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ON THE FRATTINI SUBALGEBRA OF A STONE ALGEBRA

BY

K. M. KOH (SINGAPORE)

1. Introduction. The *Frattini* subalgebra $\Phi(\mathfrak{A})$ of an algebra $\mathfrak{A} = \langle A; F\rangle$ is defined as the intersection of all maximal subalgebras of $\mathfrak{A}$. The Frattini sublattices of lattices have recently been studied in several classes of lattices ([4], [7] and [8]). In this paper * we proceed one step further to investigate the Frattini subalgebra $\Phi(L)$ of a Stone algebra $\langle L; \vee, \wedge, *, 0, 1\rangle$. Since every Stone lattice is distributive, our investigation is based quite naturally on the results for Frattini sublattices of distributive lattices [4].

In Section 2 of this paper we shall introduce some basic concepts and state some fundamental results which will be used in the sequel. The study of maximal subalgebras and, consequently, that of the Frattini subalgebra of a Stone algebra, is contained in Section 3. In Section 4, we restrict ourselves to consider the Frattini subalgebra of a finite Stone algebra. It turns out that this subalgebra can be determined completely in the finite case.

2. Preliminaries. For a given lattice L, let $L(\vee)$ and $L(\wedge)$ be the sets of all $\vee$*-reducible* elements and all $\wedge$*-reducible* elements of L, respectively. Let $\mathrm{Irr}(L)$ be the set of all elements of L which are both $\vee$-irreducible and $\wedge$-irreducible. Then we have $L-\mathrm{Irr}(L) = L(\vee)\cup L(\wedge)$. A sublattice N of a lattice L is called *prime* if $L-N$ is either empty or is a sublattice of L. A prime sublattice N of L is called a *minimal prime sublattice* of L if N contains no prime sublattice of L other than itself. Let $P(L)$ and $Q(L)$ be the posets of prime ideals and prime dual ideals of L, respectively. A pair $(P, Q)\in P(L)\times Q(L)$ is said to be *minimal* in $P(L)\times Q(L)$ if

(1) $P\cap Q\neq\emptyset$,

(2) $P^*\cap Q^* = \emptyset$ whenever $(P^*, Q^*)\in P(L)\times Q(L)$ and $(P^*, Q^*) < (P, Q)$.

Dually, a pair (P, Q) is said to be *maximal* in $P(L)\times Q(L)$ if

(1) $P\cup Q\subset L$,

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(2) $P^* \cup Q^* = L$ whenever $(P^*, Q^*) \in P(L) \times Q(L)$ and $(P^*, Q^*) > (P, Q)$.

Using this terminology, we can formulate

LEMMA 1 [4]. *Let L be a distributive lattice and let M be a subset of L. Then the following are equivalent:*

(1) *M is a maximal sublattice of L;*

(2) *$M = P \cup Q$, where (P, Q) is a maximal pair in $P(L) \times Q(L)$;*

(3) *$L - M$ is a minimal prime sublattice of L;*

(4) *$L - M = P \cap Q$, where (P, Q) is a minimal pair in $P(L) \times Q(L)$.*

In finite case we obtain the following

LEMMA 2 [4]. *Let L be a finite distributive lattice. A subset N of L is a minimal prime sublattice if and only if*

(1) *$N = [a, b]$, where $a = \bigwedge N$ and $b = \bigvee N$,*

(2) *$a \in L(\wedge) - L(\vee)$ and $b \in L(\vee) - L(\wedge)$,*

(3) *$(a, b) \subseteq L(\vee) \cap L(\wedge)$.*

The Frattini sublattice of a distributive lattice L was proved in [4] to be identical with $L - E$, where E is the union of all minimal prime sublattices of L. Thus, by applying Lemma 2, the Frattini sublattice of a finite distributive lattice can easily be determined.

