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On the radius of convexity of some family of functions regular in the ring $0<|z|<1$

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Introduction. Let $\mathscr{P}$ denote the family of all functions of the form

$$p(z) = 1+b_1z+\ldots,$$

regular in the circle $K = \{z: |z| < 1\}$ and such that

$$\operatorname{re} p(z) > 0 \quad \text{for every } z \in K,$$

and let $\mathscr{P}(M)$ denote its subclass of functions $P(z)$ for which

$$|P(z)-M| < M, \quad z \in K,$$

where $M \geqslant 1$ is an arbitrary fixed number.

Denote by Σ the family of all functions $F(z)$ regular and univalent in the ring $A = \{z: 0 < |z| < 1\}$ which have a single pole at the point $z = 0$ and which may be expanded in some neighbourhood of this point in a power series of the form

$$w = F(z) = \frac{1}{z}+a_0+a_1z+\ldots$$

Let Σ^* be the subclass of starlike functions of the family Σ, i.e. the subclass of functions mapping the ring A onto regions whose complements to the plane are starlike regions in relation to the point $w = 0$. If $F(z) \in \Sigma^*$, then

$$\operatorname{re}\left[-\frac{zF'(z)}{F(z)}\right] > 0, \quad z \in K;$$

thus

$$-\frac{zF'(z)}{F(z)} = p(z), \quad z \in K$$

for some function $p(z) \in \mathscr{P}$.

Finally let $\Sigma^*(M)$ denote the subclass functions of the family Σ^* for which

$$\left|-\frac{zF'(z)}{F(z)}-M\right| < M, \quad z \in K.$$

In this paper we determine the radius of convexity of the family $\Sigma^*(M)$ (part I). Hence we obtain in the limit case $M = \infty$ the result of [2].

Moreover, the estimates obtained are estimates of the modulus of the derivative $|F'(z)|$ in the class of functions acute from above and below (part II).

I.1. By the *radius of convexity* of an arbitrary subclass U of the class Σ^* we mean the upper bound of the radii of circles $|z| \leqslant r$, $0 \leqslant r \leqslant 1$, in which the functions of the family U are convex.

Since a function of the family U is convex if and only if

$$\operatorname{re}\left\{-\left[1+\frac{zF''(z)}{F'(z)}\right]\right\} > 0 \quad \text{for every } z \in K_r,$$

$K_r = \{z\colon |z| < r\}$, $0 < r \leqslant 1$, the problem of finding the radius of convexity of a compact family U can be reduced to finding the greatest value of r, $0 < r \leqslant 1$ for which

$$\operatorname{re}\left\{-\left[1+\frac{zF''(z)}{F'(z)}\right]\right\} \geqslant 0$$

for every $|z| \leqslant r$ and $F(z) \in U$.

Thus the radius of convexity r_c of the family $\Sigma^*(M)$ is equal to the smallest root r_0, $0 < r_0 \leqslant 1$, of the equation $\Omega(r) = 0$, where

$$\Omega(r) = \min_{\substack{|z|=r \\ F(z)\in\Sigma^*(M)}} \operatorname{re}\left\{-\left[1+\frac{zF''(z)}{F'(z)}\right]\right\}. \tag{1}$$

It follows from the definitions of the families $\Sigma^*(M)$ and $\mathscr{P}(M)$ that $F(z) \in \Sigma^*(M)$ if and only if

$$-\frac{zF'(z)}{F(z)} = P(z), \quad P(z) \in \mathscr{P}(M). \tag{2}$$

Differentiating (2) we obtain

$$-\left[1+\frac{zF''(z)}{F'(z)}\right] = P(z) - \frac{zP'(z)}{P(z)}.$$

Thus

$$\Omega(r) = \min_{\substack{|z|=r \\ P(z)\in\mathscr{P}(M)}} \operatorname{re}\left[P(z) - \frac{zP'(z)}{P(z)}\right].$$

It is known [1] that $p(z) \in \mathscr{P}$ if and only if

$$\frac{a}{p(z)+b} = P(z), \quad P(z) \in \mathscr{P}(M), \tag{3}$$

where

$$a = \frac{2}{1+m}, \quad b = \frac{1-m}{1+m}, \quad m = 1-\frac{1}{M}.$$

So

(4) $$\Omega(r) = \min_{\substack{|z|=r \\ p(z)\in\mathscr{P}}} \operatorname{re} \frac{a+zp'(z)}{p(z)+b}, \quad p(z)\in\mathscr{P}.$$

