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A CHARACTERIZATION OF THE ČECH HOMOLOGY THEORY

BY

S. K. KAUL (REGINA, SASKATCHEWAN)

Dedicated to the memory of my Aunt

0. Explicit reference will be made where terminology is not from [3]. Let $\mathscr{C}$ be the admissible category of compact metrizable pairs (X, A), that is X is a compact metrizable space and A is a closed subspace of X, and their maps. Let H or more explicitly $\{H, *, \partial\}$ be a homology theory on $\mathscr{C}$ satisfying the Eilenberg-Steenrod axioms (see [3], p. 10).

(0.1) Any map $f\colon (X, A) \to (Y, B)$ in C is called a *Vietoris map* if it has the following properties: (1) f is onto, that is $f_1\colon X \to Y$ and $f_2\colon A \to B$ defined by f are both onto, (2) $f^{-1}(B) = A$ and (3) for any $y \in Y, f^{-1}(y)$ is acyclic, that is, its reduced homology groups are trivial in each dimension.

(0.2) We say that H *has the Vietoris property* if for every Vietoris map $f\colon (X, A) \to (Y, B)$ in $\mathscr{C}$ the induced homomorphism $f_*\colon H(X, A) \to H(Y, B)$ is an isomorphism. By an *isomorphism* we mean an onto isomorphism (see [2] and [5]).

(0.3) We say that H is a *partially continuous theory on* $\mathscr{C}$ if, whenever a pair (X, A) in $\mathscr{C}$ is the inverse limit of an inverse sequence of simplicial pairs (X_n, A_n) and simplicial maps $\pi_n^m\colon (X_m, A_m) \to (X_n, A_n)$ for $m > n$, $m, n = 1, 2, \ldots$, and directed by the natural order, then the natural transformation $\pi_*\colon H(X, A) \to G = \operatorname{Inv Lim}\{H(X_n, A_n), \pi_n^m{}_*\}$ is a natural equivalence ([3], def. (2.3), p. 260).

(0.4) Let H and $\bar{H}$ be any two homology theories on $\mathscr{C}$. We say that H and $\bar{H}$ are *isomorphic* if for each pair (X, A) in $\mathscr{C}$ and each q, there exists an isomorphism $k(q, X, A)\colon H_q(X, A) \to \bar{H}_q(X, A)$ such that for any map $f\colon (X, A) \to (Y, B)$ in $\mathscr{C}$ the diagrams

$$\begin{array}{ccc} H_q(X, A) & \xrightarrow{k(q,X,A)} & \bar{H}_q(X, A) \\ \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle \bar{f}_*} \\ H_q(Y, B) & \xrightarrow{k(q,Y,B)} & \bar{H}_q(Y, B) \end{array}$$

and

$$\begin{array}{ccc} H_q(X, A) & \xrightarrow{k(q,X,A)} & \bar{H}_q(X, A) \\ \partial \downarrow & & \bar{\partial} \downarrow \\ H_{q-1}(A) & \xrightarrow{k(q-1,A)} & \bar{H}_{q-1}(A) \end{array}$$

commute (see [3], th. (12.2), p. 288), the homology groups being defined on the same coefficient group.

In the sequel whenever a statement is true for homology groups of H in each dimension, we shall merely write H instead of "H_q for each q".

MAIN THEOREM. *If a homology theory H on $\mathscr{C}$ has the Vietoris property and is partially continuous, then it is isomorphic to the Čech homology theory on $\mathscr{C}$ over the same coefficient group.*