Maximal subalgebras of Boolean algebras were extensively studied by Sachs [9]. For an ideal I of a Boolean algebra B, let $I' = \{x' \mid x \in I\}$. Then I' is a dual ideal of B. In [9] it was shown that if M is a maximal subalgebra of B, then $M = I \cup I'$ for some ideal I of B, and, on the other hand, if $P_1, P_2 \in P(B)$, then $(P_1 \cap P_2) \cup (P_1' \cap P_2')$ is a maximal subalgebra of B. In fact, these two results can be combined to yield the following characterization of maximal subalgebras of Boolean algebras:

LEMMA 3. *Let M be a subset of a Boolean algebra B. The following are equivalent:*

(1) *M is a maximal subalgebra of B;*

(2) *$M = (P_1 \cap P_2) \cup (P_1' \cap P_2')$ for some $P_1, P_2 \in P(B)$ with $P_1 \neq P_2$;*

(3) *$M = B - (P \cap Q) \cup (P' \cap Q')$ for some $P \in P(B)$, $Q \in Q(B)$ with $P \cap Q \neq \emptyset$.*

LEMMA 4 [9]. *Every subalgebra of a Boolean algebra B is the intersection of some maximal subalgebras of B.*

From this lemma it follows immediately that

COROLLARY. *The Frattini subalgebra of any Boolean algebra is equal to $\{0, 1\}$.*

3. Maximal subalgebras and the Frattini subalgebra. Let $\langle L; \vee, \wedge, *, 0, 1\rangle$, henceforth, more briefly, L, be a Stone algebra. In this section we shall characterize maximal subalgebras of L, and then determine the Frattini subalgebra $\Phi(L)$ of L. To achieve this, we first recall some basic concepts which can be found in [3] and [5].

Two significant subsets $D(L)$ and $S(L)$ of a Stone algebra L, called the *dense set* and *skeleton* of L, respectively, are defined by

$$D(L) = \{a \mid a^* = 0\} \quad \text{and} \quad S(L) = \{a^* \mid a \in L\}.$$

The elements of $D(L)$ are called *dense*, and those of $S(L)$ are called *skeletal*. The dense set $D(L)$ is a dual ideal of L with $1 \in D(L)$. Thus $D(L)$ is a distributive lattice with 1. The skeleton $S(L)$ of L is a subalgebra of L and $\langle S(L); \vee, \wedge, *, 0, 1\rangle$ forms itself a Boolean algebra. L is called *dense* if $S(L) = \{0, 1\}$. For each $a \in S(L)$, put

$$F_a = \{x \mid x^{**} = a\}.$$

Then the set $\{F_a \mid a \in S(L)\}$ forms a partition of L. Moreover, $F_0 = \{0\}$ and $F_1 = D(L)$.

The mapping a, defined by $a(x) = x^{**}$, is clearly a homomorphism of the Stone algebra L onto $S(L)$ with $aa^{-1} = F_a$ for each $a \in S(L)$. We are now in a position to consider maximal subalgebras of L. First of all, we have the following

LEMMA 5. *Let L be a Stone algebra and let K be a maximal subalgebra of $S(L)$. Then the set $M = \bigcup \{F_x \mid x \in K\}$ is a maximal subalgebra of L such that $S(L) \nsubseteq M$.*

Proof. Since $M = \bigcup \{F_x \mid x \in K\} = Ka^{-1}$, it is a proper subalgebra of L with $S(L) \nsubseteq M$. To show that M is maximal, let $u \in L - M$ and consider the subalgebra $A = [M \cup \{u\}]$ of L generated by $M \cup \{u\}$. Observe that as $u^{**} \in A - K$ and K is a maximal subalgebra of $S(L)$, we have

$$S(L) = [K \cup \{u^{**}\}]_{S(L)} = [K \cup \{u^{**}\}] \subseteq A.$$

Clearly, $D(L) = F_1 \subseteq M \subseteq A$. Thus, we have $L = [D(L) \cup S(L)] = A$. Therefore, M is a maximal subalgebra of L.

The following result is the converse of Lemma 5:

LEMMA 6. *Let M be a maximal subalgebra of a Stone algebra L with $S(L) \nsubseteq M$. Then $M \cap S(L)$ is a maximal subalgebra of $S(L)$ and*

$$M = \bigcup \{F_x \mid x \in M \cap S(L)\}.$$

Proof. Evidently, $M \cap S(L) = Ma$ is a proper subalgebra of $S(L)$. Since $S(L)$ is a Boolean algebra, $Ma \subseteq K$ for some maximal subalgebra K of $S(L)$. Thus we have $Ka^{-1} \supseteq M$. By Lemma 5, this implies that $M = Ka^{-1}$. Hence $M \cap S(L) = Ma = K$ is a maximal subalgebra of $S(L)$ and

$$M = Ka^{-1} = \big(M \cap S(L)\big)a^{-1} = \bigcup \{F_x \mid x \in M \cap S(L)\}.$$