2. According to Theorem 1 of [2] we have

(5) $$\min_{\substack{|z|=r \\ p(z)\in\mathscr{P}}} \operatorname{re} \frac{a+zp'(z)}{p(z)+b} = \min_{\lambda,\vartheta,\varphi} \operatorname{re} \frac{a+zp^{*\prime}(z)}{p^*(z)+b},$$

where

(6) $$p^*(z) = \frac{1+\lambda}{2}\frac{1+\varepsilon z}{1-\varepsilon z} + \frac{1-\lambda}{2}\frac{1+\bar{\varepsilon}z}{1-\bar{\varepsilon}z},$$

$$\varepsilon = e^{i\vartheta}, \ z = re^{i\varphi}, \ -1 \leqslant \lambda \leqslant 1, \ 0 \leqslant \vartheta \leqslant 2\pi, \ 0 \leqslant \varphi \leqslant 2\pi.$$

We transform for $z = re^{i\varphi}$ the fraction appearing on the right-hand side of equality (5).

First we have

$$p^*(re^{i\varphi}) = \frac{1+\lambda}{2}(c+\varrho\eta_1) + \frac{1-\lambda}{2}(c+\varrho\eta_2)$$

with

(7) $$c = c(r) = \frac{1+r^2}{1-r^2}, \quad \varrho = \varrho(r) = \frac{2r}{1-r^2},$$
$$\eta_1 = \eta_1(r, \vartheta, \varphi) = \varepsilon e^{i\varphi}\frac{1-r\bar{\varepsilon}e^{-i\varphi}}{1-r\varepsilon e^{i\varphi}}, \quad \eta_2 = \eta_2(r, \vartheta, \varphi) = \bar{\varepsilon}e^{i\varphi}\frac{1-r\varepsilon e^{-i\varphi}}{1-r\bar{\varepsilon}e^{i\varphi}}$$

(comp. [3]). Thus

(8) $$p^*(re^{i\varphi}) = c+\varrho(\lambda_1\eta_1+\lambda_2\eta_2),$$

where

(9) $$\lambda_1 = \frac{1+\lambda}{2}, \quad \lambda_2 = \frac{1-\lambda}{2}.$$

Let

$$\varkappa = \varkappa(r) = \varrho|\lambda_1\eta_1+\lambda_2\eta_2|.$$

Thus

(10) $$\varrho(\lambda, \eta_1+\lambda_2\eta_2) = \varkappa\gamma, \quad |\gamma| = 1.$$

(10) imples the equality

$$\varrho^2|\lambda_1\eta_1+\lambda_2\eta_2|^2 = \varkappa^2.$$

Assuming $\eta_j = e^{i\beta_j}$, $j = 1, 2$, we find hence the relationship

$$\varrho^2(\lambda_1^2+\lambda_2^2+2\lambda_1\lambda_2 \operatorname{re}\eta_1\bar{\eta}_2) = \varkappa^2$$

and because of

$$\lambda_1^2 = \lambda_1(1-\lambda_2), \qquad \lambda_2^2 = \lambda_2(1-\lambda_1)$$

we have

$$\varrho^2\left(1-4\lambda_1\lambda_2\sin^2\frac{\beta_1-\beta_2}{2}\right) = \varkappa^2; \tag{11}$$

thus $0 < \varkappa \leqslant \varrho$.

Coming back to (8) we may thus write $p^*(re^{i\varphi})$ in the form

$$p^*(re^{i\varphi}) = c+\varkappa\gamma, \qquad \text{where } |\gamma| = 1,\ 0<\varkappa\leqslant\varrho. \tag{12}$$

Now we express $(re^{i\varphi})p^{*\prime}(re^{i\varphi})$ in terms of $p^*(re^{i\varphi})$ and η_1, η_2. Assume

$$p_1^*(z) = \frac{1+\varepsilon z}{1-\varepsilon z}, \qquad p_2^*(z) = \frac{1+\bar{\varepsilon} z}{1-\bar{\varepsilon} z}. \tag{13}$$

Differentiating function (6) in relation to z, we obtain because of (9) and (13)

$$zp^{*\prime}(z) = \lambda_1 z p_1^{*\prime}(z)+\lambda_2 z p_2^{*\prime}(z) = \tfrac{1}{2}\lambda_1[p_1^{*2}(z)-1]+\tfrac{1}{2}\lambda_2[p_2^{*2}(z)-1]. \tag{14}$$

