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ON THE DENSITY MAXIMA OF A FUNCTION

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It is well known that the set of points at which a function of a real variable takes on a strict relative maximum or minimum is at most countable. O'Malley [1] showed that the set of points at which a real-valued function defined on a Euclidean n-space takes on a strict density maximum or minimum (defined below) is of measure 0. While a slightly stronger result is given in this paper, it will be shown that very few additional restrictions can be placed on the behavior of f on its set of density maxima even if f is supposed continuous. For example, the image of the density maxima of a continuous function of a real variable can contain an interval. This will follow from the theorem which states that, given any continuous function f of a real variable and a closed set P of measure 0, there is a continuous function g such that $g = f$ on P, and P is the set of density maxima for g.

The following definitions will be needed:

Given a set A, a point x, and a real-valued function f,

1. $m^*(A)$ is the Lebesgue outer measure of A, and $m(A)$ is the Lebesgue measure of A in the event that A is measurable.

2. $B(x, r) = \{t \mid \mathrm{dist}(x, t) \leqslant r\}$.

3. $\bar{D}_x(A) = \overline{\lim}\, m^*\big(A \cap B(x, h)\big)/m\big(B(x, h)\big)$; similarly, $\underline{D}_x(A)$ is defined using $\underline{\lim}$.

4. For $0 < a \leqslant 1$, $\bar{M}_a(f) = \big\{x \mid \bar{D}_x\big(f(t) \geqslant f(x)\big) < a\big\}$; similarly, $\underline{M}_a(f)$ is defined using $\underline{D}_x$.

5. x is a *strict density maximum* [*minimum*] of f provided

$$\bar{D}_x\big(f(t) \geqslant f(x)\big) = 0 \qquad [\bar{D}_x\big(f(t) \leqslant f(x)\big) = 0].$$

The set of strict density maxima of f will be denoted by $M_0(f)$.

A point x is a strict density maximum if and only if there exists a set E such that $x \in E$, $D_x(E) = 1$, and the function $f|E$ has a proper maximum at x.

Note. While the main concern will be with functions of a real variable, the following theorem generalizes readily to a Euclidean n-space and will be proved in that context. On the real line the definition of $\bar{D}_x$ [$\underline{D}_x$] refers to the upper [lower] symmetric density. This agrees with the usual definition of density when $\bar{D}_x = \underline{D}_x = 0$ or $\bar{D}_x = \underline{D}_x = 1$.

THEOREM 1. *If $f(x)$ is any real-valued function defined on a Euclidean n-space, then $m\big(\bar{M}_{2^{-n}}(f)\big) = 0$.*

Proof. Suppose not. Then there exists an a with $0 < a < 2^{-n}$ such that $m^*(\bar{M}_a) > 0$. There also exists an $\varepsilon > 0$ such that if

$$(1) \qquad A = \Big\{x \in \bar{M}_a \,|\, 0 < h < \varepsilon \text{ implies } m^*\Big(\big(f(t) \geqslant f(x)\big) \cap B(x, h)\Big) < am\big(B(x, h)\big)\Big\},$$

then $m^*(A) > 0$. Given A, let E be a measurable set with $A \subset E$ and $m^*(A) = m(E)$. The collection of all balls $B(x, h/2)$ with $x \in A$ and $0 < h < \varepsilon$ covers the set A in the sense of Vitali. So there exists an at most countable collection of balls $B_k = B(x_k, h_k/2)$ which come from this collection, cover almost all of A, are pairwise disjoint, and satisfy

$$m\big(\bigcup B_k\big) < m(E)\cdot(2^n a)^{-1}.$$

For each k, let

$$y_k = \inf\big\{y \,|\, m^*\{t \in B_k \,|\, f(t) \geqslant y\} \leqslant 2^n a m(B_k)\big\}.$$

It follows that

$$(2) \qquad m^*\{x \in B_k \,|\, f(x) > y_k\} \leqslant 2^n a m(B_k)$$

and

$$(3) \qquad m^*\{x \in B_k \,|\, f(x) \geqslant y_k\} \geqslant 2^n a m(B_k).$$

If $x \in B_k$, then

$$B_k = B(x_k, h_k/2) \subset B(x, h_k).$$

Thus, from (1) and (3) it follows that if $x \in A \cap B_k$, then $f(x) > y_k$ (since $m\big(B(x, h_k)\big) = 2^n m(B_k)$). But then (2) implies that

$$m^*(A \cap B_k) \leqslant 2^n am(B_k).$$

Thus

$$m^*(A) = \sum m^*(A \cap I_k) \leqslant 2^n a \sum m(B_k) = 2^n am\big(\bigcup B_k\big) < m(E).$$

This contradiction implies the theorem.