Lemmas 5 and 6 deal with those maximal subalgebras M of L which have the property that $S(L) \nsubseteq M$. For maximal subalgebras of L containing $S(L)$, we have the following result:

LEMMA 7. *Let M be a subset of a Stone algebra L. Then the following are equivalent*:

(1) *M is a maximal subalgebra of L with $S(L) \subseteq M$;*

(2) *M is a maximal sublattice of $\langle L; \vee, \wedge \rangle$ with $S(L) \subseteq M$;*

(3) *$L-M = P \cap Q$ for some minimal pair $(P, Q) \in P(L) \times Q(L)$ with $P \cap Q \cap S(L) = \emptyset$.*

Remark. The sublattice of $\langle L; \vee, \wedge \rangle$ generated by A will be denoted by $[A]_{(\vee, \wedge)}$ in order to distinguish it from the subalgebra $[A]$ of a Stone algebra L generated by a subset A of L. Obviously, we have $[A]_{(\vee, \wedge)} \subseteq [A]$.

Proof. The equivalence of (2) and (3) follows from Lemma 1. Thus it suffices to prove that (1) is equivalent to (2).

(1) $\Rightarrow$ (2). Assume that M is a maximal subalgebra of L. To show that M is a maximal sublattice of $\langle L; \vee, \wedge \rangle$, pick an element $z \in L-M$. Since $S(L) \subseteq M$, it follows that

$$[M \cup \{z\}] \subseteq [M \cup \{z\}]_{(\vee, \wedge)}.$$

Thus $[M \cup \{z\}]_{(\vee, \wedge)} = [M \cup \{z\}] = L$, which shows that M is a maximal sublattice of $\langle L; \vee, \wedge \rangle$.

(2) $\Rightarrow$ (1). Suppose that M is a maximal sublattice of $\langle L; \vee, \wedge \rangle$. Since $S(L) \subseteq M$, M is a subalgebra of L. Take an arbitrary element $z \in L-M$, and observe that

$$[M \cup \{z\}] \supseteq [M \cup \{z\}]_{(\vee, \wedge)} = L.$$

Thus M is a maximal subalgebra of L.

Summarizing the results of Lemmas 5, 6 and 7, we arrive at the following characterization of maximal subalgebras of a Stone algebra L.

THEOREM 1. *Let M be a subset of a Stone algebra L. Then M is a maximal subalgebra of L if and only if one of the following conditions holds*:

(i) *$M \cap S(L)$ is a maximal subalgebra of $S(L)$ and*

$$M = \bigcup \{F_x \mid x \in M \cap S(L)\};$$

(ii) *$M \cap S(L) = S(L)$ and $M = L - P \cap Q$ for some minimal pair $(P, Q) \in P(L) \times Q(L)$.*

Maximal subalgebras of the Boolean algebra $S(L)$ can easily be visualized by Lemma 3, and hence the same can be done for maximal subalgebras of Stone algebras according to Theorem 1.

By means of Theorem 1, the determination of the Frattini subalgebra of a Stone algebra is almost immediate.

THEOREM 2. *Let $\Phi(L)$ be the Frattini subalgebra of a Stone algebra L. Then $\Phi(L) = \{0\} \cup (D(L) - E)$, where*

$$E = \bigcup \{P \cap Q \mid (P, Q) \text{ is a minimal pair in } P(L) \times Q(L) \text{ with } (P \cap Q) \cap S(L) = \emptyset\}.$$

Proof. Let I be the intersection of all maximal subalgebras of L not containing $S(L)$. Then, by Theorem 1,

$$I = \bigcap\{\bigcup\{F_x \mid x \in K\} \mid K \text{ is a maximal subalgebra of } S(L)\}$$
$$= \bigcup\{F_x \mid x \in \Phi(S(L))\}.$$

Hence, by the corollary to Lemma 4, we have $I = F_0 \cup F_1$, and thus $I = \{0\} \cup D(L)$.

Let J be the intersection of all maximal subalgebras of L containing $S(L)$. Then, by Theorem 1, $J = L - E$.

Thus, by the definition and Theorem 1,

$$\Phi(L) = I \cap J = (\{0\} \cup D(L)) \cap (L - E) = \{0\} \cup (D(L) - E)),$$

as required.