After some transformations we get

$$zp^{*\prime}(z) = \tfrac{1}{2}\{p^{*2}(z)+\lambda_1\lambda_2[p_1^*(z)-p_2^*(z)]^2-1\};$$

thus

$$(re^{i\varphi})p^{*\prime}(re^{i\varphi}) = \tfrac{1}{2}[p^{*2}(re^{i\varphi})-1]+\tfrac{1}{2}\lambda_1\lambda_2[p_1^*(re^{i\varphi})-p_2^*(re^{i\varphi})]^2.$$

The second term of the last sum may be written in an equivalent form $-2\lambda_1\lambda_2\varrho^2\eta\sin^2\dfrac{\beta_1-\beta_2}{2}$, where, $\eta = \eta_1\cdot\eta_2$, or by (11) in the form: $\frac{1}{2}(\varkappa^2-\varrho^2)\eta$. Thus

$$(re^{i\varphi})p^{*\prime}(re^{i\varphi}) = \tfrac{1}{2}[p^{*2}(re^{i\varphi})-1]-\tfrac{1}{2}(\varrho^2-\varkappa^2)\eta. \tag{15}$$

By (5) and (15) equality (4) assumes the form

$$\Omega(r) = \tfrac{1}{2}\min_{\lambda,\vartheta,\varphi}\operatorname{re} U\big(p^*(re^{i\varphi})\big), \tag{16}$$

where

$$U\big(p^*(re^{i\varphi})\big) = \frac{2a-1+p^{*2}(re^{i\varphi})-(\varrho^2-\varkappa^2)\eta}{p^*(re^{i\varphi})+b}. \tag{17}$$

3. Let

$$p^*(re^{i\varphi})+b = se^{it}, \qquad \operatorname{im} t = 0. \tag{18}$$

Then

$$\Omega(r) = \tfrac{1}{2}\min_{s,t}\operatorname{re}\left\{\frac{e^{-it}}{s}\left[2a-1+(se^{it}-b)^2-(\varrho^2-\varkappa^2)\eta\right]\right\}.$$

Since $\operatorname{re}\eta e^{-it} \leqslant 1$,

$$\Omega(r) \geqslant \tfrac{1}{2}\min_{s,t}\left(\operatorname{re}\left\{\frac{e^{-it}}{s}[2a-1+(se^{it}-b)^2]\right\}-\frac{\varrho^2-\varkappa^2}{s}\right). \tag{19}$$

From (12) we have $\varkappa^2 = |p^*(re^{i\varphi})-c|^2$ and because of (18)

$$\varkappa^2 = |se^{it}-b-c|^2. \tag{20}$$

Substituting $\varkappa^2$ from (20) into (19), we obtain after some transformations

$$\Omega(r) \geqslant \min_{s,t}\Phi(s, t),$$

where

$$\Phi(s, t) = \tfrac{1}{2}\left\{\left[s-2(b+c)+\frac{a^2}{s}\right]\cos t+s-2b+\frac{b^2+2bc+1}{s}\right\}. \tag{21}$$

By (7), (12) and (18) the function $\Phi(s, t)$ is defined in the region

$$D = \{(s, t):\ c-\varrho+b < s < c+\varrho+b,\ -\Psi(s) < t < \Psi(s)\}$$

and on its boundary ∂D, where

$$\Psi(s) = \arccos-\frac{s^2+b^2+2bc+1}{2(b+c)s}, \quad 0 \leqslant \Psi(s) \leqslant \Psi(s_0), \tag{22}$$

with $s_0 = |\sqrt{b^2+2bc+1}|$ (1).

4. If, at some point (s^*, t^*) of the region D, $\Phi(s^*, t^*) = \min_{(s,t)\in D}\Phi(s, t)$, then s^*, t^* are the solutions of the system of equations $\dfrac{\partial\Phi(s, t)}{\partial s} = 0$, $\dfrac{\partial\Phi(s, t)}{\partial t} = 0$ with the unknowns s and t, i. e. of the system

$$\left(1-\frac{a^2}{s^2}\right)\cos t+1-\frac{b^2+2bc+1}{s^2} = 0, \quad \left[s-2(b+c)+\frac{a^2}{s}\right]\sin t = 0.$$

Because of (22) this system has the following solutions:

$$s_1 = \sqrt{ab+bc+1}, \quad t_1 = 0$$

and

$$s_2 = b+c+\chi\sqrt{(b+c)^2-a^2}, \quad \cos t_2 = \frac{b^2+2bc+1-s_2^2}{s_2^2-a^2}, \quad \chi = \pm 1$$

(1) In the sequel $|\sqrt{a}|$, for $a > 0$ will be denoted by $\sqrt{a}$.