Note that it follows from the theorem that the set of density maxima [minima] of any function is of measure 0. On the real line the constant $1/2$ cannot be improved, since any strictly monotone f satisfies $\bar{D}_x\big(f(t) \geqslant f(x)\big)$

$= 1/2$ at every point. That $\bar{D}_x$ cannot be replaced by $\underline{D}_x$ is shown by the example to follow.

PROBLEM. Can 2^{-n} in Theorem 1 be improved for a Euclidean n-space $(n \geqslant 2)$? (**P 1019**)

Example. There is a continuous function f defined on $[0,1]$ and satisfying

$$m\{x \mid \underline{D}_x(f(t) \geqslant f(x)) = 0\} = 1.$$

Construction. Each real number $x \in [0,1]$ can be written uniquely as $\sum_{n=2}^{\infty} q_n(x)/n!$, where each $q_n(x)$ is an integer, $0 \leqslant q_n(x) < n$, and the sequence $q_n(x)$ has infinitely many non-zero values. For all real numbers x, set $g_2(x) = f_2(x) = 0$. For $n > 2$ define $g_n(x)$ and $f_n(x)$ inductively as follows:

$$g_n(x) = \begin{cases} 2^{-n} & \text{if } n \text{ is odd and } q_n(x) = (n+1)/2, \\ 2^{-n} & \text{if } q_n(x) = 0 \text{ and } f_{n-1}\big(x - q_n(x)/n!\big) > f_{n-1}(x), \\ 2^{-n} & \text{if } q_n(x) = n-1 \text{ and } f_{n-1}\big(x + q_n(x)/n!\big) > f_{n-1}(x), \\ 2^{-n} & \text{if } n \text{ is even, } q_n(x) \text{ is even, and } q_n(x) \neq 0, \\ -2^{-n} & \text{otherwise,} \end{cases}$$

$$f_n(x) = g_n(x) + f_{n-1}(x).$$

Clearly, this sequence of functions $\{f_n(x)\}$ converges, at every point x, to a function $f(x)$. To prove that $f(x)$ is continuous, it will be shown that, for every $m, n \ni m \geqslant n$, if $|x-y| < 1/n!$, then

$$|f_m(x) - f_m(y)| \leqslant 4 \cdot 2^{-n} - 2 \cdot 2^{-m}.$$

This is clearly true if $m = n = 2$. Suppose that it is true for $n = k$ and $m = k, k+1, \ldots, k+j$. Then, since for all t

$$|g_{k+j+1}(t)| \leqslant 2^{-(k+j+1)},$$

we have

$$|f_{k+j+1}(x) - f_{k+j+1}(y)| \leqslant 4 \cdot 2^{-k} - 2 \cdot 2^{-(k+j)} + 2 \cdot 2^{-(k+j+1)} \leqslant 4 \cdot 2^{-k} - 2 \cdot 2^{-(k+j+1)},$$

and the statement is true for $n = k$ and $m = k+j+1$.

Now suppose that it is true for $n = k$ and $m = k$. Then, if

$$|x-y| < \frac{1}{(k+1)!} \quad \text{and} \quad f_k(x) = f_k(y),$$

clearly

$$|f_{k+1}(x) - f_{k+1}(y)| \leqslant 2 \cdot 2^{-(k+1)}.$$

On the other hand, if $f_k(x) \neq f_k(y)$, by hypothesis we have

$$|f_k(x) - f_k(y)| \leqslant 4 \cdot 2^{-k} - 2 \cdot 2^{-k} = 2 \cdot 2^{-k}.$$

By the definition of g_{k+1}, we obtain

$$|f_{k+1}(x)-f_{k+1}(y)| = |f_k(x)-f_k(y)|-2\cdot 2^{-(k+1)} \leqslant 2\cdot 2^{-(k+1)}.$$