Remark. Let $\bar{H}$ denote the Čech homology theory on $\mathscr{C}$. Let $f: (X, A) \to (Y, B)$ in $\mathscr{C}$ be a Vietoris map. Consider the following diagram:

$$\begin{array}{ccccccccccc} \ldots \to \bar{H}_q(A) & \xrightarrow{\bar{i}_*} & \bar{H}_q(X) & \xrightarrow{\bar{j}_*} & \bar{H}_q(X, A) & \xrightarrow{\bar{\partial}} & \bar{H}_{q-1}(A) & \to & \bar{H}_{q-1}(X) & \to \ldots \\ \downarrow \bar{f}_{2*} & & \downarrow \bar{f}_{1*} & & \downarrow \bar{f}_* & & \downarrow \bar{f}_{2*} & & \downarrow \bar{f}_{1*} & \\ \ldots \to \bar{H}_q(B) & \xrightarrow{\bar{i}_*} & \bar{H}_q(Y) & \xrightarrow{\bar{j}_*} & \bar{H}_q(Y, B) & \xrightarrow{\bar{\partial}} & \bar{H}_{q-1}(B) & \to & \bar{H}_{q-1}(Y) & \to \ldots \end{array}$$

If the coefficient group is either an elementary compact topological group ([4], p. 672) or a field, then since $f_1^{-1}(y)$ and $f_2^{-1}(z)$, for any $y \in Y$ and any $z \in B$, are acyclic from condition (3) of (0.1) and f_1 and f_2 are onto maps, it follows from theorem 2 of [2], p. 538, that f_{1*} and f_{2*} are isomorphisms for each dimension. Also then $\bar{H}$ is exact ([3], th. (7.6), p. 248). Thus from the Five Lemma ([3], lemma (4.3), p. 16) f_{*q}: $\bar{H}_q(X, A) \to \bar{H}_q(Y, B)$ is an isomorphism for each integer q. Hence $\bar{H}$ has the Vietoris property. $\bar{H}$ also has the partial continuity property ([3], th. (3.1), p. 261). Hence on the above-mentioned coefficient groups the main theorem above gives a characterization of $\bar{H}$.

1. Let X be a compact metrizable space, and $\{U_n\}$ be a cofinal sequence of finite open coverings of X, cofinal in the family of all open coverings of X, directed by refinement.

(1.0) We say that $\{U_n\}$ is a *special sequence* if (1) $U_{n+1} > U_n$, where $>$ means refines, and (2) for $u \in U_{n+1}$, $u \subset v \in U_n$, then

$$\bar{u} \subset v \quad \text{for each } n = 1, 2, \ldots$$

Whenever these two conditions are satisfied we write $\overline{U_{n+1}} > U_n$. For a compact metrizable space such a sequence always exists (see section 4).

Let then $\{U_n\}$ be a special sequence of coverings of X. Let K_n denote the nerve of U_n, and Π_n^{n+1}: $K_{n+1} \to K_n$ be a projection map. Then $\{K_n, \Pi_n^m\}$, where for $m > n$, $\Pi_n^m = \Pi_{m-1}^m \ldots \Pi_n^{n+1}$ and Π_n^n is the identity, is an inverse sequence of simplicial complexes and simplicial maps. Let K be its inverse limit. For any x in X, let $\sigma_n(x)$ denote the simplex in K_n corresponding to all the members of U_n containing x. Since Π_n^m are projection maps, $\Pi_n^m(\sigma_m(x)) \subset \sigma_n(x)$. Hence $\{\sigma_n(x), '\Pi_n^m\}$, where $'\Pi_n^m = \Pi_n^m | \sigma_m(x)$, is an inverse sequence. Let $\sigma(x)$ denote its inverse limit. We may assume for convenience that a metric is given in X and that with respect to this metric mesh $U_n < 1/n$ for each $n = 1, 2, \ldots$, since $\{U_n\}$ is cofinal.

(1.1) *If $x, y \in X$ are distinct, then $\sigma(x) \cap \sigma(y) = \emptyset$.*

Since mesh $U_n < 1/n$, there exists a positive integer N such that no member of U_N containing x intersects any of its members containing y. Hence $\sigma_N(x) \cap \sigma_N(y) = \emptyset$ and consequently $\sigma(x) \cap \sigma(y) = \emptyset$.