($s \neq a$, for if the contrary was the case, we would have $r = 0$). It is easy to verify that $s_2 \notin J$, where $J = \{s\colon c-\varrho+b < s < c+\varrho+b\}$. Since

$$\frac{\partial^2\Phi(s_1, t_1)}{\partial s^2} > 0 \quad \text{and} \quad \frac{\partial^2\Phi(s_1, t_1)}{\partial s^2}\cdot\frac{\partial^2\Phi(s_1, t_1)}{\partial t^2} - \left(\frac{\partial^2\Phi(s_1, t_1)}{\partial s\partial t}\right)^2 > 0,$$

we have $s^* = s_1$, $t^* = t_1$. Thus

$$\min_{(s,t)\in D}\Phi(s, t) = \Phi(s^*, t^*) = 2\sqrt{ab+bc+1}-2b-c.$$

If $(s, t)\in\partial D$, then because of $\Phi(s, t) = \Phi(s, -t)$ we have

$$\Phi[s, \Psi(s)] = \Phi[s, -\Psi(s)] = \Delta(s),$$

where

$$\Delta(s) = \frac{1}{4(b+c)}\left[s^2+2(a-bc-b^2)+a^2\frac{b^2+2bc+1}{s^2}\right],$$

with $s\in\{c-\varrho+b\}\cup J\cup\{c+\varrho+b\}$.

If at some point s^{**} of the interval J we have $\Delta(s^{**}) = \min\limits_{s\in J}\Delta(s)$, then s^{**} is a solution of the equation $\Delta'(s) = 0$. Since

$$\Delta'(s) = \frac{s^4-a^2(b^2+2bc+1)}{2(b+c)s^3},$$

we have

$$s^{**} = \sqrt[4]{a^2(b^2+2bc+1)}.$$

Thus

$$\min_{s\in J}\Delta(s) = \Delta(s^{**}) = \frac{a\sqrt{b^2+2bc+1}+a-bc-b^2}{2(b+c)}.$$

Since $\Delta(s^{**}) < \Delta(c-\varrho+b)$ and $\Delta(s^{**}) < \Delta(c+\varrho+b)$,

$$\Delta(s^{**}) = \min_{s\in\partial D}\Delta(s).$$

With a fixed M we assume

$$H(c) = \Phi(s^*, t^*), \qquad T(c) = \Delta(s^{**}),$$

where because of $0 \leqslant r < 1$ we have $c \geqslant 1$.

We apply the Taylor formula to the function $T(c)-H(c)$ for an arbitrary $c \geqslant 1$:

$$T(c)-H(c) = T(1)-H(1)+(c-1)[T'_+(1)-H'_+(1)]+$$
$$+\frac{(c-1)^2}{2}\left[T''\big(1+\delta\cdot(c-1)\big)-H''\big(1+\delta\cdot(c-1)\big)\right], \qquad 0 < \delta < 1.$$

Since $T(1) = H(1) = 1$ and $T'_+(1) = H'_+(1)$, because of

$$T'(c) = -a\frac{bc+1+\sqrt{b^2+2bc+1}}{2(b+c)^2\sqrt{b^2+2bc+1}}, \quad H'(c) = \frac{b}{\sqrt{ab+bc+1}} - 1,$$

we have

$$T(c)-H(c) = \frac{(c-1)^2}{2}\left[T''\left(1+\delta\cdot(c-1)\right)-H''\left(1+\delta\cdot(c-1)\right)\right].$$

Differentiating the functions $H(c)$ and $T(c)$ twice in relation to c, we obtain

$$H''(c) = -\frac{b^2}{2}A(c)^{-3/2},$$

$$T''(c) = \frac{aB(c)^{-3/2}}{2(b+c)^3}\left[(2+bc-b^2)B(c)+b(b+c)(bc+1)+2B(c)^{3/2}\right],$$

where

$$A(c) = ab+bc+1, \qquad B(c) = b^2+2bc+1.$$

Because of $2+bc-b^2 > 0$ we find $T(c) > H(c)$ for every $c \geqslant 1$. Thus

(23) $$\min_{(s,t)\in D\cup\partial D} \Phi(s, t) = \Phi(s^*, t^*) = 2\sqrt{ab+bc+1}-2b-c.$$

