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REFINABLE MAPS

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1. Introduction. All spaces considered are compacta (compact metric spaces), and all maps are continuous. Let $\varepsilon > 0$, and let f be a map from X onto Y. Then f is said to be an *ε-map* if $\operatorname{diam}(f^{-1}(y)) < \varepsilon$ for each $y \in Y$. If r is a map from X onto Y, then f is called an *ε-refinement* of r if f is an ε-map from X onto Y ε-near r (i.e., $d(f(x), r(x)) < \varepsilon$ for each $x \in X$ or, more concisely, $d(f, r) < \varepsilon$). The map r is *refinable* if r has an ε-refinement for each $\varepsilon > 0$ or, equivalently, r is a uniform limit of ε-maps for every $\varepsilon > 0$. Refinable maps clearly include near homeomorphisms (uniform limits of homeomorphisms from X onto Y), but the notions are not equivalent: the map from the $\sin(1/x)$-continuum onto an arc obtained by shrinking the limiting interval to a point is refinable, but not a near homeomorphism, since the domain and range are not homeomorphic. The notions are not even equivalent when the range and domain are both polyhedra. Indeed, consider the map from a pair of tangent disks onto a single disk obtained by shrinking one of the disks in the domain to a point.

Although X and Y need not be homeomorphic, given a refinable map from X onto Y, they are closely related, since X clearly must be Y-like (i.e., there is an ε-map from X onto Y for every $\varepsilon > 0$). It follows from this, e.g., that there can be no refinable map from a circle or a triod onto an arc. Requiring the existence of a refinable map from X onto Y is a stronger condition than requiring that X be Y-like, however, for we will show in the next section that there is no refinable map from the pseudo-arc onto the arc.

In Section 3 we will see that if the range and domain of a refinable map are both ANR's, then they are quasi-homeomorphic and have the same homotopy type.

2. Some general properties of refinable maps. Theorem 1 shows that refinable maps satisfy a property somewhat stronger than weak confluence; strong enough to prevent the mapping of indecomposable continua onto decomposable ones and to imply monotonicity if Y is locally connected.

THEOREM 1. *If r is a refinable map from X onto Y and H is a subcontinuum of Y, then there is a continuum C in X such that $r(C) = H$ and C contains $r^{-1}(\text{int}(H))$, where $\text{int}(H)$ denotes the interior of H.*

Proof. For each i, let h_i be a $(1/i)$-refinement of r. Some subsequence of $\{h_i^{-1}(H)\}$ converges to a closed subset C of X which is connected (even though each $h_i^{-1}(H)$ may not be). If $r(x)$ is in $\text{int}(H)$, then, for all but finitely many positive integers j, $h_j(x)$ is also in $\text{int}(H)$, since $\{h_i(x)\} \to r(x)$. This means that $x \in h_j^{-1}(H)$ for almost all positive integers j, and hence $x \in C$.

Definition (Lelek [4]). The map f from X onto Y is *weakly confluent* if for each subcontinuum H of Y some component of $f^{-1}(H)$ is mapped onto H by f.

COROLLARY 1.1. *Every refinable map from X onto Y is weakly confluent.*

Hence, if X is finitely Suslinian, Suslinian, or hereditarily locally connected, then Y has the same property, and if X is acyclic and 1-dimensional, then Y is 1-dimensional [5].

QUESTION 1. Do refinable maps preserve the property of being rational? **(P 1033)**

COROLLARY 1.2. *If r is a refinable map from X onto Y and Y is connected im kleinen at p, then $r^{-1}(p)$ is connected; hence r is monotone if Y is locally connected.*

Proof. Suppose that Y is connected im kleinen at p, but $r^{-1}(p)$ is not connected. There are mutually exclusive open sets U and V in X such that $r^{-1}(p)$ intersects both and lies in their union. The compactness of X implies that $O = Y - r(X - U \cup V)$ is open, and $p \in O$. By hypothesis, O contains a continuum H with p in its interior. By Theorem 1, there is a continuum C in X which contains $r^{-1}(\text{int}(H)) \supseteq r^{-1}(p)$ and such that $r(C) = H$. Hence

$$C \subseteq r^{-1}(H) \subseteq r^{-1}(O) \subseteq U \cup V.$$

But the continuum C cannot intersect both U and V and lie in their union.