(1.2) *If $p \in K$, then there exists an $x \in X$ such that $p \in \sigma(x)$.*

Let $p = \{p_n\colon n = 1, 2, \ldots\}$, where $p_n \in K_n$. Let σ_n denote the smallest simplex in K_n containing p_n. Let V_n denote the carrier of σ_n ([3], def. (2.2), p. 234). Since $p \in K$, $\Pi_n^{n+1}(p_{n+1}) = p_n$. Again, since Π_n^{n+1} is a simplicial map and σ_n is the smallest simplex containing p_n, $\Pi_n^{n+1}(\sigma_{n+1}) = \sigma_n$ for each n. Since $\bar{U}_{n+1} > U_n$, it follows from the last assertion and the fact that Π_n^{n+1} is a projection map that $\bar{V}_{n+1} \subset V_n$ for each n. Lastly, since mesh $U_n \to 0$ as $n \to \infty$, $\bigcap_{n=1}^{\infty} V_n$ is a single point, say x in X. Then by the definition of $\sigma_n(x)$ it follows that σ_n is a face of $\sigma_n(x)$. Hence $p_n \in \sigma_n(x)$ and, consequently, $p \in \sigma(x)$. This proves (1.2).

From (1.1) and (1.2) it follows that for any $p \in K$ there exists a unique x in X such that $p \in \sigma(x)$. Define a function

$$\alpha\colon K \to X$$

by setting $\alpha(p) = x$. Note that $\alpha^{-1}(x) = \sigma(x)$ for $x \in X$.

(1.3) *α is onto.*

Since $\sigma(x)$ is non-empty for any x in X, being the inverse limit of an inverse sequence of compact spaces and maps ([1], th. (3.6), p. 217), α is onto. This proves (1.3).

Let A be a closed subset of X, and L_n denote the subcomplex of K_n consisting of those simplices of K_n whose carriers intersect A. Then $\{L_n, \tilde{\Pi}_n^m\}$, where $\Pi_n^m | L_m = \tilde{\Pi}_n^m$ is an inverse sequence of simplicial complexes and simplicial maps. Let L denote its inverse limit. Then L is compact, since each L_n is ([3], th. (3.6), p. 217) and, therefore, is a closed subset of K.

(1.4) $\alpha^{-1}(A) = L$.

If $x \in A$, then $\sigma_n(x) \subset L_n$ for each n, and hence $\sigma(x) \subset L$. If $x \notin A$, then there exists a positive integer N such that no member of U_N containing x intersects A. Hence $\sigma_N(x) \cap L_N = \emptyset$, and consequently $\sigma(x) \cap L = \emptyset$. Thus,

$$L = \bigcup_{x \in A} \sigma(x) = \bigcup_{x \in A} \alpha^{-1}(x) = \alpha^{-1}(A).$$

(1.5) *α is continuous.*

Follows immediately from (1.4).

2. Let (X, A) be any pair in C. Let U be any finite open covering of X, and R be its nerve. Let S be the subcomplex of R defined as follows: A simplex in R is in S if and only if the intersection of the members of U defining the simplex in R has a non-empty intersection with A. Thus any finite open covering U of X defines a simplicial pair (R, S) corresponding to the pair (X, A).

Now let (X, A) be any pair in $\mathscr{C}$, and $\{U_n\}$ be a special sequence of coverings of X (see (1.0)). Let (K_n, L_n) be the simplicial pair corresponding to U_n and (X, A), $n = 1, 2, \ldots$ It is easy to see that any projection map $\Pi_n^{n+1}\colon K_{n+1} \to K_n$ takes L_{n+1} into L_n, and thus defines a simplicial map, which we denote by the same symbol, $\Pi_n^{n+1}\colon (K_{n+1}, L_{n+1}) \to (K_n, L_n)$. Thus a special sequence of coverings of X gives rise to an inverse sequence of simplicial pairs and simplicial maps corresponding to the pair (X, A). We call such an inverse sequence an *expansion* of (X, A).