The induction is complete, and thus the statement is true. But then letting m approach ∞, it follows that $|x-y|<1/n!$ implies

$$|f(x)-f(y)| \leqslant 4\cdot 2^{-n};$$

and thus f is a continuous function. By observing $g_n(x)$ when n is odd at points x for which $k_n(x) = (n+1)/2$, one can see that $f(x)>f(t)$ provided

$$\frac{1}{n!} < \operatorname{dist}(x,t) < \frac{n-3}{2\cdot n!}.$$

That is, the relative density of the set of t in

$$\left[x-\frac{n-3}{2\cdot n!},\ x+\frac{n-3}{2\cdot n!}\right]$$

such that $f(t)\geqslant f(x)$ is no larger than $1/(n-3)$. Consequently, each x in $A = \{x \mid \text{for infinitely many odd } n,\ k_n(x) = (n+1)/2\}$ satisfies

$$\underline{D}_x\big(f(t)\geqslant f(x)\big) = 0.$$

Let

$$B_n = \{x \mid m\geqslant n \text{ implies } k_{2m+1}(x) \neq m+1\}.$$

Then $[0,1]\cap A^c = \bigcup B_n$. However,

$$m(B_n) = \prod_{m\geqslant n}\frac{2m}{2m+1} = 0.$$

Thus $m(A) = 1$, and $f(x)$ satisfies the hypotheses claimed.

The following result shows that any property which is satisfied by the graph of some continuous function on a closed set of measure 0 can be satisfied by the set of strict density maxima of a continuous function.

THEOREM 2. *If $f\colon \boldsymbol{R}\to\boldsymbol{R}$ is a continuous function and P is a closed set of measure 0, then there is a continuous function g which agrees with f on P such that P is the set of strict density maxima of g.*

Proof. Without loss of generality, P is compact, $P\subset[0,1]$, $\{0,1\}\subset P$, and

$$f(x) = \begin{cases} f(0)+x & \text{if } x<0,\\ f(1)+1-x & \text{if } x>1.\end{cases}$$

(Heuristically speaking, $f(x)$ will be diminished on the intervals contiguous to P so as to be small enough on these intervals so that each

$x \in P$ is a strict density maximum of g but not so small as to make the resulting function g discontinuous.)

For $x \in P^c \cap [0, 1]$, let I_x be the interval contiguous to P which contains x. Let $d_x = \text{dist}(x, P)$. Let $\{I_k\}_{k=1}^{\infty}$ be an enumeration of the intervals I_x. For each natural number n, let $k(n)$ be the least number such that

$$\sum_{k \geqslant k(n)} |I_k| \leqslant n^{-2}.$$

Let

$$\mathscr{I}_n = \{I_k \mid k < k(n)\}, \qquad E_n = \{x \mid x \in I_k \in \mathscr{I}_n\},$$

and

$$\mathscr{J}_n = \mathscr{I}_{n+1} \setminus \mathscr{I}_n.$$

Note that if $I \subset [0, 1]$ and $|I| \geqslant n^{-1}$, then

$$|E_n^c \cap I| \leqslant n^{-2} \leqslant n^{-1}|I|,$$

and thus

$$|E_n \cap I| \geqslant (1 - n^{-1})|I|.$$

Let $h_0(x) = x + \sup[f(t_1) - f(t_2)]$, where the supremum is taken over all t_1, t_2 with $|t_1 - t_2| \leqslant x$. Define $h(x)$ on each interval $I_x \in \mathscr{J}_n$ by

$$h(x) = \min[2h_0(|I_x|^{-2} d_x), h_0(2n^{-1})]$$

and put

$$h(x) = 0 \quad \text{ if } x \in P \cup (-\infty, 0) \cup (1, \infty).$$

Note that

$h_0(x)$ is continuous and strictly increasing on $[0, \infty)$;

$h_0(px) \geqslant p h_0(x)$ if $0 \leqslant p \leqslant 1$, $h_0(px) \leqslant ph_0(x)$ if $p > 1$;

$h(x)$ is continuous (it is clearly continuous on each interval contiguous to P and, if $\{x_n\} \subset P^c$ and $x_n \to x_0 \in P$, at least one of $2h_0(|I_{x_n}|^{-2} d_{x_n})$ or $h(2n^{-1})$ approaches $0 = h(x_0)$).