Remark. On the contrary, to see that a refinable map need not in general be monotone, consider the chainable continuum M, made up of a sequence of $\sin(1/x)$-continua, as indicated in Fig. 1. If points of M on and between the dotted lines A and B are identified whenever they lie on the same vertical line, the resulting map is refinable but clearly not monotone; its range is a single $\sin(1/x)$-continuum.

COROLLARY 1.3. *Let r be a refinable map from X onto Y. Then X is decomposable if and only if Y is decomposable.*

Proof. Since X is Y-like, it is easily seen that if X is decomposable, then so is Y. If Y is decomposable, then Y contains a proper subcontinuum H with non-empty interior. By Theorem 1, there is a continuum C in X containing $r^{-1}(\mathrm{int}(H))$, an open set in X. Since C has a non-empty interior, X is decomposable.

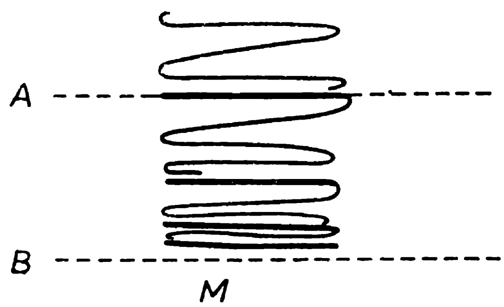

Fig. 1

THEOREM 2. *If r is a refinable map from X onto Y, and the point q separates Y, then some point of $f^{-1}(q)$ is a weak cut point of X.*

Proof. Pick $h \in r^{-1}(H)$ and $k \in r^{-1}(K)$, where H and K are mutually separated sets whose union is $Y-\{q\}$. There is a sequence $\{g_i\}$ such that, for each i, g_i is a $(1/i)$-refinement of r and such that the sequence $\{g_i^{-1}(q)\}$ converges to a point p of X. Note that $p \in r^{-1}(q)$.

Now, suppose that M is a continuum in X containing h and k, but not p. Then there is a positive integer n so large that $g_n^{-1}(q)$ misses M, $g_n(h) \in H$, and $g_n(k) \in K$. So the continuum M intersects both of the mutually exclusive open sets $g_n^{-1}(H)$ and $g_n^{-1}(K)$, and lies in their union, a contradiction.

COROLLARY 2.1. *If r is a refinable map from X onto Y, and X is locally connected and has no cut point, then, if $y \in Y$, $X-r^{-1}(y)$ is connected. Hence Y has no cut point.*

Proof. If, for some $y \in Y$, $r^{-1}(y)$ separates X, then since r is monotone (Corollary 1.2), y separates Y. By Theorem 2, some point of $r^{-1}(y)$ is a weak cut point of X, and hence a cut point, since X is locally connected. But this contradicts the hypothesis.

QUESTION 2. Suppose that r is a refinable map from X onto Y and Y is locally connected at y. Does it follow that if $r^{-1}(y)$ separates X, then y is a cut point of Y?[1]

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[1] E. E. Grace has recently answered Question 2 in the negative.

QUESTION 3. Suppose that r is a refinable map from X onto Y and K is a subcontinuum of X. Under what conditions is $r|K$ refinable? (**P 1034**)

QUESTION 4. Suppose that there exist maps, f from X onto Y and g from Y onto X, such that both compositions fg and gf are refinable. Need there exist a refinable map from one of X and Y onto the other (see [3], Theorem 5)? (**P 1035**)

3. Refinable maps on ANR's. If the domain of a refinable map is assumed to be an ANR, then a number of interesting results follow, many as corollaries to the next theorem. We will need a lemma and a definition.

