ASSERTION $Q$ DISTINGUISHES TOPOLOGICALLY $\omega^* AND m^*$
WHEN $m$ REGULAR AND $m > \omega$

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We consider the following question: can $\omega^*$ be homeomorphic to $m^*$ for $m > \omega$, $\omega$ denoting the set of non-negative integers? Here $m^*$ stands for $\beta m - m$, where $\beta m$ is the Čech-Stone compactification of the set $m$ with the discrete topology. Only the case $2^\omega = 2^m$ is of interest, since only in that case the weights of $\omega^*$ and $m^*$ are equal. We answer the question for regular $m$ negatively in ZFC + non CH + Q, where CH stands for the continuum hypothesis and Q is the following assertion (Rothberger [4]):

For each family $\mathcal{X}$ of less than $2^\omega$ functions $f: \omega \to \omega$ there exists a function $g: \omega \to \omega$ such that for each $f \in \mathcal{X}$ the set $\{n \in \omega: f(n) \geq g(n)\}$ is finite.

Assertion $Q$ is known to be a theorem in the theory ZFC + non CH + Martin’s Axiom (Kunen and Tall [2]) which is consistent if ZFC is (Martin and Solovay [3]); in that theory, $2^m = 2^\omega$ if $\omega \leq m < 2^\omega$ (see papers [2] and [3]).

The problem to distinguish $\omega^*$ and $m^*$ by means of ZFC axioms only seems to be open. (1021)

A free ultrafilter in a set is a maximal filter on that set not containing finite subsets.

The space $m^*$ consists of all free ultrafilters on $m$; the topology on $m^*$ is generated by sets $U^* = \{p: U \in p, \ p \in m^*\}$, where $U$ is an infinite subset of $m$.

An uncountable cardinal $\kappa$ is called measurable if there exists a free ultrafilter $q$ on $\kappa$ such that $\bigcap \mathcal{U} \in q$ for each countable subfamily $\mathcal{U}$ of $q$.

An ultrafilter $q \in m^*$ is called a $P$-ultrafilter if for every sequence consisting of members of $q$ there exists a member $U$ of $q$ such that $U - V$ is finite for each $V$ from that sequence.

LemMA. Let $m$ be non-measurable and let $q$ be a $P$-ultrafilter on $m$. Then there exists a countable subset $a$ of $m$ such that $a \in q$. 
Proof. Since $m$ is non-measurable, there exists a sequence $\{a_i : i \in \omega\}$ such that
\[ a_i \in q, \ a_i \supseteq a_{i+1}, \ \text{and} \ \bigcap_{i \in \omega} a_i = 0. \]

Since $q$ is a $P$-ultrafilter, there exists an infinite $a, a \in q$, such that $a - a_i$ is finite for all $i$ and $a \subseteq a_1$. Hence the sets $a \cap (a_i - a_{i+1})$ are finite. But $a$ is infinite and equal to
\[ \bigcup_{i < \omega} (a \cap (a_i - a_{i+1})), \]
thus $a$ is countable.

Let $V = \{m - a : a \subseteq m, \ \text{card} \ a \leq \omega\}$. Since $\text{cf}(m) > \omega$, $m$ being regular, there exists a filterbase of cardinality $m$ such that the filter generated by that filterbase contains $V$.

For instance, the set
\[ \mathcal{F} = \{m - a : a \text{ is an ordinal, } a < m\} \]
in such a filterbase.

**Theorem (ZFC + non CH + Q), Szymański [6], Solomon [5] for $m = \omega_1$.** Let $m$ be a regular cardinal, $m < 2^\omega$, and let $\{a_\xi : \xi < m\}$ be a family of closed-open subsets of $\omega^*$ linearly ordered by inclusion. Then there exists a $P$-ultrafilter $q$ on $\omega$ such that
\[ q \in \bigcap_{\xi < m} a_\xi. \]

**Theorem (ZFC + non CH + Q).** $\omega^*$ is not homeomorphic to $m^*$ whenever $m$ is regular and such that $m > \omega$.

Proof. Only the case $m < 2^\omega$ requires a proof. Let $f$ be a homeomorphism from $\omega^*$ onto $m^*$ and let
\[ \mathcal{F}^* = \{a^* : a \in \mathcal{F}\}, \quad \text{where} \quad a^* = \text{cl}_{m^*} a - m. \]

The family $\mathcal{F}$ is linearly ordered by inclusion and $\text{card} \ \mathcal{F}^* = m$.

The family
\[ \mathcal{G} = \{f^{-1}(a^*) : a \in \mathcal{F}\} \]
is also linearly ordered by inclusion, and each element of $\mathcal{G}$ is a closed-open subset of $\omega^*$.

From the above-quoted result of Szymański we infer that there exists a $P$-ultrafilter $q \in \omega^*$ such that $q \in \bigcap \mathcal{G}$. The ultrafilter $f(q)$ is a $P$-ultrafilter on $m$, $f$ being a homeomorphism. We can see that $f(q) \in \bigcap \mathcal{F}^*$. This means that each element of $\mathcal{F}$, and, therefore, each element of $V$, belongs to $f(q)$. The ultrafilter $f(q)$ is not a $P$-ultrafilter on $m$. In fact,
m is non-measurable by Ulam’s theorem (Ulam [7], and Jech [1], p. 167), m being not greater than 2^α.

In other case, by the Lemma, a countable a would exist with a ∈ f(q); however, we know that m − a belongs to V and, therefore, to f(q); a contradiction.

REFERENCES


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