Let $g(x) = f(x) - h(x)$. Then $g(x)$ is continuous and, in order to show that each $x \in P$ is a strict density maximum for $g(x)$, it will suffice to show that, for each $x \in P$,

$$\lim_{h \to 0+} m\{t \in [x, x+h] \mid g(t) \geqslant g(x)\} \cdot h^{-1} = 0. \tag{4}$$

A parallel proof will yield that, for each $x \in P$,

$$\lim_{h \to 0+} m\{t \in [x-h, x] \mid g(t) \geqslant g(x)\} \cdot h^{-1} = 0,$$

and thus that each $x \in P$ is a strict density maximum for g.

Case (i). Let x be a left-hand end point of a contiguous interval $I_t \in \mathscr{I}_{\bar{n}}$. Choose $h < 0$ so small that $h < |I_t|$ and

$$2h_0(|I_t|^{-2}d_t) \leqslant h_0(2\bar{n}^{-1}) \tag{5}$$

whenever $x < t < x+h$. Then

$$h(t) = 2h_0(|I_t|^{-2}d_t) > h_0(d_t) > |f(t)-f(x)|$$

and it follows that

$$g(t) = f(t)-h(t) < f(x) = g(x)$$

whenever $t \in (x, x+h)$.

Note. Whenever x is either end point of $I_t \in \mathscr{I}_n$ and (5) holds, then

$$h(t) > h_0(|I_t|^{-1}nd_t) \geqslant |f(t)-f(x)|.$$

Clearly, (4) holds for the point x; thus (4) holds for each left-hand end point of intervals contiguous to P.

Case (ii). Let x be a point of P which is not a left-hand end point of any contiguous interval. Let m be a large natural number and choose $\varepsilon > 0$ so small that $\varepsilon < 1/2$, $\varepsilon < 1-x$, and no interval of $\mathscr{I}_m$ meets $[x, x+h]$ whenever $0 < h < \varepsilon$. Given h with $0 < h < \varepsilon$, determine n so that $(n+1)^{-1} \leqslant h < n^{-1}$. Consider any interval $I_k \in \mathscr{I}_{n+1}$ such that $I_k \subset [x, x+h]$. If $I_k \in \mathscr{I}_{\bar{n}}$, then $n \geqslant \bar{n} \geqslant m$ and, for $t \in I_k$ satisfying $\bar{n}d_t \geqslant 2|I_t|$, it follows that

$$\bar{n}d_t \geqslant |I_t|^2, \qquad |I_t|^{-2}d_t \geqslant \bar{n}^{-1}, \qquad h_0(2|I_t|^{-2}d_t) \geqslant h_0(2\bar{n}^{-1}),$$

and

$$2h_0(|I_t|^{-2}d_t) \geqslant h_0(2\bar{n}^{-1}). \tag{6}$$

Consequently, $h(t) = h_0(2\bar{n}^{-1})$ on an interval $\tilde{I}_k \subset I_k$, where

$$|\tilde{I}_k| \geqslant (1-\bar{n}^{-1})|I_k| \geqslant (1-m^{-1})|I_k|.$$

Now, since $|t-x| \leqslant n^{-1}$, we have

$$h(t) = h_0(2\bar{n}^{-1}) > h_0(n^{-1}) \geqslant |f(t)-f(x)|$$

at each point $t \in \tilde{I}_k$. (Note that $t \in I \in \mathscr{I}_{\bar{n}+1}$ and $I \cap [x, x+h] \neq \emptyset$ and (6) holding for t imply $h(t) > |f(t)-f(x)|$.)

Now consider an interval $J = (u, v)$, if such an interval exists, satisfying $J \in \mathscr{I}_{\bar{n}}$, $n \geqslant \bar{n} \geqslant m$ and $x < u < x+h < v$. As noted above, there is an interval $\tilde{J} \subset J$ on which (6) holds and, consequently, $g(t) < g(x)$ on $\tilde{J}$. Let $(a, b) \supset \tilde{J}$ be the largest such interval on which (6) holds. If $t \in (u, a)$, then, since

$$f(u)-f(x) < h(u-x) \qquad \text{and} \qquad g(x) = f(x),$$

we have

$$0 \leqslant g(t)-g(x) = f(t)-f(u)+f(u)-f(x)-h(t)$$
$$< f(t)-f(u)+h_0(u-x)-2h_0(|I_t|^{-2}d_t).$$