COROLLARY. *Let L be a Stone algebra. Then*

(i) $\Phi(L) \cap S(L) = \{0, 1\}$;

(ii) $\Phi(L)$ *is dense.*

According to Lemma 4, every subalgebra of a Boolean algebra is the intersection of some maximal subalgebras. The same statement, however, does not hold for Stone algebras. Indeed, it is not difficult to construct a Stone algebra L with $\{0, 1\} \subset \Phi(L)$, and hence the subalgebra $\{0, 1\}$ of L cannot be any intersection of maximal subalgebras of L. Furthermore, if L is a dense Stone algebra satisfying A.C.C. or D.C.C., then, by Corollary 1 to Theorem 1 in [7] and our Lemma 7, one can show that every subalgebra of L is an intersection of maximal subalgebras of L if and only if L is a chain. In spite of this, we do have the following result:

THEOREM 3. *Let A be a proper subalgebra of a Stone algebra L with $D(L) \subseteq A$. Then A is the intersection of some maximal subalgebras of L.*

Proof. Let A be a proper subalgebra of L with $D(L) \subseteq A$. We first prove that $A = (Aa)a^{-1}$. The inclusion $A \subseteq (Aa)a^{-1}$ is trivial. To prove the converse, let $x \in (Aa)a^{-1}$. Then there exists $a \in A$ with $xa = au$, i.e., $x^{**} = a^{**} \in A$. Since $x = x^{**} \wedge (x \vee x^*)$, $x^{**} \in A$, and $x \vee x^* \in D(L) \subseteq A$, it follows that $x \in A$. Hence $A = (Aa)a^{-1}$, as was to be shown.

Now, observe that $A \cap S(L) = Aa$ is a subalgebra of $S(L)$. If $Aa = S(L)$, then $L = S(L)a^{-1} = (Aa)a^{-1} = A$, a contradiction. Hence Aa is a proper subalgebra of $S(L)$. By Lemma 4,

$$Aa = \bigcap_{i \in I} K_i,$$

where, for each $i \in I$, K_i is a maximal subalgebra of $S(L)$. Let $M_i = K_i a^{-1}$ for each $i \in I$. Then, by Lemma 5, M_i is a maximal subalgebra of L. Our proof will be complete if we shall show that

$$A = \bigcap_{i \in I} M_i.$$

This is indeed the case for

$$A = (Aa)a^{-1} = \Big(\bigcap_{i\in I} K_i\Big)a^{-1} = \bigcap_{i\in I} (K_i a^{-1}) = \bigcap_{i\in I} M_i .$$

4. Finite Stone algebras. In this section we shall confine ourselves to the study of Frattini subalgebras of finite Stone algebras. Grätzer and Schmidt [6] proved that a finite distributive lattice L is a Stone lattice if and only if

$$L = \prod_{i=1}^{n} K_i,$$

where, for each $i = 1, \ldots, n$, K_i is a finite distributive dense lattice. This result enables us to obtain from Theorem 2 a description of the Frattini subalgebra of a finite Stone algebra L in terms of the Frattini sublattices of its components K_i. Since there is a definite way to determine completely the Frattini sublattice of a finite distributive lattice [4], our result enables us to determine the Frattini subalgebra of a finite Stone algebra in a definite way. As a corollary to our result, we can provide necessary and sufficient conditions for a finite Stone algebra to be represented as the Frattini subalgebra of some finite Stone algebra. For any Stone algebra L, $\Phi(L)$ always contains $\{0, 1\}$ as a subalgebra. Thus, it is interesting to inquire what necessary and sufficient conditions should be imposed on L so that $\Phi(L)$ attains the lowest bound. A solution to this question will be given for the finite case.

We first state the following

LEMMA 8. *Let* $L = \prod_{i=1}^{n} L_i$ *be a finite product of lattices and let* $a \in L$, $a \neq 1$. *Then* a *is* $\wedge$*-irreducible if and only if*

$$a = \langle 1_1, \ldots, 1_{i-1}, a_i, 1_{i+1}, \ldots, 1_n \rangle,$$

where a_i *is* $\wedge$*-irreducible in* L_i *and* $a_i \neq 1_i$ *for some* i.