LEMMA (see [6], Lemma 1). *Let f be a map from X onto the ANR A, and let $c_1 > 0$. Then there is a positive number c_2 such that if g_1 is any c_2-map from X onto any compactum Y, then there is a map g_2 from Y onto A such that $d(f, g_2g_1) < c_1$.*

Definition. If f is a map from X onto Y, and $\varepsilon > 0$, then

$$L(f, \varepsilon) = \sup\{c \mid \text{ if } H \subseteq X \text{ and } \operatorname{diam}(H) < c, \text{ then } \operatorname{diam}(f(H)) < \varepsilon\}.$$

That $L(f, \varepsilon) > 0$ follows from the uniform continuity of f.

THEOREM 3. *Let r be a refinable map from X onto Y and let $\varepsilon > 0$. Then there exists a positive number δ such that if f is a δ-map from X onto an ANR A, then there exist a map g_1 from X onto Y ε-near r and an ε-map g_2 from Y onto A such that $d(f, g_2g_1) < \varepsilon$.*

Proof. Pick $\delta < L(r, \varepsilon)$, and let f be any δ-map from X onto an ANR A. There is a positive number $c_1 < \varepsilon$ such that any map c_1-near f is also a δ-map. By the Lemma, there is a positive number c_2 such that if g_1 is a c_2-map from X onto Y, then there exists a map g_2 from Y onto A such that $d(f, g_2g_1) < c_1$. Since r is refinable, there is a c_2-map g_1 so close to r that $d(g_1, r) < \varepsilon$, and $\delta < L(g_1, \varepsilon)$. Let g_2 have the properties mentioned above. Then g_2g_1 is sufficiently close to f so that $d(f, g_2g_1) < \varepsilon$ and g_2g_1 is a δ-map. To see that g_2 is an ε-map, suppose that $x \in A$. Then

$$\operatorname{diam}\big((g_2g_1)^{-1}(x)\big) < \delta < L(g_1, \varepsilon),$$

whence $\operatorname{diam}\big(g_2^{-1}(x)\big) < \varepsilon$.

Definition. Let $\mathscr{P}$ be a collection of spaces. The space X is said to be *$\mathscr{P}$-like* if, for each $\varepsilon > 0$, X can be ε-mapped onto some element of $\mathscr{P}$.

COROLLARY 3.1. *Let r be a refinable map from X onto Y and let $\mathscr{P}$ be some collection of ANR's. Then X is $\mathscr{P}$-like if and only if Y is.*

The special case of this corollary, in which $\mathscr{P}$ consists of an arc, was first shown by E. E. Grace.

COROLLARY 3.2. *If r is a refinable map from the ANR X onto Y, then X and Y are quasi-homeomorphic, i.e., X is Y-like and Y is X-like.*

COROLLARY 3.3. *If r is a refinable map from the ANR X onto Y and $\varepsilon > 0$, then there exist maps g_1 from X onto Y and g_2 from Y onto X such that*

(i) $d(rg_2, \mathrm{id}_Y) < \varepsilon$ *(in a sense, r almost has a right inverse),*

(ii) $d(g_1g_2, \mathrm{id}_Y) < \varepsilon$ *and* $d(g_2g_1, \mathrm{id}_X) < \varepsilon$.

Proof. Pick $c > 0$ so that $c < \varepsilon/4$ and $c < L(r, \varepsilon/4)$. By Theorem 3 (letting $A = X$ and $f = \mathrm{id}_X$), there exist maps g_1 from X onto Y and g_2 from Y onto X so that

(a) $d(g_1, r) < c$,

(b) $d(g_2g_1, \mathrm{id}_X) < c$.