Applying the Note from case (i) to the interval $[u, t]$ yields

$$0 \leqslant g(t)-g(x) < h_0(u-x)-h_0(|I_t|^{-2}d_t)$$
$$< h_0(u-x)-h_0\big((v-u)^{-1}\bar{n}(t-u)\big).$$

Whence, a is at least as small as the number $\bar{a}$ which satisfies

$$h_0(u-x) = h_0\big((v-u)^{-1}\bar{n}(\bar{a}-u)\big)$$

and, since h_0 is strictly increasing, we have

$$a-u \leqslant \bar{a}-u = (u-x)(v-u)\bar{n}^{-1}.$$

Thus

$$\frac{a-u}{u-x} \leqslant m^{-1} \quad \text{and} \quad \frac{a-u}{v-u} \leqslant m^{-1},$$

and this implies that

$$|[u, a] \cap [x, x+h]| \leqslant hm^{-1}.$$

Now, if $t \in (b, v)$ and if (5) holds for t, since

$$f(v)-f(x) < h_0(v-x) \quad \text{and} \quad g(x) = f(x),$$

by applying the Note of case (i) to $[t, v]$ we obtain

$$0 < h_0(|I_t|^{-1}d_t)-[f(t)-f(v)]$$

and

$$0 \leqslant g(t)-g(x) = f(t)-f(v)+f(v)-f(x)-h(t)$$
$$< f(t)-f(v)+h_0(v-x)-2h_0(|I_t|^{-2}d_t)$$
$$< h_0(v-x)-h_0(|I_t|^{-2}d_t) < h_0(v-x)-h_0\big((v-u)^{-1}\bar{n}(v-t)\big).$$

So b is at least as large as the number $\bar{b}$ satisfying

$$v-x = (v-u)^{-1}\bar{n}(v-\bar{b}).$$

Thus

$$v-b \leqslant (v-x)(v-u)\bar{n}^{-1} \quad \text{and} \quad \frac{v-b}{v-x} \leqslant m^{-1}.$$

Now, if $[b, v] \cap [x, x+h] \neq \emptyset$, then $x+h \in [b, v]$ and it is easily checked that

$$\frac{x+h-b}{h} \leqslant \frac{v-b}{v-x},$$

so that $|[b, v] \cap [x, x+h]| \leqslant hm^{-1}$.

Putting these observations together with the fact that J and the intervals I_k are the totality of intervals in $\mathscr{I}_{n+1}$ which meet $[x, x+h]$, we have

$$m\{t \in [x, x+h] \mid f(t) < f(x)\} \geqslant (1-m^{-1})(1-m^{-1})h - 2hm^{-1}.$$

Since m is an arbitrarily large number, (4) holds for x. Since x was an arbitrary element of P, each element $x \in P$ is a strict density maximum for g.

In order to have each density maximum of g belong to P it is sufficient to follow through with the above construction after having redefined f on the intervals contiguous to P so as to make f linear on each such interval. It is then readily observed that the construction does not yield any density maxima for g other than the points $x \in P$. Thus Theorem 2 is proved.

Next it is shown that any F_σ of measure 0 can be contained in the set of density maxima for a continuous function.

THEOREM 3. *If E is an F_σ of measure 0, F is a closed subset of E, and f is a continuous function, then there is a continuous function g which agrees with f on F such that each point of E is a strict density maximum of g. Moreover, given $\varepsilon > 0$, g can be chosen so that $|f(x)-g(x)| \leqslant \varepsilon$.*

Proof. Again, without loss of generality, $E \subset [0, 1]$, and $\{0, 1\} \in F$. Write E as $\bigcup F_n$ with $F_1 = F$ and $F_n\uparrow$. As in Theorem 1, let

$$f_0(x) = \begin{cases} f(x) & \text{if } x \in [0,1], \\ f(0) - x & \text{if } x < 0, \\ f(1)+1-x & \text{if } x > 1. \end{cases}$$

Now, let f_{k+1} $(k = 0, 1, \ldots)$ be the function determined by applying the method of Theorem 1 to the function f_k with $P = F_{k+1}$ and the following alteration of the function h to the function h_k:

if $x \in I_n \in \mathscr{I}_n$,

$$h_k = \min[2h_0(|I_x|^{-2}d_x), h_0(2n^{-1}), \varepsilon\cdot 2^{-k}];$$

if $x \in F_k$, set $h_k(x) = 0$.

