ANNALES
POLONICI MATHEMATICI
XXV (1971)

On the proximate order of $f^{(s)}(z)*g^{(s)}(z)$ and $[f(z)*g(z)]^{(s)}$

by M. K. Sen (Calcutta)

A function $\varrho(r)$ that satisfies the condition

(i) $\varrho(r)$ is differentiable for $r > r_0$ except at isolated points at which $\varrho'(r-0)$ and $\varrho'(r+0)$ exist,

(ii) $\limsup_{r\to\infty} \varrho(r) = \varrho$,

(iii) $\lim_{r\to\infty} r\varrho'(r)\log r = 0$,

(iv) $\lim_{r\to\infty} \dfrac{\log M(r)}{r^{\varrho(r)}} = 1$,

is called a *proximate order* for the integral function $f(z)$ of order ϱ, $0 < \varrho < \infty$.

Shah ([3], p. 326-328) has proved the existence of a proximate order for every integral function. In this paper we are going to construct a proximate order for $f^{(s)}(z)*g^{(s)}(z)$ and also we shall construct a function which plays the role of a proximate order for both $f^{(s)}(z)*g^{(s)}(z)$ and $[f(z)*g(z)]^{(s)}$.

We assume that

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad g(z) = \sum_{n=0}^{\infty} b_n z^n$$

are two integral functions.

We define

$$f(z)*g(z) = \sum_{n=0}^{\infty} a_n b_n z^n,$$

$$f^{(s)}(z)*g^{(s)}(z) = \sum_{n=s}^{\infty} n^2(n-1)^2 \dots (n-s+1)^2 a_n b_n z^{n-s}$$

and $\big(f(z)*g(z)\big)^{(s)}$ denotes the s-th derivative of $f(z)*g(z)$. It is known ([1], p. 318) that $f(z)*g(z)$ and $f^{(s)}(z)*g^{(s)}(z)$ are of same order. Throughout this paper we shall assume that $f(z)$ and $g(z)$ are integral functions of regular growth and the order of $f(z)*g(z)$ is finite and different from zero.

1. Let $\mu(r, s)$ denote the maximum term of

$$\sum_{n=s}^{\infty} n^2 (n-1)^2 \ldots (n-s+1)^2 a_n b_n z^{n-s}$$

and let $\nu(r, s)$ be the rank of this term.

Lemma 1. *If $f(z)$ and $g(z)$ be two integral functions of regular growth and $\lim_{r\to\infty} [\nu(r, s)-\nu(r, 0)]$ exists, then*

$$\lim_{r\to\infty} [\nu(r, s)-\nu(r, 0)] = 2s\varrho,$$

where ϱ is the order of $f(z) * g(z)$.

Proof. We have

$$\log\mu(r, s) = \log\mu(r_0, s)+\int_{r_0}^{r} \frac{\nu(x, s)-s}{x}\, dx, \tag{1.1}$$

$$\log\mu(r, 0) = \log\mu(r_0, s)+\int_{r_0}^{r} \frac{\nu(x, 0)}{x}\, dx. \tag{1.2}$$

Hence

$$\log[r^{1/2}\{\mu(r, s)/\mu(r, 0)\}^{1/2s}] = \frac{1}{2s}\int_{r_0}^{r} \frac{\nu(x, s)-\nu(x, 0)}{x}\, dx+O(1).$$

Dividing both sides by $\log r$ and making $r\to\infty$, we have

$$\operatorname*{Lim}_{r\to\infty}\frac{\log[r^{1/2}\{\mu(r, s)/\mu(r, 0)\}^{1/2s}]}{\log r} = \lim_{r\to\infty}\frac{1}{2s\log r}\int_{r_0}^{r}\frac{\nu(x, s)-\nu(x, 0)}{x}\, dx. \tag{1.3}$$

But we have (see [2])

$$\lim_{r\to\infty}\frac{\log\left[r\left\{\frac{\mu(r, s)}{\mu(r, 0)}\right\}^{1/s}\right]}{\log r} = 2\varrho.$$

Hence from (1.3)

$$\lim_{r\to\infty}\frac{1}{2s\log r}\int_{r_0}^{r}\frac{\nu(x, s)-\nu(x, 0)}{x}\, dx = \varrho. \tag{1.4}$$

Now the lemma follows from (1.3) when $\lim_{r\to\infty}[\nu(r, s)-\nu(r, 0)]$ exists.