For $\vee$*-irreducible elements, the dual statement is true.*

We are now going to prove the following main result:

THEOREM 4. *Let* L *be a finite Stone algebra and let*

$$L = \prod_{i=1}^{n} K_i,$$

where, for each $i = 1, \ldots, n$, K_i *is a finite distributive dense lattice. Let* $\Phi(L)$ *be the Frattini subalgebra of* L *and let* $\Phi(K_i)$ *be the Frattini sublattice of* K_i *for* $i = 1, \ldots, n$. *Then*

$$\Phi(L) = \prod_{i=1}^{n} \big(\Phi(K_i) \cup \{1_i\}\big) \cup \{0\}.$$

Proof. Let $a \in \Phi(L)$. If a is 0 or 1, then, clearly,

$$a \in \prod_{i=1}^{n} \big(\Phi(K_i) \cup \{1_i\}\big) \cup \{0\}.$$

Thus we may assume $0 < a < 1$. By Theorem 2, $\Phi(L) \subseteq D(L) \cup \{0\}$. Thus

$$a = \langle a_1, \ldots, a_i, \ldots, a_n \rangle,$$

where $a_i > 0_i$ for each $i = 1, \ldots, n$. Now suppose, to the contrary, that

$$a \notin \prod_{i=1}^{n} \big(\Phi(K_i) \cup \{1_i\}\big) \cup \{0\}.$$

Then there exists an i such that $a_i \notin \Phi(K_i) \cup \{1_i\}$. Therefore $0_i < a_i < 1_i$, and there is a maximal sublattice A_i of K_i such that $a_i \notin A_i$. Obviously, $S(K_i) = \{0_i, 1_i\} \subseteq A_i$. Hence A_i is a maximal subalgebra of the dense Stone algebra K_i by Lemma 7. Let

$$A = K_1 \times \ldots \times K_{i-1} \times A_i \times K_{i+1} \times \ldots \times K_n.$$

It is not difficult to check that A is a maximal subalgebra of L. Since

$$a = \langle a_1, \ldots, a_i, \ldots, a_n \rangle \notin A,$$

we have $a \notin \Phi(L)$, a contradiction. Hence

$$\Phi(L) \subseteq \prod_{i=1}^{n} \big(\Phi(K_i) \cup \{1_i\}\big) \cup \{0\}.$$

Conversely, let

$$a = \langle a_1, \ldots, a_i, \ldots, a_n \rangle \in \prod_{i=1}^{n} \big(\Phi(K_i) \cup \{1_i\}\big) \cup \{0\}.$$

If a is 0 or 1, then $a \in \Phi(L)$. Thus, assume that $0 < a < 1$. Observe that as $a_i \in \Phi(K_i) \cup \{1_i\} \subseteq K_i - \{0_i\}$, it follows that $a \in D(L)$.

Suppose that $a \notin \Phi(L)$. Then there is a maximal subalgebra M of L such that $a \notin M$. If $M = \bigcup \{F_x \mid x \in K\}$ for some maximal subalgebra K of $S(L)$, then, clearly, $a \in D(L) = F_1 \subseteq M$, which is impossible. Therefore, by Theorem 1, $M = L - P \cap Q$ for some minimal pair $(P, Q) \in P(L) \times Q(L)$ with $(P \cap Q) \cap S(L) = \emptyset$. Since L is finite, by Lemma 2, $a \in [x, y]$ for some $x \in L(\wedge) - L(\vee)$ and $y \in L(\vee) - L(\wedge)$ with

$$(x, y) \subseteq L(\vee) \cap L(\wedge) \quad \text{and} \quad [x, y] \cap S(L) = \emptyset.$$

Obviously, $0 < x \leqslant y \leqslant 1$.

As x is $\vee$-irreducible in L, by Lemma 8,

$$x = \langle 0_1, \ldots, 0_{i-1}, x_i, 0_{i+1}, \ldots, 0_n \rangle,$$

where x_i is $\vee$-irreducible in K_i and $x_i > 0_i$. Since y is $\wedge$-irreducible in L, by Lemma 8 again,

$$y = \langle 1_1, \ldots, y_j, 1_{j+1}, \ldots, 1_i, \ldots, 1_n \rangle,$$

where y_j is $\wedge$-irreducible in K_i and $y_j < 1_j$. (Here, without loss of generality, we assume $j \leqslant i$.)