Here, h_0, I_x, and $\mathscr{I}_n$ are determined by F_k and f_k. It follows that each point of F_k is a strict density maximum of f_{k+1} for $k = 0, 1, \ldots$ (by minor modification of the proof of Theorem 1). Now, at each point $x \in [0, 1]$, $f_k(x) \geqslant f_{k+1}(x)$ and, since

$$f_k(x) - f_{k+1}(x) = h_k(x) \leqslant \varepsilon\cdot 2^{-k},$$

it follows that $g(x) = \lim f_k(x)$ is a continuous function and that

$$|f(x)-g(x)| \leqslant \sum \varepsilon\cdot 2^{-k} = \varepsilon.$$

Moreover, $g(x) = f_k(x)$ on F_k and $g(x) \leqslant f_k(x)$. This, along with the fact that each point of F_k is a strict density maximum of f_{k+1}, implies that each point of F_k is a strict density maximum of g. Thus the theorem is proved.

One might suspect that the functions produced in Theorems 2 and 3 might have been constructed so as to be piecewise linear on each interval contiguous to the compact set. Theorem 4 shows that, in general, this cannot be accomplished. This result and the method for proving it were suggested to me by Richard Fleissner.

THEOREM 4. *Let P be a perfect set of measure 0. Suppose that f is a continuous function and the set of end points a_n of intervals contiguous to P which satisfy*

$$\varlimsup_{\substack{x\in P\\ x\to a_n}} \left| \frac{f(x)-f(a_n)}{x-a_n} \right| = \infty$$

is dense in $(a,b)\cap P$, where $(a,b)\cap P \neq \emptyset$. Then, for every function g such that $g = f$ on P and g is piecewise linear on each interval contiguous to P, there exists a point $x \in P$ such that $x \notin M_0(g)$.

Proof. Without loss of generality, suppose that on a dense set of left-hand end points c_n in P, f satisfies

$$\varlimsup_{\substack{x\in P\\ x\to c_n}} \left| \frac{f(x)-f(c_n)}{x-c_n} \right| = \infty,$$

f is continuous, and f is piecewise linear on intervals contiguous to P.

It remains to show that not every point of P belongs to $M_0(f)$. Let m^n be the slope of the first linear piece of the graph originating at c_n. Choose $h_n > 0$ so that this piece is defined on (c_n, c_n+h_n). Then each m_n is negative (otherwise, $c_n \notin M_0(f)$). Choose $\varepsilon_n \downarrow 0$ and consider the set A_n consisting of all $x \in P$ which satisfy

$$f(c_n)-f(x) < |m_n| h_n \tag{7}$$

and

$$\big(f(c_n)-f(x)\big)(c_n-x)^{-1} > |m_n|\varepsilon_n^{-1}. \tag{8}$$

Each A_n is open in the relative topology on P and, for each k,

$$E_k = \bigcap_{n=k}^{\infty} A_n$$

is dense in P. By the Baire Category Theorem, $\bigcap E_k \neq \emptyset$. Let $x \in \bigcap E_k$. Then $x \in A_n$ for infinitely many n. Consider such an n and the set of h satisfying

$$c_n < x+h < c_n+\big(f(c_n)-f(x)\big)|m_n|^{-1}. \tag{9}$$

Since, by (7),

$$\big(f(c_n)-f(x)\big)\,|m_n|^{-1} < h_n,$$

we have

$$f(x+h) = f(c_m) - |m_n|\,(x+h-c_n).$$

But, by (9),

$$x+h-c_n < \big(f(c_n)-f(x)\big)\,|m_n|^{-1}.$$

Consequently, $f(x+h) > f(x)$. Finally, the relative measure of

$$\Big(c_n,\, c_n+\big(f(c_n)-f(x)\big)\,|m_n|^{-1}\Big)$$

in the interval $\Big(x,\, c_n+\big(f(c_n)-f(x)\big)\,|m_n|^{-1}\Big)$ is given by

$$\Big(1+(c_n-x)\,|m_n|\,\big(f(c_n)-f(x)\big)^{-1}\Big)^{-1},$$

which, by (8), is larger than $(1+\varepsilon_n)^{-1}$.

Thus, since this is true for arbitrarily large n, it follows that $\bar{D}_x\big(f(t) > f(x)\big) \geqslant 1/2$. Thus $x \notin M_0(f)$ and the theorem is proved.

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