5. It follows from (1), (19), (21) and (23) that the radius of convexity r_c of the family $\Sigma^*(M)$ is at least equal to the smallest root r_0, $0 \leqslant r_0 < 1$ of the equation

(24) $$2\sqrt{ab+bc(r)+1}-2b-c(r) = 0,$$

where

(25) $$c(r) = \frac{1+r^2}{1-r^2}.$$

Because of (25) we find from (24)

(26) $$r_0 = \sqrt{\frac{2\sqrt{a}-1}{2\sqrt{a}+1}}$$

with

(27) $$c_0 = c(r_0) = 2\sqrt{a}.$$

Thus we have proved that the radius of convexity is

$$r_c \geqslant r_0,$$

where r_0 is given by (26).

6. We shall prove that

$$r_c \leqslant r_0.$$

To this aim we observe first that if a function $p_0^*(z)$ of form (6) satisfies condition (18) at some point $z_0 = r_0 e^{i\varphi_0}$ with $s = s^*$ and $t = t^*$, then

$$p_0^*(z_0) = \sqrt{a}. \tag{28}$$

Denote by λ_0 and ϑ_0 the values of the parameters λ and ϑ corresponding to the function $p_0^*(z)$. Thus

$$p_0^*(z) = \frac{1+\lambda_0}{2} p_{1,0}^*(z) + \frac{1-\lambda_0}{2} p_{2,0}^*(z), \tag{29}$$

with

$$p_{1,0}^*(z) = \frac{1+\varepsilon_0 z}{1-\varepsilon_0 z}, \quad p_{2,0}^*(z) = \frac{1+\bar{\varepsilon}_0 z}{1-\bar{\varepsilon}_0 z}, \quad |\varepsilon_0| = 1.$$

Next we have

$$p_{1,0}^*(z_0) = c_0 + \varrho_0 \eta_{1,0}, \quad p_{2,0}^*(z_0) = c_0 + \varrho_0 \eta_{2,0}, \tag{30}$$

where

$$c_0 = c(r_0), \quad \varrho_0 = \varrho(r_0), \quad \eta_{j,0} = \eta_j(r_0, \vartheta_0, \varphi_0), \quad j = 1, 2$$

(comp. (17)).

Because of $\eta = \eta_1 \cdot \eta_2$ and the assumption that $\eta = 1$, we obtain

$$\eta_{j,0} = \bar{\eta}_{k,0} \quad \text{for } j \neq k, \ j, k = 1, 2; \tag{31}$$

thus

$$p_{2,0}^*(z_0) = \overline{p_{1,0}^*(z_0)}.$$

Thus by equating $p_0^*(z_0)$ in (28) and in (29) we get

$$\operatorname{re} p_{1,0}^*(z_0) = \sqrt{a}, \quad \lambda_0 \operatorname{im} p_{1,0}^*(z_0) = 0. \tag{32}$$

From (15) we obtain

$$z_0 p_0^{*\prime}(z_0) = \tfrac{1}{2}[p_0^{*2}(z_0) - 1] - \tfrac{1}{2}(\varrho_0^2 - \varkappa_0^2),$$

where

$$\varkappa_0 = |p_0^*(z_0) - c_0| = \sqrt{a}.$$

Thus because of $c^2(r) - \varrho^2(r) \equiv 1$ we have

$$z_0 p_0^{*\prime}(z_0) = -a. \tag{33}$$

Simultaneously we have from (14) for $z = z_0$, because of (9)

$$z_0 p_0^{*\prime}(z_0) = \frac{1+\lambda_0}{4}[p_{1,0}^{*2}(z_0) - 1] + \frac{1-\lambda_0}{4}[p_{2,0}^{*2}(z_0) - 1].$$

