1.1. Definition. A Lie algebra L is a *locally convex topological Lie algebra* if there is a family of seminorms $\{|\ |_i | i \in I\}$ defining a locally convex Hausdorff vector space topology of L such that, for each $i \in I$, $|[x, y]|_i \leqslant C_i |x|_i |y|_i$ for some $C_i \geqslant 0$. Let Ω^- denote the category of locally convex topological Lie algebras with continuous Lie algebra homomorphisms.

Let Ω denote the full subcategory of Ω^- consisting of objects L which satisfies the following property:

(*) The set $D_L = \{(x, y) | x \circ y \in L\}$ contains a convex set K such that $(x, y) \in K$ implies $x \circ y$ converges uniformly and absolutely on K, where

$$(**) \qquad x \circ y = x + y + \tfrac{1}{2}[x, y] + \tfrac{1}{12}[x, [x, y]] + \ldots$$

is the Campbell-Hausdorff formula.

A locally convex topological Lie algebra homomorphism $f\colon L_1 \to L_2$ is said to be *isometry* if L_1 and L_2 have families of seminorms indexed by a same index set I and, for $i \in I$ and $x \in L_1$, $|x|_i^1 = |f(x)|_i^2$. We say L_1

is a *subobject of* L_2 if there exists an isometrical Lie algebra monomorphism $j: L_1 \to L_2$.

1.2. Definition. Let $\mathrm{Hom}(\boldsymbol{R}, G)$ be the set of continuous group homomorphisms from $\boldsymbol{R}$ to G with scalar multiplication defined by $r \cdot x: \boldsymbol{R} \to G$ as $r \cdot x(t) = x(rt)$ for $r \in \boldsymbol{R}$ and $x \in \mathrm{Hom}(\boldsymbol{R}, G)$. A Hausdorff topological group G is said to be a *generalized Lie group* if $\mathrm{Hom}(\boldsymbol{R}, G)$ is an object in Ω and

(1) the map $\exp: \mathrm{Hom}(\boldsymbol{R}, G) \to G$ defined as $\exp(x) = x(1)$ is continuous,

$$\text{(2)} \qquad \exp(x+y) = \lim\left(\exp\frac{x}{n} \cdot \exp\frac{y}{n}\right)^n,$$

$$\exp[xy] = \lim\left(\left(\exp\frac{x}{n}\right) \cdot \left(\exp\frac{y}{n}\right) \cdot \left(\exp\frac{x}{n}\right)^{-1} \cdot \left(\exp\frac{y}{n}\right)^{-1}\right)^{n^2},$$

(3) on the convex set K (see (*)) of $D_{\mathrm{Hom}(\boldsymbol{R}, G)}$, $(x, y) \in K$ implies $\exp(x \circ y) = \exp x \cdot \exp y$.

A Hausdorff topological group G is a *Lie group* if $\mathrm{Hom}(\boldsymbol{R}, G)$ has a structure of a Lie algebra and a Banach space such that

(a) there is an open ball B around 0 of $\mathrm{Hom}(\boldsymbol{R}, G)$ such that $\exp|_B: B \to \exp(B)$ is a homeomorphism onto an open neighborhood of the identity of G,

(b) on the ball B of (a), the Campbell-Hausdorff multiplication (**) is defined as a continuous function from $B \times B \to \mathrm{Hom}(\boldsymbol{R}, G)$ such that $\exp(x \circ y) = \exp x \cdot \exp y$ for $x, y \in B$.

We remark here that a Lie group is a generalized Lie group [3]. We denote by Γ the category of generalized Lie groups with morphism $f: G \to G'$ which is a topological group homomorphism and $\mathrm{Hom}(\boldsymbol{R}, f): \mathrm{Hom}(\boldsymbol{R}, G) \to \mathrm{Hom}(\boldsymbol{R}, G')$ is in Ω.

We say H is a *generalized Lie subgroup* of G if H and G are in Γ and there exists a monomorphism $j: H \to G$, and $\mathrm{Hom}(\boldsymbol{R}, j)$ is an isometrical monomorphism in Ω.