Suppose that $y \in Y$, and pick x so that $g_1(x) = y$. Then, using (a) and (b), we have

$$\begin{aligned} d(rg_2(y), y) &= d(rg_2g_1(x), g_1(x)) \\ &\leqslant d(rg_2g_1(x), r(x)) + d(r(x), g_1(x)) < \tfrac{1}{4}\varepsilon + \tfrac{1}{4}\varepsilon = \tfrac{1}{2}\varepsilon. \end{aligned}$$

This verifies (i), and (ii) follows since

$$d(g_1g_2(y), y) \leqslant d(g_1g_2(y), rg_2(y)) + d(rg_2(y), y) < \tfrac{1}{4}\varepsilon + \tfrac{1}{2}\varepsilon < \varepsilon.$$

COROLLARY 3.4. *If r is a refinable map from X onto Y and X is an ANR, then Y-homotopy dominates X; if Y is also an ANR, then r is a homotopy equivalence.*

Proof. Since, for ANR's, maps sufficiently close together are homotopic, this follows immediately from Corollary 3.3 (ii).

QUESTION 5. Do refinable maps preserve shape? (**P 1036**)

QUESTION 6. If r is a refinable map from X onto Y and X is an ANR, need Y also be an ANR? (**P 1037**)

As observed earlier, refinable maps are closely related to near homeomorphisms. Bing's Shrinking Criterion, extracted from [1], gives conditions that imply that a map is a near homeomorphism. There are several versions; a typical one is

THEOREM (Bing [1]). *The map r from X onto Y is a near homeomorphism if and only if, for each $\varepsilon > 0$, there is a homeomorphism f from X onto X such that rf is an ε-map ε-near r.*

Dropping the condition that f be a homeomorphism gives a necessary and sufficient condition for refinability, as the next corollary shows.

COROLLARY 3.5. *The map r from the ANR X onto Y is refinable if and only if, for each $\varepsilon > 0$, there is a map f from X onto X such that rf is an ε-map ε-near r.*

Proof. The condition clearly implies that r is refinable, since rf is an ε-refinement for r. Conversely, suppose that r is refinable and $\varepsilon > 0$. There are an $(\varepsilon/2)$-refinement h of r, and a positive number $\delta < \varepsilon$ such that if d is any δ-map from Y to Y, then dh is still an $(\varepsilon/2)$-map (and hence an ε-map). By Corollary 3.3 (i), there is a map g_2 from Y onto X such that $d(rg_2, \mathrm{id}_Y) < \delta/2$. Write $f = g_2 h$, and note that rg_2 is a δ-map, so that $rf = (rg_2)h$ is an ε-map. Also,

$$d(rf, r) \leqslant d(rg_2 h, h) + d(h, r) < \tfrac{1}{2}\delta + \tfrac{1}{2}\varepsilon < \varepsilon.$$

Remark. Note that if X is not an ANR, then there need not exist maps such as f in Corollary 3.5. Indeed, consider the map on the $\sin(1/x)$-continuum described in the Introduction.

THEOREM 4. *Assume that r maps the k-sphere S^k onto Y, where $k = 1$ or 2. Then r is refinable if and only if r is a near homeomorphism.*

Proof. We give only the argument for S^2; that for S^1 is easier. Y must be locally connected since S^2 is, so r is monotone by Corollary 1.2. It follows from Corollary 2.1 that no point-preimage separates S^2, so Y must be homeomorphic to S^2 by a theorem of Moore [8]. But Youngs has shown [9] that every monotone map from S^2 onto itself is a near homeomorphism.

QUESTION 7. Is Theorem 4 true for $k > 2$? (**P 1038**)

4. Refinable maps on arc-like continua. Assume that r is a refinable map from X onto Y. Then X is arc-like (or chainable) if and only if Y is (Corollary 3.1). Since the only locally connected arc-like continuum is the arc (see [7], Theorem 6), the assumption in this section that X is an ANR implies that r is a (monotone) near homeomorphism on an arc, by essentially the same argument as in Theorem 4. Assuming instead that Y is an arc proves more fruitful.