In view of (32) and (33) we obtain hence the equality

(34) $$3a-1 = \operatorname{im}^2 p_{1,0}^*(z_0).$$

From (32) we get $\lambda_0 = 0$ or $\operatorname{im} p_{1,0}^*(z_0) = 0$. Supposing that the second of these equalities holds, we would have because of (34) $a = \frac{1}{3}$, which is impossible. Thus $\lambda_0 = 0$ and

$$p_0^*(z_0) = \tfrac{1}{2}[p_{1,0}^*(z_0)+\overline{p_{1,0}^*(z_0)}].$$

We shall find ϑ_0. Assuming

$$\eta_{1,0} = e^{i\beta_{1,0}}, \quad \operatorname{im}\beta_{1,0} = 0,$$

we shall have by (30)

$$p_{1,0}^*(z_0) = c_0+\varrho_0\cos\beta_{1,0}+i\varrho_0\sin\beta_{1,0},$$

thus because of (34) and (27) and of the identity $c^2(r)-\varrho^2(r)\equiv 1$ we get hence

$$\cos\beta_{1,0} = -\sqrt{\frac{a}{4a-1}} \quad \text{and} \quad \sin\beta_{1,0} = \pm\sqrt{\frac{3a-1}{4a-1}}.$$

From condition (31) because of (7) we find $\varphi_0 = 0$ or $\varphi_0 = \pi$. Assuming $\varphi_0 = 0$, we obtain

$$e^{i\beta_{1,0}} = \frac{\varepsilon_0 - r_0}{1-\varepsilon_0 r_0};$$

thus

(35) $$\varepsilon_0 = \frac{2a-1}{\sqrt{a(4a-1)}} \pm i\sqrt{\frac{3a-1}{a(4a-1)}}.$$

Thus we have proved that if $p^*(z) = p_0^*(z)$, where

$$p_0^*(z) = \frac{1}{2}\left[\frac{1+\varepsilon_0 z}{1-\varepsilon_0 z}+\frac{1+\bar{\varepsilon}_0 z}{1-\bar{\varepsilon}_0 z}\right]$$

and ε_0 is defined by (35), then the function $U\big(p^*(re^{i\varphi})\big)$ given by formula (17) attains at the point $z = r_0$ a minimum equal to zero.

Thus

$$\Omega(r_0) = 0$$

(comp. (16)).

The function $p_0^*(z)$ has been assigned by condition (3) to the function

(36) $$P_0^*(z) = a\,\frac{z^2-(\varepsilon_0+\bar{\varepsilon}_0)z+1}{(a-2)z^2-(a-1)(\varepsilon_0+\bar{\varepsilon}_0)z+a}$$

of the family $\mathcal{P}(M)$, and to this function we assign a function $F_0^*(z)$ from the family $\Sigma^*(M)$ satisfying the differential equations

$$(37)\qquad \frac{zF_0^{*\prime}(z)}{F_0^*(z)} = -P_0^*(z), \qquad \frac{zF_0^{*\prime}(z)}{F_0^*(z)}\bigg|_{z=0} = -1,$$

(comp. (2)).

Taking into account (36) we may write equations (37) in the form

$$(38)\qquad \frac{F_0^{*\prime}(z)}{F_0^*(z)} + \frac{1}{z} = -\frac{2z-(\varepsilon_0+\bar{\varepsilon}_0)}{(a-2)z^2-(a-1)(\varepsilon_0+\bar{\varepsilon}_0)z+a}.$$

We distinguish two cases: 1° $a \neq 2$, 2° $a = 2$.

1° The rational function of the variable z which appears on the right-hand side of equation (38) has the poles z_1, z_2 ($|z_1| < |z_2|$) with $z_1 < -1$ and $z_2 > 1$ and the left-hand side expression of (38) is a regular function in the circle K, thus the integrals of these functions exist along any regular curve $\Gamma \subset K$ with the origin and with the end-point at 0 and z, respectively, where $z \in K$.

Thus we conclude from (38) that

$$(39)\qquad \log zF_0^*(z) = \int_z^0 \frac{2\mathfrak{z}-(\varepsilon_0+\bar{\varepsilon}_0)}{(a-2)\mathfrak{z}^2-(a-1)(\varepsilon_0+\bar{\varepsilon}_0)\mathfrak{z}+a}\,d\mathfrak{z},$$

where $\log zF_0^*(z) = l(z)$ denotes the unique branch of a multivalent function $L(z) = \log zF_0^*(z)$ such that $l(0) = 0$.