THEOREM (2.1). *Suppose that H is a partially continuous homology theory on $\mathscr{C}$ and (K, L) is the inverse limit of an expansion of a pair (X, A) in $\mathscr{C}$. Then there exists a Vietoris map from (K, L) to (X, A).*

Proof. Let $\alpha_1\colon K \to X$ be the map defined as in Section 1 corresponding to an expansion of (X, A). From (1.4), for any $x \in A$, $\sigma(x) \subset L$ and is non-empty, and also $\alpha_1^{-1}(A) = L$. Thus $\alpha_1 | L\colon L \to A$ is an onto map. Thus α_1 defines a map $\alpha\colon (K, L) \to (X, A)$ which satisfies conditions (1) and (2) of (0.1). Again, since, for any $x \in X$, $\alpha^{-1}(x) = \operatorname{Inv Lim}\{\sigma_n(x), {}'\Pi_n^m\}$, where $\sigma_n(x)$ is a simplex and therefore acyclic in any homology theory ([3], th. (10.2), p. 119), from partial continuity of H it follows that $\alpha^{-1}(x)$ is acyclic. Thus condition (3) of (0.1) is also satisfied, and α is a Vietoris map. This completes the proof.

As an immediate consequence of (0.2) we have

THEOREM (2.2). *If, in theorem (2.1), H has furthermore the Vietoris property, then*

$$\alpha_*\colon H(K, L) \to H(X, A)$$

is an isomorphism.

3. Let $\{(K_n, L_n), \Pi_n^m\}$ be an expansion of a pair (X, A) in $\mathscr{C}$. For each n, let θ_n: $X_n \to K_n$ be a barycentric map, that is, for any $x \in X$, $\theta_n(x) \in \operatorname{Int} \sigma_n(x)$ ([1], p. 175). It is easy to check that for any $x \in A$, $\theta_n(x) \in \operatorname{Int} \sigma_n(x) \subset L$, and thus defines a barycentric map $\theta_n | A$: $A \to L_n$. Let us continue to denote the map of the pair (X, A) into (K_n, L_n) by θ_n. Let $F = \operatorname{Inv Lim}\{H(K_n, L_n), \Pi_{n*}^m\}$ and θ_*: $H(X, A) \to F$ be defined by $[\theta_*(Z)]_n = \theta_{n*}(Z)$ for any $Z \in H(X, A)$.

THEOREM (3.1). *θ_* is a homomorphism.*

Proof. It is enough to check that θ_* is a well-defined function, that is $\theta_*(Z) \in F$ for $Z \in H(X, A)$, or $\Pi_{n*}^m \cdot \theta_{m*}(Z) = \theta_{n*}(Z)$ for all $m > n$. Let $x \in X$. Now $\theta_n(x) \in \operatorname{Int} \sigma_n(x)$ for each n and, for $m > n$, Π_n^m maps $\sigma_m(x)$ into $\sigma_n(x)$, hence $\Pi_n^m \theta_m(x)$, $\theta_n(x)$ lie in the same simplex $\sigma_n(x)$. Thus $\Pi_n^m \theta_m$ and θ_n for $m > n$ are homotopic maps, and consequently $\Pi_{n*}^m \cdot \theta_{m*} = \theta_{n*}$. This completes the proof.

For each n let Π_n: $(K, L) \to (K_n, L_n)$ be the natural projection, and Π_*: $H(K, L) \to F$ be the natural transformation defined by $[\Pi_*(Z)]_n = \Pi_{n*}(Z)$ for $Z \in H(K, L)$ ([3], th. (2.1), p. 259). Let (K, L) and α be as in Section 2.

THEOREM (3.2). *If H is partially continuous and has the Vietoris property on $\mathscr{C}$, then $\theta_* = \Pi_* \alpha_*$ is an isomorphism.*

Proof. Let $p \in K$ and $\alpha(p) = x$. Then from the definition of α, $p \in \sigma(x)$. Hence $\Pi_n(p) \in \sigma_n(x)$; also $\theta_n \cdot \alpha(p) = \theta_n(x) \in \sigma_n(x)$. Hence $\theta_n \cdot \sigma$ and Π_n are homotopic maps, and therefore $\theta_{n*} \cdot \alpha_* = \Pi_{n*}$ for each n. Consequently, $\theta_* \cdot \alpha_* = \Pi_*$. Now α_* is an isomorphism from theorem (2.2) and Π_* is an isomorphism since H is partially continuous, hence θ_* is an isomorphism and the proof is complete.