Two generalized Lie groups G and G' are said to be *isomorphic* if there exists a homeomorphism (isomorphism) $f: G \to G'$ such that $\mathrm{Hom}(\boldsymbol{R}, f)$ is an isometrical isomorphism in Ω.

In the category of Hausdorff topological groups, an object Q is called an *$\boldsymbol{R}$-quotient* of object G if there exists a surjective homomorphism $q: G \to Q$ such that $\mathrm{Hom}(\boldsymbol{R}, q): \mathrm{Hom}(\boldsymbol{R}, G) \to \mathrm{Hom}(\boldsymbol{R}, Q)$ is onto.

Remark. The $\boldsymbol{R}$-quotient of a generalized Lie group is a generalized Lie group [3].

For other results of generalized Lie groups, readers should check [3] and [5].

2. $\boldsymbol{R}$-variety of a generalized Lie group.

2.1. Definition. $\mathscr{V}^{\boldsymbol{R}}$ is called an *$\boldsymbol{R}$-variety* in Γ if $\mathscr{V}^{\boldsymbol{R}}$ is a full subcategory of Γ and $\mathscr{V}^{\boldsymbol{R}}$ is closed under taking isomorphic objects of products, generalized Lie subgroups and $\boldsymbol{R}$-quotients, denoted by $\mathbf{C}$, $\mathbf{S}$ and $\mathbf{Q}^*$, respectively.

2.2. Theorem. *If $\mathscr{C}$ is a class of objects in Γ, then the $\boldsymbol{R}$-variety in Γ generated by $\mathscr{C}$ is $\mathbf{Q}^*\mathbf{SC}(\mathscr{C})$.*

Proof. Since an $\boldsymbol{R}$-quotient of Q which is an $\boldsymbol{R}$-quotient of G is an $\boldsymbol{R}$-quotient of G, $\mathbf{Q}^*\mathbf{SC}(\mathscr{C})$ is closed under $\mathbf{Q}^*$. If $\{Q_i \mid i \in I\}$ is a family of $\boldsymbol{R}$-quotients of $\{G_i \mid i \in I\}$ and $q_i\colon G_i \to Q_i$ is the canonical map for each $i \in I$, then the induced map $\Pi q_i\colon \Pi G_i \to \Pi Q_i$ is onto. We show it is $\boldsymbol{R}$-surjective. For any topological group homomorphism $f\colon \boldsymbol{R} \to \Pi Q_i$, we have $\pi_i f\colon \boldsymbol{R} \to \Pi Q_i \to Q_i$ for each $i \in I$, where π_i is the projection map for $i \in I$. Since $G_i \to Q_i$ is an $\boldsymbol{R}$-quotient, we have the lift $\overline{\pi_i f}\colon \boldsymbol{R} \to G_i$ such that $q_i \cdot \overline{\pi_i f} = \pi_i f$. The family of maps $\{\overline{\pi_i f} \mid i \in I\}$ induces $\bar{f}\colon \boldsymbol{R} \to \Pi G_i$ which satisfies $(\Pi q_i)\bar{f} = f$. Furthermore, for each $G_i \in \mathbf{SC}(\mathscr{C})$, we have $\Pi G_i \in \mathbf{SC}(\mathscr{C})$ and $\Pi Q_i \in \mathbf{Q}^*\mathbf{SC}(\mathscr{C})$. Last, we show that if $Q \in \mathbf{Q}^*\mathbf{SC}(\mathscr{C})$ and H is a generalized Lie subgroup of Q, then $H \in \mathbf{Q}^*\mathbf{SC}(\mathscr{C})$. Now, $Q \in \mathbf{Q}^*\mathbf{SC}(\mathscr{C})$ says that $Q = G/K$, where $G \in \mathbf{SC}(\mathscr{C})$. Let $q\colon G \to Q$ be the canonical map and let M be the pullback of the diagram

$$\begin{array}{ccc} & & G \\ & & \big\downarrow{\scriptstyle q} \\ H & \xrightarrow[j]{} & Q \end{array}$$