We shall now prove that $i = j$. Firstly, let us observe that an element $e = \langle e_1, \ldots, e_i, \ldots, e_n \rangle$ is in $S(L)$ if and only if e_k is 0_k or 1_k for each $k = 1, \ldots, n$. Thus, if $i \neq j$, we then consider the element

$$u = \langle u_1, \ldots, u_i, \ldots, u_n \rangle,$$

where $u_i = 1_i$ and $u_k = 0_k$ for each $k \neq i$. Clearly, $u \in S(L)$ and $x \leqslant u \leqslant y$. Therefore, $u \in [x, y] \cap S(L) \neq \emptyset$, a contradiction. Hence $i = j$ and we have

$$x = \langle 0_1, \ldots, 0_{i-1}, x_i, 0_{i+1}, \ldots, 0_n \rangle,$$
$$y = \langle 1_1, \ldots, 1_{i-1}, y_i, 1_{i+1}, \ldots, 1_n \rangle,$$

where $x_i \in K_i(\wedge) - K_i(\vee)$, $y_i \in K_i(\vee) - K_i(\wedge)$ and $0_i < x_i \leqslant y_i < 1_i$.

We now claim that $a_i \notin \Phi(K_i)$. Since $a \in [x, y]$, we have $x_i \leqslant a_i \leqslant y_i$. If $x_i = y_i$, then $a_i = x_i = y_i \in \mathrm{Irr}(K_i)$, and so $K_i - \{a_i\}$ is a maximal sublattice of K_i. Thus $a_i \notin \Phi(K_i)$. Assume $x_i < y_i$. Our aim is to prove that $(x_i, y_i) \subseteq K_i(\vee) \cap K_i(\wedge)$. Let $z_i \in (x_i, y_i)$ and write

$$z = \langle 1_1, \ldots, 1_{i-1}, z_i, 1_{i+1}, \ldots, 1_n \rangle,$$
$$z' = \langle 0_1, \ldots, 0_{i-1}, z_i, 0_{i+1}, \ldots, 0_n \rangle.$$

Clearly, $z, z' \in (x, y)$. Since $[x, y]$ is a minimal prime sublattice, we have $z, z' \in L(\vee) \cap L(\wedge)$. Therefore, by Lemma 8, $z_i \in K_i(\vee) \cap K_i(\wedge)$. Consequently, we have $(x_i, y_i) \subseteq K_i(\vee) \cap K_i(\wedge)$, as required. Hence $[x_i, y_i]$ is a minimal prime sublattice of K_i with $a_i \in [x_i, y_i]$. By Lemma 2, $a_i \notin \Phi(K_i)$, which is a contradiction. Hence

$$\Phi(L) \supseteq \prod_{i=1}^{n} \big(\Phi(K_i) \cup \{1_i\}\big) \cup \{0\}$$

and, therefore, Theorem 4 follows.

COROLLARY 1. *Let $\{L_i \mid i = 1, \ldots, n\}$ be a finite family of finite Stone algebras. Then*

$$\Phi\Big(\prod_{i=1}^{n} L_i\Big) = \Big(\prod_{i=1}^{n} \big(\Phi(L_i) - \{0_i\}\big)\Big) \cup \{0\}.$$

COROLLARY 2. *Let L be a finite Stone algebra. Then $\Phi(L) = \{0, 1\}$ if and only if L is a direct product of finite chains.*

Proof. If $L = \prod_{i=1}^{n} K_i$, where, for each $i = 1, \ldots, n$, K_i is a chain, then $\Phi(K_i) = \emptyset$ and hence, by Theorem 4,

$$\Phi(L) = \Big(\prod_{i=1}^{n} \big(\Phi(K_i) \cup \{1_i\}\big)\Big) \cup \{0\} = \{1\} \cup \{0\} = \{0, 1\}.$$

Conversely, assume that

$$L = \prod_{i=1}^{n} K_i \quad \text{and} \quad \Phi(L) = \{0, 1\}.$$

Then, by Theorem 4,

$$\prod_{i=1}^{n} \big(\Phi(K_i) \cup \{1_i\}\big) = \{1\}.$$

If there exists an i such that K_i is not a chain, then, by a corollary to Theorem 1 in [7], we have $|\Phi(K_i)| \geqslant 2$, and thus

$$\prod_{i=1}^{n} \big(\Phi(K_i) \cup \{1_i\}\big) \supset \{1\},$$

a contradiction. Hence L is a direct product of finite chains.

Recall that a Stone algebra I is *injective* if, whenever B is a subalgebra of a Stone algebra A, then any homomorphism of B into I can be extended to a homomorphism of A into I. Balbes and Grätzer [1] pointed out that a finite Stone algebra I is injective if and only if $I = C_2^m \times C_3^n$ for some m, n. Thus we have

COROLLARY 3. *If I is a finite injective Stone algebra, then $\Phi(I) = \{0, 1\}$.*

COROLLARY 4. *Let L be a finite Stone algebra and let*

$$L = \prod_{i=1}^{n} K_i.$$

Then $\Phi(L)$ is a chain if and only if there exists $k \in \{1, \ldots, n\}$ such that $\Phi(K_k)$ and K_i for each $i \neq k$ are chains.