Denoting the integral in (39) by $I(z)$, we get

$$I(z) = \frac{1}{2-a}h_1(z) + \frac{\varepsilon_0+\bar{\varepsilon}_0}{a(2-a)}\int_0^z \frac{d\mathfrak{z}}{(1-\mathfrak{z}/z_1)(1-\mathfrak{z}/z_2)}$$

with

$$h_1(z) = \log\left[\frac{a-2}{a}z^2 - \frac{a-1}{a}(\varepsilon_0+\bar{\varepsilon}_0)z+1\right] \equiv \log\left(1-\frac{z}{z_1}\right)\left(1-\frac{z}{z_2}\right),$$
$$h_1(0) = 0.$$

Next

$$(40)\qquad I(z) = \frac{h_1(z)}{2-a} + \frac{\varepsilon_0+\varepsilon_0}{(2-a)^2(z_2-z_1)}h_2(z),$$

where

$$h_2(z) = \log\frac{1-z/z_1}{1-z/z_2}, \qquad h_2(0) = 0.$$

Because of (40) we obtain from equation (39)

$$F_0^*(z) = \frac{(1-z/z_1)^{(1+k)/(1-b)}(1-z/z_2)^{(1-k)/(1-b)}}{z},$$

where

$$z_1 = \frac{-b\,\mathrm{re}\,\varepsilon_0 - \sqrt{\Delta}}{1-b}, \qquad z_2 = \frac{-b\,\mathrm{re}\,\varepsilon_0 + \sqrt{\Delta}}{1-b}.$$

$$K = \frac{\mathrm{re}\,\varepsilon_0}{\sqrt{\Delta}}, \qquad \Delta = b^2\mathrm{re}^2\,\varepsilon_0 + 1 - b^2.$$

2° If $a = 2$, then equation (39) becomes

$$\log z \cdot F_0^*(z) = \frac{2z}{\varepsilon_0 + \bar{\varepsilon}_0} - \left(\frac{\varepsilon_0 - \bar{\varepsilon}_0}{\varepsilon_0 + \bar{\varepsilon}_0}\right)^2 h_3(z),$$

where

$$h_3(z) = \log\left(1 - \frac{\varepsilon_0 + \bar{\varepsilon}_0}{2} z\right), \qquad h_3(0) = 0.$$

Therefore

$$F_0^*(z) = \frac{\left(1 - \frac{3}{\sqrt{14}} z\right)^{5/9} \cdot e^{(\sqrt{14}/3)z}}{z}.$$

Summing, we obtain

$$(41) \qquad F_0^*(z) = \begin{cases} (1 - zz_1^{-1})^{(1+k)(1-b)^{-1}} \cdot (1 - zz_2^{-1})^{(1-k)(1-b)^{-1}} \cdot z^{-1} & \text{for } M > 1, \\ (1 - 3\cdot 14^{-1/2} z)^{5/9} [\exp(14^{1/2}\cdot 3^{-1}\cdot z)] z^{-1} & \text{for } M = 1. \end{cases}$$

It can easily be verified that $F_0^*(z) \in \Sigma^*(M)$. Since

$$\mathrm{re}\left(1 + \frac{zF_0^{*\prime\prime}(z)}{F_0^{*\prime}(z)}\right) = 0$$

for $z = r_0$, thus the function $F_0^*(z)$ is not convex in the circle $|z| < r$ for $r > r_0$. Thus because of $r_c \geqslant r_0$ and $r_c \leqslant r_0$ we obtain $r_c = r_0$.

Thus we have

THEOREM 1. *The radius of convexity of the family $\Sigma^*(M)$ is given by the formula*

$$(42) \qquad r_c = \frac{\sqrt{6M+1}}{2\sqrt{2M} + \sqrt{2M-1}}.$$

Passing to the limit in (42) as $m \to 1$ $(M \to \infty)$, we obtain $r_c = 3^{-1/2}$, [2].

II.1. Basing ourselves on the results obtained in part I of this paper, we obtain an acute estimate from above of the modulus of the derivative $|F'(z)|$, $F(z) \in \Sigma^*(M)$ at an arbitrary fixed point z_0 of the ring A.