Lemma 2. *For the function $f^{(s)}(z) * g^{(s)}(z)$*

$$\lim_{r\to\infty}\frac{\log M(r, s)}{r^{\varrho(r)}} = 1,$$

where

$$\varrho(r) = \frac{\log\left\{\frac{r^{1/2}}{\varrho}\left(\mu(r, s)/\mu(r, 0)\right)^{1/2s}\right\}}{\log r}$$

and

$$M(r, s) = \max_{|z|=r} |f^{(s)}(z) * g^{(s)}(z)|.$$

Proof. Since

$$r^{\varrho(r)} = \frac{r^{1/2}}{\varrho}\{\mu(r, s)/\mu(r, 0)\}^{1/2s}$$

and ([2], Theorem 6), $\mu(r, s) \sim \mu(r, 0)[\nu(r, 0)^{2s}]/r^s$, when $r \to \infty$ through values excluding a set of measure zero, we have

$$r^{\varrho(r)} \sim \frac{1}{\varrho}\nu(r, 0), \quad \text{as } r \to \infty.$$

Hence

$$\frac{\log M(r, s)}{r^{\varrho(r)}} \sim \frac{\varrho \log M(r, s)}{\nu(r, 0)} \sim \frac{\varrho \log M(r, s)}{\nu(r, s)} \to 1,$$

since it is known that ([4], p. 32)

$$\log M(r, s) \sim \log \mu(r, s) \quad \text{and} \quad \nu(r, s) \sim \nu(r, 0)$$

and therefore

$$\lim_{r\to\infty} \frac{\log M(r, s)}{r^{\varrho(r)}} = \frac{1}{\varrho}.$$

THEOREM 1. *If* ϱ *is the order of* $f(z) * g(z)$ *and* $\lim_{r\to\infty}[\nu(r, s) - \nu(r, 0)]$ *exists, then*

$$\varrho(r) = \frac{\log\left\{\frac{r^{1/2}}{\varrho}\left(\mu(r, s)/\mu(r, 0)\right)^{1/2s}\right\}}{\log r}$$

will be a proximate order of $f^{(s)}(z) * g^{(s)}(z)$.

Proof. Condition (i) of proximate order follows from the properties of $\mu(r, s)$. From the known result (see [2])

$$\lim_{r\to\infty} \frac{\log\{r^{1/2}(\mu(r, s)/\mu(r, 0))^{1/2s}\}}{\log r} = \varrho \tag{1.5}$$

we have $\lim_{r\to\infty} \varrho(r) = \varrho$. Thus $\varrho(r)$ satisfies condition (ii).

For (iii) we calculate $r\varrho'(r)\log r$. Form (1.1) and (1.2)

$$(1.6)\quad r\varrho'(r)\log r = \frac{r}{2s}\left[\frac{\nu(r,s)}{r} - \frac{\nu(r,0)}{r}\right] - \frac{\log\left\{\frac{r^{1/2}}{\varrho}\big(\mu(r,s)/\mu(r,0)\big)^{1/2s}\right\}}{\log r}.$$

Since at the points of existence

$$\frac{\mu_1(r,s)}{\mu(r,s)} = \frac{\nu(r,s)-s}{r} \quad \text{and} \quad \frac{\mu_1(r,0)}{\mu(r,0)} = \frac{\nu(r,0)}{r},$$

where $\mu_1(r,s)$ and $\mu_1(r,0)$ are derivatives of $\mu(r,s)$ and $\mu(r,0)$ respectively, the right-hand side of (1.6) tends to zero by Lemma 1 and (1.5).