COROLLARY 5. *Let $F_S(n)$ be the free Stone algebra on n generators. Then*

$$\Phi\big(F_S(n)\big) \cong \prod \big\{F_D(|X|) \cup \{1_X\} - X \mid X \subseteq \{1, \ldots, n\}\big\} \cup \{0\},$$

where $F_D(|X|)$ is the free distributive lattice on $|X|$ generators.

Proof. By a result of Balbes and Horn [2], we have

$$F_S(n) \cong \prod \big\{F_{D(0,1)}(|X|) \mid X \subseteq \{1, \ldots, n\}\big\}.$$

Thus, by Theorem 4,

$$\Phi\big(F_S(n)\big) \cong \prod\{\Phi\big(F_{D(0,1)}(|X|)\big)\cup\{1_X\}\,|\,X \subseteq \{1, \ldots, n\}\}\cup\{0\}$$
$$= \prod\{\Phi\big(F_D(|X|)\big)\cup\{1_X\}\,|\,X \subseteq \{1, \ldots, n\}\}\cup\{0\}.$$

Hence, by Lemma 1 in [7],

$$\Phi\big(F_S(n)\big) = \prod\{\big(F_D(|X|)-X\big)\cup\{1_X\}\,|\,X \subseteq \{1, \ldots, n\}\}\cup\{0\},$$

so that the assertion follows.

THEOREM 5. *Let K be a finite Stone algebra. Then $K \cong \Phi(L)$ for some finite Stone algebra L if and only if*

(1) *K is dense,*

(2) *$K-\{0\}$ can be represented as $\prod_{i=1}^{n} K_i$, where, for each $i = 1, \ldots, n$, either $|K_i| = 1$ or $K_i \cong \Phi(L_i)$ for some finite distributive lattice L_i with $1_i \in L_i(\vee)$.*

Proof. We first prove the necessity. $\Phi(L)$ is clearly dense. By Theorem 4,

$$\Phi(L)-\{0\} = \prod_{i=1}^{n}\big(\Phi(L_i)\cup\{1_i\}\big), \qquad \text{where } L = \prod_{i=1}^{n} L_i.$$

For each $i = 1, \ldots, n$, let $K_i = \Phi(L_i)\cup\{1_i\}$.

If $\Phi(L_i) = \emptyset$, then $K_i = \{1_i\}$ is a singleton.

If $\Phi(L_i) \neq \emptyset$, we have two cases. If $1_i \in \Phi(L_i)$, then $K_i = \Phi(L_i)$ and $1_i \in L_i(\vee)$. If $1_i \notin \Phi(L_i)$, then $1_i \in \mathrm{Irr}(L_i)$. Let m be the maximal element of $L_i(\vee)$. Then $m \in \Phi(L_i)$ by Lemma 3 in [7]. Let $C = \{c \in L_i \,|\, m < c < 1_i\}$. Construct a new lattice $L_i^* = (L_i - C)\cup\{a, b\}$ by adjoining two new incomparable elements a, b to the lattice $L_i - C$ in such a way that $a \vee b = 1_i$ and $a \wedge b = m$. Obviously, L_i^* is finite, distributive and $1_i \in L_i^*(\vee)$. Moreover, $\Phi(L_i^*) \cong \Phi(L_i)\cup\{1_i\} = K_i$.

To prove the sufficiency, let K be a finite Stone algebra satisfying (1) and (2). For each $i = 1, \ldots, n$, if $K_i = \{1_i\}$, we set $S_i = \{0_i\}\oplus\{1_i\} \cong C_2$; if $K_i \cong \Phi(L_i)$, we set $S_i = \{0_i\}\oplus L_i$, where 0_i is the new zero element of S_i. Let

$$S = \prod_{i=1}^{n} S_i.$$

As S_i is a finite distributive dense lattice for each $i = 1, \ldots, n$, S is a Stone algebra. Observe that if $S_i = \{0_i\}\oplus\{1_i\}$, then

$$\Phi(S_i)\cup\{1_i\} \cong K_i,$$

and if $S_i = \{0_i\}\oplus L_i$, then

$$\Phi(S_i)\cup\{1_i\} = \Phi(L_i)\cup\{1_i\} = \Phi(L_i) \cong K_i \quad \text{as } 1_i \in L_i(\vee).$$

Hence, by Theorem 4,

$$\Phi(S) = \prod_{i=1}^{n} \big(\Phi(S_i) \cup \{1_i\}\big) \cup \{0\} = \Big(\prod_{i=1}^{n} K_i\Big) \cup \{0\} = (K - \{0\}) \cup \{0\} = K.$$

This completes the proof of Theorem 5.