where j is the inclusion map. Since Γ has finite limits, $M \in |\Gamma|$, and the map $p\colon M \to H$ is an onto generalized Lie group morphism. For any $g\colon \boldsymbol{R} \to H$, we have a map $jg\colon \boldsymbol{R} \to Q$ which factors across q as $jg = q\bar{g}$. But M is a pullback, g factors as $g = pg'$. Hence H is an $\boldsymbol{R}$-quotient of M, and from the fact that $M \subset G$, we infer that $H \in \mathbf{Q}^*\mathbf{SC}(\mathscr{C})$.

2.3. Definition. Let Ω be the category of the locally convex Hausdorff topological Lie algebra. A full subcategory $\mathscr{V}$ of Ω is called a *variety* if $\mathscr{V}$ is closed under taking products, subobjects (isometrical) and quotients, denoted by $\mathbf{C}$, $\mathbf{S}$ and $\mathbf{Q}$, respectively.

2.4. Corollary. *Let $\mathscr{C}$ be a class of objects in Γ. Then, for each generalized Lie group G in $\mathscr{V}(\mathscr{C})$,* $\mathrm{Hom}(\boldsymbol{R}, G)$ *is in* $\mathbf{QSC}\{\mathrm{Hom}(\boldsymbol{R}, X) \mid X \in \mathscr{C}\}$.

3. *-Variety of a generalized Lie group.

3.1. Definition. Let $\mathfrak{G}$ be the category of Hausdorff topological groups. A full subcategory $\mathscr{V}^*$ of $\mathfrak{G}$ is called a **-variety* in $\mathfrak{G}$ if $\mathscr{V}^*$ is closed under operations taking products, closed subgroups and $\boldsymbol{R}$-quotients, denoted by $\mathbf{C}$, $\bar{\mathbf{S}}$ and $\mathbf{Q}^*$, respectively.

3.2. THEOREM. *Let $\mathscr{C}$ be a class of objects in $\mathfrak{G}$. Then the *-variety generated by $\mathscr{C}$, denoted by $\mathscr{V}^*(\mathscr{C})$, is $\mathbf{Q}^*\bar{\mathbf{S}}\mathbf{C}(\mathscr{C})$.*

Proof. Clearly, $\mathbf{Q}^*\bar{\mathbf{S}}\mathbf{C}(\mathscr{C})$ is closed under $\mathbf{Q}^*$. Let $\{Q_i \mid i \in I\}$ be a family of objects in $\mathbf{Q}^*\bar{\mathbf{S}}\mathbf{C}(\mathscr{C})$, by the same argument as in 2.2. We infer that $\prod_i Q_i$ is an $\boldsymbol{R}$-quotient of an object in $\bar{\mathbf{S}}\mathbf{C}(\mathscr{C})$. Finally, if Q is $\mathbf{Q}^*\bar{\mathbf{S}}\mathbf{C}(\mathscr{C})$ and H is a closed subgroup of Q, then $Q = G/K$, where $G \in \bar{\mathbf{S}}\mathbf{C}(\mathscr{C})$. Let M be the pullback of diagram

$$\begin{tikzcd} & G \arrow[d, "q"] \\ H \arrow[r, "j"'] & G/K \end{tikzcd}$$

where j is the inclusion map and q is the canonically quotient map. Then M is a closed subgroup of G, whence $M \in \bar{\mathbf{S}}\mathbf{C}(\mathscr{C})$. Again, by the use the same argument as in 2.2, we have $H \in \mathbf{Q}^*\bar{\mathbf{S}}\mathbf{C}(\mathscr{C})$.