THEOREM (3.3). *θ_* is independent of the choice of a metric in X.*

Proof. Let $\{(K_n, L_n), \Pi_n^m\}$ be an expansion of (X, A) and θ_n, θ_n' be barycentric maps from (X, A) into (K_n, L_n), $n = 1, 2, \ldots$ Since for any $x \in X$, $\theta_n(x)$ and $\theta_n'(x)$ both lie in the same simplex $\sigma_n(x)$, they are homotopic maps. Hence $\theta_{n*} = \theta_{n*}'$, and consequently $\theta_* = \theta_*'$ and the proof is complete.

(3.4) Let $\{(K_n, L_n), \Pi_n^m\}$ and $\{(P_n, Q_n), \psi_n^m\}$ be any two expansions of (X, A) in $\mathscr{C}$, corresponding to two special coverings $\{U_n\}$ and $\{V_n\}$ respectively. Let $\operatorname{Inv Lim}\{H(K_n, L_n), \Pi_{n*}^m\} = F$ and $\operatorname{Inv Lim}\{H(P_n, Q_n), \psi_{n*}^m\} = G$ and $\bar{F}, \bar{G}$ be the corresponding inverse limits for the Čech homology theory $\bar{H}$. Let $\bar{H}(X, A)$ be the Čech homology group of (X, A) corresponding to the set of all finite open coverings of the pair (X, A). Then there exist isomorphisms h_u: $\bar{H}(X, A) \to \bar{F}$ and h_v: $\bar{H}(X, A) \to \bar{G}$ ([3], corol. (3.16), p. 220). Let θ_*: $H(X, A) \to F$ and φ_*: $H(X, A) \to G$ be the homomorphisms defined by barycentric maps (see theorem 3.2).

Since $\{U_n\}$ is a special covering, it has a subsequence $\{U_{n_k}\}$ such that $U_{n_k} > V_k$, $k = 1, 2, \ldots$ We may assume without loss of generality that $\{U_n\}$ itself has this property. Let λ_n: $(K_n, L_n) \to (P_n, Q_n)$ be any projection maps, $n = 1, 2, \ldots$ Now $\psi_n^m \cdot \lambda_m$ and $\lambda_n \cdot \Pi_n^m$: $(K_m, L_m) \to (P_n, Q_n)$, being projection maps, are homotopic. Hence for $\bar{H}$ $\bar{\psi}_{n*}^m \cdot \bar{\lambda}_{m*} = \bar{\lambda}_{n*} \cdot \bar{H}_{n*}^m$, and $\{\bar{\lambda}_{n*}\}$: $\{\bar{H}(K_n, L_n), \Pi_{n*}^m\} \to \{\bar{H}(P_n, Q_n), \bar{\psi}_n^m\}$ is a map. Let λ_*: $\bar{F} \to \bar{G}$ be its inverse limit ([3], def. 3.10, p. 218). From the definitions of h_u, h_v and $\bar{\lambda}_*$ it follows that $\bar{\lambda}_* \cdot h_u = h_v$.

Again, because λ_n is a projection map, $\lambda_n \cdot \theta_n$ and φ are homotopic for the image of each x in X lies in the same simplex in P_n. Hence $\lambda_{n*} \cdot \theta_{n*} = \varphi_{n*}$ for each n and, therefore, $\lambda_* \cdot \theta_* = \varphi_*$, where λ_* is defined for H just as $\bar{\lambda}_*$ was for $\bar{H}$. Since the inverse sequences above are in the category of triangulable pairs and their maps, there exist natural isomorphisms $H(K_n, L_n) \simeq \bar{H}(K_n, L_n)$ and $H(P_n, Q_n) \simeq \bar{H}(P_n, Q_n)$ ([3], th. (10.2), p. 119). Let l_u: $F \to \bar{F}$ and l_v: $G \to \bar{G}$ be their inverse limits. Then the diagram

$$(*) \qquad \begin{array}{ccc} \bar{F} & \xrightarrow{\bar{\lambda}_*} & \bar{G} \\ \uparrow{\scriptstyle l_u} & & \uparrow{\scriptstyle l_v} \\ F & \xrightarrow{\lambda_*} & G \end{array}$$

commutes. Thus we have

THEOREM (3.5). $h_u^{-1} \cdot l_u \cdot \theta_* = h_v^{-1} \cdot l_v \cdot \varphi_*$.