Finally, as a by-product of the proof of Theorem 4, we have the following result:

THEOREM 6. *Let $\{L_i \mid i = 1, \ldots, n\}$ be a family of finite distributive lattices and let $\bar{L}_i = \{0_i\} \oplus L_i$ for $i = 1, \ldots, n$. If $\Phi\big(\prod_{i=1}^{n} \bar{L}_i\big)$ is the Frattini sublattice of $\prod_{i=1}^{n} \bar{L}_i$, and $\Phi(\bar{L}_i)$ is the Frattini sublattice of $\bar{L}_i$, then*

$$\Phi\Big(\prod_{i=1}^{n} \bar{L}_i\Big) \cap \Big(\prod_{i=1}^{n} L_i\Big) = \prod_{i=1}^{n} \big(\Phi(\bar{L}_i) \cup \{1_i\}\big).$$

Proof. Let

$$\bar{L} = \prod_{i=1}^{n} \bar{L}_i \quad \text{and} \quad L = \prod_{i=1}^{n} L_i.$$

Then $\bar{L}$ is a Stone lattice with $D(\bar{L}) = L$. By modifying slightly the argument of the proof of Theorem 4, we can show that the left-hand set is contained in the right-hand set. The reverse inclusion follows from the following lemmas and the proof of Theorem 4.

LEMMA 9. *Let a be a maximal element of $S(\bar{L})$. Then a is $\wedge$-irreducible in $\bar{L}$.*

Proof. As a is maximal in $S(\bar{L})$, there is a unique i such that

$$a = \langle 1_1, \ldots, 1_{i-1}, 0_i, 1_{i+1}, \ldots, 1_n \rangle.$$

Then, by Lemma 8, a is $\wedge$-irreducible in $\bar{L}$.

LEMMA 10. *Let $[x, y]$ be a minimal prime sublattice of $\bar{L}$ and let $y \in D(\bar{L})$. Then $[x, y] \cap S(\bar{L}) = \emptyset$.*

Proof. Assume that $u \in [x, y] \cap S(\bar{L})$. Then

$$u = \bigwedge_{i=1}^{m} e_i,$$

where, for each $i = 1, \ldots, m$, e_i is a maximal element of $S(\bar{L})$. Since

$$\bigwedge_{i=1}^{m} e_i = u \in [x, y]$$

and $[x, y]$ is prime, it follows that $e_i \in [x, y]$ for some i. Clearly, $y < 1$ and $y \in D(\bar{L})$ by assumption. Thus $y \notin S(L)$, and so we have $x \leqslant e_i < y$. But this contradicts Lemma 9 since every element of $[x, y)$ is $\wedge$-reducible.

REFERENCES

[1] R. Balbes and G. Grätzer, *Injective and projective Stone algebras*, Duke Mathematical Journal 38 (1971), p. 339-347.
[2] R. Balbes and A. Horn, *Stone lattices*, ibidem 37 (1970), p. 537-546.
[3] C. C. Chen and G. Grätzer, *Stone lattices, I. Construction theorems*, Canadian Journal of Mathematics 21 (1969), p. 884-894.
[4] C. C. Chen, K. M. Koh and S. K. Tan, *Frattini sublattices of distributive lattices*, Algebra Universalis 3 (1973), p. 294-303.
[5] G. Grätzer, *Lattice theory. First concepts and distributive lattices*, San Francisco 1971.
[6] — and E. T. Schmidt, *On a problem of H. M. Stone*, Acta Mathematica Academiae Scientiarum Hungaricae 8 (1957), p. 455-460.
[7] K. M. Koh, *On the Frattini sublattice of a lattice*, Algebra Universalis 1 (1971), p. 104-116.
[8] — *On the Frattini sublattice of a finite modular planar lattice*, ibidem 3 (1973), p. 304-317.
[9] D. Sachs, *The lattice of subalgebras of a Boolean algebra*, Canadian Journal of Mathematics 14 (1962), p. 451-460.

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