Let $F(z)\in\Sigma^*(M)$. Since

$$\log\left(z^2F'(z)\right) = \log|z^2F'(z)| + i\arg\left(z^2F'(z)\right),$$

we have

$$2+\operatorname{re}\left(z\frac{F''(z)}{F'(z)}\right) = r\frac{\partial}{\partial r}\log|z^2F'(z)|, \quad |z| = r, \tag{43}$$

because of which

$$r\frac{\partial}{\partial r}\log(r^2|F'(z)|) \leqslant 1 - \min_{\substack{|z|=r \\ F(z)\in\Sigma^*(M)}} \operatorname{re}\left\{-\left[1+\frac{zF''(z)}{F'(z)}\right]\right\}.$$

Hence by (23) we have

$$\frac{\partial}{\partial r}\log\left(r^2|F'(z)|\right) \leqslant \frac{1+2b+c(r)-2\sqrt{ab+bc(r)+1}}{r}, \quad |z| = r.$$

Thus

$$\log\left(r^2|F'(z)|\right) \leqslant \int_0^r \frac{1+2b+c(s)-2\sqrt{ab+bc(s)+1}}{s}\,ds, \quad |z| = r.$$

Carrying out the integration and replacing z by z_0 we obtain the following estimates from above of $|F'(z)|$ in the family of functions $\Sigma^*(M)$:

$$|F'(z_0)| \leqslant \mathfrak{B}(|z_0|, M), \tag{44}$$

where

$$\mathfrak{B}(|z_0|, M) = \frac{r^{-2}}{1-r^2}\left(\frac{a\sqrt{1-r^2}+\sqrt{a^2-v^2r^2}}{2a}\right)^{2a} \times \tag{45}$$

$$\times\left(\frac{a+v}{v\sqrt{1-r^2}+\sqrt{a^2-v^2r^2}}\right)^{2v}, \quad |z| = r,$$

with

$$v = \sqrt{a^2-2b}, \quad a = \frac{2}{1+m}, \quad b = \frac{1-m}{1+m}, \quad m = 1-\frac{1}{M}.$$

Estimate (44) is acute, the equality sign holds for function (41).

2. Let

$$\theta(r) = \max_{\substack{|z|=r \\ F(z)\in\Sigma^*(M)}} \operatorname{re}\left\{-\left[1+\frac{zF''(z)}{F'(z)}\right]\right\}.$$

Proceeding similarly as in part I of this paper and preserving the notation adopted there, we obtain first

$$\theta(r) \leqslant \max_{s,t} \psi(s, t),$$

where

$$\psi(s, t) = \tfrac{1}{2}\left[s+2\big(b+c(r)\big)+\frac{a^2}{s}\right]\cos t - s - 2b - \frac{b^2+2bc(r)+1}{s}.$$

Since

$$\max_{(s,t)\in D\cup\partial D} \psi(s, t) = \frac{c(r)[c(r)+\varrho(r)]+b}{c(r)+\varrho(v)+b},$$

we obtain because of (43)

$$(46)\qquad r\frac{\partial}{\partial r}\log\big(r^2|F'(z)|\big) \geqslant 1 - \frac{c(r)[c(r)+\varrho(r)]+b}{c(r)+\varrho(r)+b}.$$

Dividing both sides of inequality (46) by r and then integrating in the interval $[0, r]$, we have

$$\log\big(r^2|F'(z)|\big) \geqslant \int\limits_0^r \frac{[1-c(s)][c(s)+\varrho(s)]}{c(s)+\varrho(s)+b}\,ds;$$

integrating and replacing z by z_0, we obtain the estimation

$$(47)\qquad |F'(z_0)| \geqslant \mathfrak{A}(|z_0|, M),$$

where

$$(48)\quad \mathfrak{A}(|z_0|, M) = \begin{cases} \dfrac{1}{r^2}\left[(1-r)\left(1+\dfrac{2-a}{a}r\right)\right]^{a/(2-a)} & \text{for } a \neq 2 \ (r = |z_0|), \\ \dfrac{1-r}{r^2}e^r & \text{for } a = 2. \end{cases}$$

Estimation (47) is acute, the equality sign holds for the function

$$F_*(z) = \begin{cases} \dfrac{1}{z}\left(1+\dfrac{2-a}{a}\varepsilon z\right)^{2/(2-a)} & \text{for } a \neq 2 \ \left(\varepsilon = \dfrac{|z_0|}{z_0}\right), \\ \dfrac{1}{z}e^{\varepsilon z} & \text{for } a = 2. \end{cases}$$

Thus we have proved

THEOREM 2. *If $F(z)\in\Sigma^*(M)$, then at every point z of the ring $0 < |z| < 1$ the following acute estimate of the functional $|F'(z)|$ holds:*

$$\mathfrak{A}(|z|, M) \leqslant |F'(z)| \leqslant \mathfrak{B}(|z|, M),$$

where the functions $\mathfrak{A}(|z|, M)$ and $\mathfrak{B}(|z|, M)$ are defined by (45) *and* (48).

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