THEOREM (3.6). *If H satisfies the Vietoris property and is partially continuous, then for any pair (X, A) in $\mathscr{C}$ the isomorphism $k(X, A)$: $H(X, A) \to \bar{H}(X, A)$ given by $k(X, A) = h_u^{-1} \cdot l_u \cdot \theta_* = h_u^{-1} \cdot l_u \cdot \Pi_* \cdot a_*^{-1}$ is independent of the expansion.*

Proof. $k(X, A) = h_u^{-1} \cdot l_u \cdot \theta_*$ is an isomorphism since θ_* is an isomorphism from theorem (3.2). It is independent of the expansion from theorem (3.5). $k(X, A) = h_u^{-1} \cdot l_u \cdot \Pi_* \cdot a_*^{-1}$, since $\Pi_* \cdot a_*^{-1} = \theta_*$ from theorem (3.2). This proves the theorem.

We may remark that, in view of theorem (3.3), $k(X, A)$ is also independent of the choice of a metric in X, since θ_* is so.

4. THEOREM (4.1). *If H has the Vietoris property and is partially continuous, then, for any map f: $(X, A) \to (Y, B)$ in $\mathscr{C}$, $k(Y, B) \cdot f_* = \bar{f}_* \cdot k(X, A)$.*

To prove this theorem we choose metrics in the given pairs for convenience and construct special sequences as follows.

Let V_0 be any finite open covering of Y. Let $x \in X$ and $f(x) = y$. Let $v(y)$ be an open set of diameter < 1 containing y and such that $\bar{v}(y)$ is contained in each member of V_0 containing y. Let $u(x)$ be an open set containing x such that diam $u(x) < 1$ and $f[u(x)] \subset v(x)$. This construction carried out for each x in X gives rise to an open covering of X which has a finite subcover $U_1 = \{u(x_i): i \in I_1\}$. Let $V_1' = \{v(f(x_i)): i \in I_1\}$. If V_1' does not cover Y, let V_1'' be a finite open covering of $Y - \cup V_1'$, each member of which is such that its diameter is < 1 and its closure is contained in each member of V_0 containing it. Then U_1 refines both $f^{-1}(V_0) = \{f^{-1}(v): v \in V_0\}$ and $f^{-1}(V_1)$, where $V_1 = V_1' \cup V_1''$. Also by construction $\bar{V}_1 > V_0$ and the mesh of both U_1 and V_1 is less than 1. We agree to denote a vertex in a nerve by the open set to which it corresponds.

Let (K_1, L_1) be the nerve corresponding to (X, A) of the covering U_1, and (P_0, Q_0) and (P_1, Q_1) be the nerves corresponding to (Y, B) and coverings V_0 and V_1 respectively. Let α_0: $(K_1, L_1) \to (P_0, Q_0)$ be any *carrier map* with respect to f. That is α_0 is the simplicial map obtained by extending linearly the vertex correspondence defined as follows: The vertex in K_1, corresponding to $u \in U_1$, is mapped into a vertex v in P_0 for $v \in V_0$, provided $f(u) \subset v$. We note then that:

(4.2) *For any* $x \in X$, $\alpha_0(\sigma_n(x)) \subset \sigma_n(f(x))$.