THEOREM 5. *Assume that r is a map from the arc-like continuum X onto $[0, 1]$. Then r is refinable if and only if r is monotone.*

Proof. If r is refinable, then r is monotone by Corollary 1.2. Suppose that r is monotone and $\varepsilon > 0$. By a *chain* in this argument, we will mean a sequence $c_1, c_2, \ldots, c_n$ of sets, not necessarily open, such that c_i intersects c_j if and only if $|i-j| \leqslant 1$. Let $s_1, s_2, \ldots, s_n$ denote a chain of non-overlapping intervals of length less than $\varepsilon/2$ covering $[0, 1]$ with 0 in s_1. For each i, let $K_i = r^{-1}(s_i)$, and let x_i denote a point of K_i such that $r(x_1) = 0$, $r(x_n) = 1$, and $r(x_i)$ lies in the interior of s_i for $1 < i < n$. Since r is monotone, K_i is a continuum for each i, and $K_1, K_2, \ldots, K_n$ is a chain. Pick $\delta > 0$ so that $\delta < \varepsilon$, if $i \neq j$, then $\delta < d(x_i, K_j)$, and if $|i-j| > 1$, then $\delta < d(K_i, K_j)$. We know that X is arc-like; let f denote a δ-map from X onto $[0, 1]$, and note that $f(K_1), f(K_2), \ldots, f(K_n)$ is a chain of intervals covering $[0, 1]$, and if $i \neq j$, then $f(x_i)$ is not in $f(K_j)$. It follows that the sequence $f(x_1), f(x_2), \ldots, f(x_n)$ is either increasing or decreasing; assume the former. Then there is a homeomorphism h from

$[0,1]$ onto $[0,1]$ such that $hf(x_i) = r(x_i)$ for $1 < i < n$. Now, hf is a δ-map, and hence an ε-map. Also we have

$$hf(K_i) \subseteq \big(r(x_{i-1}), r(x_{i+1})\big) \subseteq s_{i-1} \cup s_i \cup s_{i+1} \quad \text{for } 1 < i < n,$$

and similar relationships if $i = 1$ or n, while $r(K_i) = s_i$ for each i. It follows that $d(hf, r) < \varepsilon$, and hf is an ε-refinement of r.

Remark. The equivalence between monotonicity and refinability in Theorem 5 breaks down if the range of the map is not an arc, even if it is a hereditarily decomposable, arc-like continuum. The map described earlier on the continuum M in Fig. 1 is refinable, but not monotone. On the other hand, the map from M onto the continuum Y, obtained by shrinking only the lowest limiting interval of M to a point p, is monotone but not refinable. Indeed, if f is any map from M onto Y, then $f^{-1}(p)$ contains some arc-component of M, which is large; so M is not even Y-like.

COROLLARY 5.1. *Every hereditarily decomposable arc-like continuum admits a refinable map onto an arc.*

This follows since every such continuum has an upper semicontinuous decomposition to an arc (see [2], Theorem 8).

REFERENCES

[1] R. H. Bing, *A homeomorphism between the 3-sphere and the sum of two solid horned spheres*, Annals of Mathematics 56 (1952), p. 354-362.

[2] — *Snakelike continua*, Duke Mathematical Journal 18 (1951), p. 653-663.

[3] M. Brown, *Some applications of an approximation theorem for inverse limits*, Proceedings of the American Mathematical Society 11 (1960), p. 478-483.

[4] A. Lelek, *A classification of mappings pertinent to curve theory*, Proceedings of the University Oklahoma Topology Conference 1972, p. 97-103.

[5] — *Report on weakly confluent mappings*, Proceedings of the Virginia Polytechnic Institute and State University Topology Conference 1973, p. 168-170.

[6] I. Lončar and S. Mardešić, *A note on inverse sequences of ANR's*, Glasnik Matematički 3 (23) (1968), p. 41-48.

[7] S. Mardešić and J. Segal, *ε-mappings onto polyhedra*, Transactions of the American Mathematical Society 109 (1963), p. 146-164.

[8] R. L. Moore, *Concerning upper semi-continuous collections of continua*, ibidem 27 (1925), p. 416-428.

[9] J. W. T. Youngs, *Homeomorphic approximations to monotone mappings*, Duke Mathematical Journal 15 (1948), p. 87-94.

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