Finally condition (iv) of proximate order follows from Lemma 2.

2. In this section we are going to construct a function $\bar{\varrho}(r)$ which will play the role of a proximite order for both $f^{(s)}(z)*g^{(s)}(z)$ and $[f(z)*g(z)]^{(s)}$, $s = 1, 2, \ldots$

Let $\mu^*(r,s)$ denote the maximum term of

$$[f(z)*g(z)]^{(s)} = \sum_{n=s}^{\infty} n(n-1)\ldots(n-s+1)a_n b_n z^{n-s}$$

and let $\nu^*(r,s)$ be the rank of this term.

LEMMA 3. *If $f(z)$ and $g(z)$ are two integral functions and $\lim_{r\to\infty}[\nu(r,s)-\nu^*(r,s)]$ exists, then*

$$(2.1)\qquad \lim_{r\to\infty}[\nu(r,s)-\nu^*(r,s)] = s\varrho,$$

when ϱ $(0<\varrho<\infty)$ is the order of $f(z)*g(z)$.

Proof. We have

$$(2.2)\qquad \log\mu(r,s) = \log\mu(r_0,s) + \int_{r_0}^{r}\frac{\nu(x,s)-s}{x}\,dx,$$

and

$$(2.3)\qquad \log\mu^*(r,s) = \log\mu^*(r_0,s) + \int_{r_0}^{r}\frac{\nu^*(x,s)-s}{x}\,dx.$$

Therefore

$$(2.4)\qquad \frac{\log[\mu(r,s)/\mu^*(r,s)]}{\log r} = \frac{1}{\log r}\int_{r_0}^{r}\frac{\nu(x,s)-\nu^*(x,s)}{x}\,dx + o(1).$$

Now it is known ([2], Theorem 5) that

$$\lim_{r\to\infty}\frac{\log[\mu(r,s)/\mu^*(r,s)]}{\log r} = s\varrho.$$

Hence the lemma follows from (2.4)

LEMMA 4. *If $f(z)$ and $g(z)$ are two integral functions of regular growth, then*

$$\lim_{r\to\infty}\frac{\log M(r, s)}{r^{\bar{\varrho}(r)}} = \lim_{r\to\infty}\frac{\log M^*(r, s)}{r^{\bar{\varrho}(r)}} = 1,$$

where

$$\bar{\varrho}(r) = \frac{\log\left[\frac{1}{\varrho}\{\mu(r, s)/\mu^*(r, s)\}^{1/s}\right]}{\log r}, \quad s = 1, 2, \ldots$$

and

$$M^*(r, s) = \max_{|z|=r}\left|\left(f(z)*g(z)\right)\right|^{(s)}.$$

Proof.

$$\mu(r, s) = \left[\nu(r, s)\left(\nu(r, s)-1\right)\ldots\left(\nu(r, s)-s+1\right)\right]^2|a_{\nu(r,s)}b_{\nu(r,s)}|r^{\nu(r,s)-s}$$
$$\leqslant \nu(r, s)\left(\nu(r, s)-1\right)\ldots\left(\nu(r, s)-s+1\right)\mu^*(r, s)$$
$$\leqslant [\nu(r, s)]^s\mu^*(r, s).$$

Similarly we can show that

$$[\nu^*(r, s)-s+1]^s\mu^*(r, s) \leqslant \mu(r, s).$$

Hence

$$(2.5)\qquad \left(\nu^*(r, s)-s+1\right)^s \leqslant \frac{\mu(r, s)}{\mu^*(r, s)} \leqslant [\nu(r, s)]^s.$$

Now from (2.5) we can show

$$(2.6)\qquad \log\mu(r, s) \sim \log\mu^*(r, s), \quad \text{when } r\to\infty.$$