Let λ_1: $(K_1, L_1) \to (P_1, Q_1)$ be the carrier map with respect to f defined by the vertex correspondence $u(x_i) \to v(f(x_i))$ for each $i \in I_1$. Now define a vertex correspondence: $(P_1, Q_1) \to (P_0, Q_0)$ as follows: If v in V_1 is a member of V_1', that is $v = v(f(x_i))$ for some $i \in I_1$, then $v \to \alpha_0(u(x_i))$. If v in V is a member of V'', then $v \to w \in V_0$ such that $v \subset w$. Let ψ_0^1: $(P_1, Q_1) \to (P_0, Q_0)$ be the linear extension of the above correspondence. Then by the construction we have: ψ_0^1 is a projection map, and if $\psi_0^1(v) = w$ for a vertex v in P_0, then $\bar{v} \subset w$. Furthermore $\psi_0^1 \cdot \lambda_1 = \alpha_0$.

Starting now with the covering V_1 of Y we construct coverings U_2 and V_2 of X and Y respectively, just as U_1, V_1 were constructed starting with V_0, except that: (1) The mesh of both the coverings is $< \frac{1}{2}$, and (2) while choosing $u(x)$ to define an open covering of X, we not only have $f(u(x)) \subset v(f(x))$, but furthermore that $\bar{u}(x)$ is contained in all the members of U_1 containing x. Note then that $\bar{U}_2 > U_1$ and as before $\bar{V}_2 > V_1$.

Let (K_2, L_2) and (P_2, Q_2) be the nerves with respect to (X, A) and (Y, B) of U_2 and V_2 respectively. Let Π_1^2: $(K_2, L_2) \to (K_1, L_1)$ be any projection map, and set $\gamma_1 = \lambda_1 \cdot \Pi_1^2$. Then clearly γ_1 is a carrier map with respect to f. Let λ_2: $(K_2, L_2) \to (P_2, Q_2)$ and ψ_1^2: $(P_2, Q_2) \to (P_1, Q_1)$ be defined in the same way as λ_1 and ψ_0^1 were above. So that $\psi_1^2 \cdot \lambda_2 = \lambda_1 \cdot \Pi_1^2$.

It is clear that this construction can be carried out inductively to get expansions $\{(K_n, L_n), \Pi_n^m\}$ and $\{(P_n, Q_n), \psi_n^m\}$, where Π_n^m and ψ_n^m are the usual compositions, and maps $\{\lambda_n\}$, where λ_n: $(K_n, L_n) \to (P_n, Q_n)$, $n = 1, 2, \ldots$, are carrier maps with respect to f such that for $m > n$,

$\lambda_n \cdot \Pi_n^m = \psi_n^m \cdot \lambda_m$. Let $g\colon (L, K) \to (P, Q)$ be the inverse limit of the inverse system

$$\{\lambda_n\}\colon \{(K_n, L_n), \Pi_n^m\} \to \{(P_n, Q_n), \psi_n^m\}$$

and $\alpha\colon (K, L) \to (X, A)$ and $\beta\colon (P, Q) \to (Y, B)$ be defined as in Section 2.

LEMMA (4.3). $f \cdot \alpha = \beta \cdot g$.

Proof. Since λ_n is a carrier map with respect to f, from (4.2) for any $x \in X$, $\lambda_n(\sigma_n(x)) \subset \sigma_n(f(x))$. Hence $g(\sigma(x)) \subset \sigma(f(x))$. Now let $p \in K$, and $\alpha(p) = x$. Then $p \in \sigma(x)$ and $g(p) \in \sigma(f(x))$. Thus $\beta(g(p)) = f(x) = f(\alpha(p))$. Hence $\beta \cdot g = f \cdot \alpha$

Proof of Theorem (4.1). Consider the diagram

$$\begin{array}{ccccccccc}
H(X, A) & \xrightarrow{\alpha_*^{-1}} & H(K, L) & \xrightarrow{\Pi_*} & F & \xrightarrow{l_u} & \bar{F} & \xrightarrow{h_u^{-1}} & \bar{H}(X, A) \\
\downarrow f_* & \text{I} & \downarrow g_* & \text{II} & \downarrow \lambda_* & \text{III} & \downarrow \bar{\lambda}_* & \text{IV} & \downarrow \bar{f}_* \\
H(Y, B) & \xrightarrow{\beta_*^{-1}} & H(P, Q) & \xrightarrow{\psi_*} & G & \xrightarrow{l_v} & \bar{G} & \xrightarrow{h_v^{-1}} & \bar{H}(Y, B)
\end{array}$$

where the symbols not defined above are defined as in (3.4).