3.3. PROPOSITION. *Let $\mathscr{C}$ be a class of objects in Γ. Then, for each object G in $\mathscr{V}^*(\mathscr{C})$,* $\mathrm{Hom}(\boldsymbol{R}, G)$ *is an object in Ω^-.*

Proof. By 3.2, we have $\mathscr{V}^*(\mathscr{C}) = \mathbf{Q}^*\bar{\mathbf{S}}\mathbf{C}(\mathscr{C})$. It is clear that the product ΠG_i for $G_i \in \mathscr{C}$ has a Lie algebra $\mathrm{Hom}(\boldsymbol{R}, \Pi G_i)$ being an object in Ω^-. Now, if H is a closed subgroup of ΠG_i, we infer that $\mathrm{Hom}(\boldsymbol{R}, H)$ is a subobject of $\mathrm{Hom}(\boldsymbol{R}, \Pi G_i)$ in Ω^- (the same argument as in Proposition 2 of [3]). Lastly, if G has a Lie algebra $\mathrm{Hom}(\boldsymbol{R}, G)$, being an object in Ω^-, and $\exp\colon \mathrm{Hom}(\boldsymbol{R}, G) \to G$ satisfying (2) of 1.2, and K is a normal closed subgroup of G, then $\mathrm{Hom}(\boldsymbol{R}, K)$ is an ideal in $\mathrm{Hom}(\boldsymbol{R}, G)$ (see [3], Proposition 7). Hence, if $G \to G/K$ is $\boldsymbol{R}$-surjective, then $0 \to \mathrm{Hom}(\boldsymbol{R}, K) \to \mathrm{Hom}(\boldsymbol{R}, G) \to \mathrm{Hom}(\boldsymbol{R}, G/K) \to 0$ is exact. $\mathrm{Hom}(\boldsymbol{R}, G/K)$ is an object in Ω^-.

3.4. COROLLARY. *Let $\mathscr{C}$ be a class of objects in Γ. Then* $\mathrm{Hom}(\boldsymbol{R}, -)$ *is a well-defined assignment to* $\mathrm{Hom}(\boldsymbol{R}, G) \in |\Omega^-|$ *for* $G \in |\mathscr{V}^*(\mathscr{C})|$ *and* $\mathrm{Hom}(\boldsymbol{R}, -)|\mathscr{V}^*(\mathscr{C})|$ *is a subclass of objects in* $\mathbf{QSC}\{\mathrm{Hom}(\boldsymbol{R}, X) \mid X \in \mathscr{C}\}$.

4. Nilpotent and solvable generalized Lie groups.

4.1. Definition. A generalized Lie group G is *nilpotent* (*solvable*) if $\mathrm{Hom}(\boldsymbol{R}, G)$ is a nilpotent (solvable) Lie algebra.

4.2. THEOREM. *If $\mathscr{C}$ is a class of nilpotent (solvable) Banach Lie groups in Γ, then, for any Banach Lie group G in the $\boldsymbol{R}$-variety $\mathscr{V}^{\boldsymbol{R}}(\mathscr{C}) \in \Gamma$, G is nilpotent (solvable).*

Proof. By 2.4, we have $\mathrm{Hom}(\boldsymbol{R}, G)$ being an object in Ω and $\mathrm{Hom}(\boldsymbol{R}, G) \in \mathbf{QSC}\{\mathrm{Hom}(\boldsymbol{R}, X) \mid X \in \mathscr{C}\}$. But G is a Banach Lie group, so $\mathrm{Hom}(\boldsymbol{R}, G)$ is a Banach Lie algebra. By the result of [2], we have $\mathrm{Hom}(\boldsymbol{R}, G) \in \mathbf{QSD}\{\mathrm{Hom}(\boldsymbol{R}, X) \mid X \in \mathscr{C}\}$, where D denotes the operation of taking finite product. Hence $\operatorname{Hom}(\boldsymbol{R}, G)$ is a quotient of an object which is a subobject of finite product of nilpotent (solvable) Lie algebras, so $\operatorname{Hom}(\boldsymbol{R}, G)$ is nilpotent (solvable).

4.3. THEOREM. *If $\mathscr{C}$ is a class of nilpotent (solvable) Banach Lie groups, then any Banach Lie group $G \in \mathscr{V}^*(\mathscr{C})$ is nilpotent (solvable).*

The proof follows by 3.4 and the result of [2].

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