Hence from

$$\varrho = \lim_{r\to\infty}\frac{\nu(r, s)}{\log\mu(r, s)} = \lim_{r\to\infty}\frac{\nu^*(r, s)}{\log\mu^*(r, s)},$$

we find that

$$(2.7)\qquad \nu(r, s) \sim \nu^*(r, s), \quad \text{when } r\to\infty.$$

Therefore, from (2.5) and (2.7) if follows that

$$(2.8)\qquad \frac{\mu(r, s)}{\mu^*(r, s)} \sim [\nu(r, s)]^s, \quad \text{when } r\to\infty.$$

Now

$$\frac{\log M(r, s)}{r^{\bar{\varrho}(r)}} = \frac{\log M(r, s)}{\frac{1}{\varrho}[\mu(r, s)/\mu^*(r, s)]^{1/s}} \sim \frac{\varrho \log \mu(r, s)}{[\mu(r, s)/\mu^*(r, s)]^{1/s}}.$$

Hence

$$\lim_{r\to\infty} \frac{\log M(r, s)}{r^{\bar{\varrho}(r)}} = \lim_{r\to\infty} \frac{\varrho \log \mu(r, s)}{[\mu(r, s)/\mu^*(r, s)]^{1/s}} = 1.$$

Again since $\nu(r, s) \sim \nu^*(r, s)$ when $r \to \infty$, we find from (2.8) that

$$\frac{\mu(r, s)}{\mu^*(r, s)} \sim [\nu^*(r, s)]^s, \quad \text{when } r \to \infty. \tag{2.9}$$

Hence, proceeding as above, we can show that

$$\lim_{r\to\infty} \frac{\log M^*(r, s)}{r^{\bar{\varrho}(r)}} = 1.$$

This completes the proof of the lemma.

THEOREM 2. *If $f(z)$ and $g(z)$ are two integral functions of regulary growth, then*

$$\bar{\varrho}(r) = \log \frac{\left[\frac{1}{\varrho}\{\mu(r, s)/\mu^*(r, s)\}\right]^{1/s}}{\log r}, \quad s = 1, 2, \ldots$$

is a proximate order of $f^{(s)}(z) * g^{(s)}(z)$ as well as of $[f(z) * g(z)]^{(s)}$ when $\lim_{r\to\infty} [\nu(r, s) - \nu^*(r, s)]$ exists.

Proof. It follows from

$$\lim_{r\to\infty} \frac{\log\{\mu(r, s)/\mu^*(r, s)\}^{1/s}}{\log r} = \varrho, \quad s = 1, 2, \ldots,$$

that $\lim_{r\to\infty} \bar{\varrho}(r) = \varrho$.

This verifies condition (ii). Again condition (i) follows from the properties of $\mu(r, s)$ and $\mu^*(r, s)$. We now verify condition (iii):

$$r\bar{\varrho}'(r)\log r$$

$$= \frac{\log \varrho}{\log r} + \frac{r}{s}\frac{\mu_1(r, s)}{\mu(r, s)} - \frac{1}{s}\frac{\log \mu(r, s)}{\log r} - \frac{r}{s}\frac{\mu_1^*(r, s)}{\mu^*(r, s)} + \frac{1}{s}\frac{\log \mu^*(r, s)}{\log r}$$

for almost all r, where $\mu_1^*(r, s)$ denotes the derivative of $\mu^*(r, s)$ with respect to r.

Therefore

$$r\bar{\varrho}'(r)\log r = \frac{\log \varrho}{\log r} + \frac{1}{s}\,[\nu(r,s) - \nu^*(r,s)] - \frac{1}{s}\,\frac{\log[\mu(r,s)/\mu^*(r,s)]}{\log r}.$$

Hence $r\bar{\varrho}'(r)\log r \to 0$ as $r \to \infty$ on using Lemma 1. Also condition (iv) satisfied owing to Lemma 2.

In conclusion I offer my grateful thanks to Prof. S. K. Bose for his guidance in the work.

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Reçu par la Rédaction le 20. 1. 1970