Now (I) commutes because of Lemma (4.3). (II) commutes since $\psi_n \cdot g = \lambda_n \cdot \Pi_n$ and from the definition of g. (III) commutes by the same argument as (*) does in (3.4). (IV) commutes because h_u and h_v are natural transformations. Hence the proof follows from the definition of $k(X, A)$ and $k(Y, B)$.

5. THEOREM (5.1). *If H has the Vietoris property and is partially continuous, then for any pair (X, A) in $\mathscr{C}$, $\bar{\partial} \cdot k(X, A) = k(A) \cdot \partial$, where ∂ and $\bar{\partial}$ are the boundary operators of H and $\bar{H}$ respectively.*

Proof. Let $\{(K_n, L_n), \Pi_n^m\}$ be an expansion of a pair (X, A) in $\mathscr{C}$ corresponding to a special sequence $\{U_n\}$. Then $\{W_n\}$ is a special sequence on A, where $W_n = \{u \cap A\colon u \in U_n\}$. From the definition of L_n it is clear that the nerve of W_n is isomorphic to L_n under the correspondence $u \to u \cap A$ for each vertex u of L_n. Hence we may regard $\{L_n, \varphi_n^m\}$, where $\varphi_n^m = \Pi_n^m | L_m$, as an expansion of A. The homomorphisms ∂_q: $H_q(K_n, L_n) \to H_{q-1}(L_n)$, $n = 1, 2, \ldots$, define a homomorphism γ_q: $G_q \to E_{q-1}$, since $(\Pi_n^m | L_m)_* \cdot \partial_q = \partial_q \cdot \Pi_{n*}^m$ ([3], axiom 3, p. 11), where $G_q = \operatorname{Inv\,Lim}\{H_q(K_n, L_n), \Pi_{n*}^m\}$ and $E_{q-1} = \operatorname{Inv\,Lim}\{H_{q-1}(L_n), (\Pi_n^m | L)_*\}$. Similarly, for $\bar{H}$ we get $\bar{\gamma}_q$: $\bar{G}_q \to \bar{E}_{q-1}$. Consider the following diagram:

$$\begin{array}{ccccccccc}
H_q(X, A) & \xrightarrow{\alpha_*^{-1}} & H_q(K, L) & \xrightarrow{\Pi_*} & G_q & \xrightarrow{l_u} & \bar{G}_q & \xrightarrow{h_u^{-1}} & \bar{H}_q(X, A) \\
\downarrow \partial_q & \text{I} & \downarrow \partial_q & & \downarrow \gamma_q & & \downarrow \bar{\gamma}_q & & \downarrow \bar{\partial}_q \\
H_{q-1}(A) & \xrightarrow{(\alpha|A)_*^{-1}} & H_{q-1}(L) & \xrightarrow{(\Pi|L)_*} & E_{q-1} & \xrightarrow{l_u} & \bar{E}_{q-1} & \xrightarrow{h_u^{-1}} & \bar{H}_{q-1}(A)
\end{array}$$

Diagram I commutes (see [3], axiom 3, p. 11), and the commutativity of the other diagrams can be established by the same arguments as in the proof of theorem (4.1) above. Hence the whole diagram commutes and completes the proof of the theorem.

6. Proof of the main theorem. The proof follows from (0.4) and theorems (3.6), (4.1) and (5.1) above.

As theorems (3.6), (4.1) and (5.1) do not depend on exactness for their proofs, we get, from the main theorem and theorem 7.6 ([3], p. 248),

THEOREM (6.1). *If H is a partially exact homology theory on $\mathscr{C}$ and satisfies the Vietoris property and is partially continuous, then H is exact provided the coefficient group is either a compact topological group or a vector space over a field.*

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UNIVERSITY OF SASKATCHEWAN
REGINA, SASKATCHEWAN

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