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TRIGONOMETRIC INTERPOLATION, III

BY

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1. Introduction. Throughout this paper the function $f(s)$ is real of period 2π, defined for all $s \in (-\infty, \infty)$. As in [1], p. 16-20, we denote by $M(u)$, $N(u)$, and $M_k(u)$, $N_k(u)$ $(k = 0, 1, 2)$ the suitable pairs of non-negative continuous functions complementary in the sense of Young. For the inverse functions the symbols $M^{-1}(v)$, $N^{-1}(v)$ etc. will be used.

Consider the n-th interpolating polynomial of f

$$\frac{a_0^{(n)}}{2} + \sum_{k=1}^{n} (a_k^{(n)} \cos ks + b_k^{(n)} \sin ks)$$

with nodes

$$s_j^{(n)} = \frac{2\pi}{2n+1} j \quad (j = 0, \pm 1, \pm 2, \ldots);$$

write

$$\sigma_{n,\nu}^{a}(x; f) = \frac{a_0^{(n)}}{2} + \sum_{k=1}^{\nu} (A_{\nu-k}^{a}/A_{\nu}^{a})(a_k^{(n)} \cos kx + b_k^{(n)} \sin kx) \quad (0 \leqslant \nu \leqslant n),$$

where

$$A_0^a = 1, \quad A_m^a = \frac{(a+1)(a+2)\ldots(a+m)}{m!} \quad (-1 < a \leqslant 0, m \geqslant 1).$$

Denote by $\omega_n(s)$ the step function which is equal to $2\pi j/(2n+1)$ for $s \in \langle s_{j-1}^{(n)}, s_j^{(n)})$ $(j = 0, \pm 1, \pm 2, \ldots)$. Introducing the integral notation as in § 1 of [5] and putting

$$K_\nu^a(t) = \frac{1}{2} + \sum_{k=1}^{\nu} (A_{\nu-k}^a/A_\nu^a) \cos kt,$$

we have

$$\sigma_{n,\nu}^a(x; f) = \frac{1}{\pi} \int_{x-\pi}^{x+\pi} f(s) K_\nu^a(s-x)\, d\omega_n(s).$$

It is well-known ([8], p. 95) that the kernel of these Cesàro means can be represented in the form

$$K_\nu^a(t) = \frac{\sin\{(\nu+\frac{1}{2}+\frac{1}{2}a)t-\frac{1}{2}\pi a\}}{A_\nu^a(2\sin\frac{1}{2}t)^{a+1}} + \frac{2\vartheta a}{\nu(2\sin\frac{1}{2}t)^2} \qquad (|\vartheta| \leqslant 1)$$

for $0 < t \leqslant \pi$. In particular, $K_\nu^0(t)$ is identical with Dirichlet's kernel.

The main theorems of this paper concern the convergence of $\sigma_{n,\nu}^a(x;f)$, as $n \geqslant \nu \to \infty$, for functions f belonging to the classes L and V_M^* which are defined below.

The class L consists of all functions $f(s)$ which satisfy the following conditions:

1° $f(s)$ has in $\langle -\pi, \pi\rangle$ at most a finite number of infinite discontinuity points (x is an infinite discontinuity point for f if, for every neighbourhood U of x, f is unbounded in U), all of the form $x_k = 2\pi r_k$ with r_k rational,

2° $f(s)$ is Riemann-integrable in every closed interval containing no x_k,

3° $f(s)$ has a 2π-periodic majorant $f^*(s)$ $\big(|f| \leqslant f^*(s)\big)$, non-decreasing in some left neighnourhood and non-increasing in some right neighbourhood of every x_k, Riemann-integrable in $\langle -\pi, \pi\rangle$ in the improper sense. Without loss of generality, we shall further suppose that the sets of points of infinite discontinuity for $f(s)$ and $f^*(s)$ are identical.

The following fact is principal. Given any $f \in L$ and an arbitrary $\varepsilon > 0$, there is a positive σ such that

$$\sum_k \int_{x_k-\sigma}^{x_k+\sigma} f^*(s)\,d\omega_n(s) < \varepsilon \qquad (x_k \in \langle 0, 2\pi\rangle)$$

uniformly in $n \geqslant 0$. Indeed, let $f(s)$ be infinitely discontinuous at the point $x_1 = 2\pi r_1$ ($r_1 = p/q$) only. Assuming that

$$s_l^{(n)} < x_1 < s_{l+1}^{(n)} \qquad (0 < p/q < 1),$$

and putting

$$H_{n,l} = \max\{f^*(s_l^{(n)}), f^*(s_{l+1}^{(n)})\},$$

we have

$$\int_{x_1-\sigma}^{x_1+\sigma} f^*(s)\,d\omega_n(s) \leqslant \int_{x_1-\sigma}^{x_1+\sigma} f^*(s)\,ds + (s_{l+1}^{(n)} - s_l^{(n)})H_{n.l}$$

for sufficiently small $\sigma > 0$. But

$$\frac{s_{l+1}^{(n)} - x_1}{x_1 - s_l^{(n)}} = -1 + \frac{q}{p(2n+1)-lq} \leqslant q-1,$$

and

$$\frac{x_1 - s_l^{(n)}}{s_{l+1}^{(n)} - x_1} = -1 + \frac{q}{q(l+1)-p(2n+1)} \leqslant q-1.$$

Consequently,

$$\int_{x_1-\sigma}^{x_1+\sigma} f^*(s)\,d\omega_n(s) \leqslant (q+1) \int_{x_1-\sigma}^{x_1+\sigma} f^*(s)\,ds < \varepsilon$$

for some positive σ. The case of $x_1 = s_l^{(n)}$ is trivial. This proof suggests also our general assertion.

Consider now partitions

$$P = \{a = t_0 < t_1 < \ldots < t_i < t_{i+1} < \ldots < t_\mu = b\}$$

of an interval $\langle a, b\rangle$ and the sequences $U = (u_0, u_1, \ldots, u_{\mu-1})$ of non-negative numbers such that

$$\sum_{i=0}^{\mu-1} N(u_i) \leqslant 1.$$

Write

$$V_M(f; a, b) = \sup_P \sum_{i=0}^{\mu-1} M\big(|f(t_{i+1}) - f(t_i)|\big)$$

and

$$V_M^*(f; a, b) = \sup_{P,U} \sum_{i=0}^{\mu-1} |f(t_{i+1}) - f(t_i)|\,u_i.$$

These quantities are called the *first* and the *second M-variation* of f in $\langle a, b\rangle$.

By the inequality of Young ([8], p. 16),

$$V_M^*(f; a, b) \leqslant V_M(f; a, b) + 1. \tag{1}$$

Also, it may be shown (see [8], p. 171) that if $V_M^*(f; a, b) < \infty$, then there is a number $\theta > 0$ such that $V_M(\theta f; a, b) < \infty$. Some other properties of M-variations can be found in [2] and [4]. The class of all f's for which $V_M^*(f; 0, 2\pi) < \infty$ will be denoted by V_M^*.

Given two ordered sets

$$A = (a_1, a_2, \ldots, a_n), \qquad B = (b_1, b_2, \ldots, b_n)$$

of n real numbers, consider the sequences

$$A' = (a_1', a_2', \ldots, a_\nu'), \qquad B' = (b_1', b_2', \ldots, b_\nu') \qquad (\nu \leqslant n)$$

in which

$$a_k' = \sum_{j=n_k+1}^{n_{k+1}} a_j \quad \text{and} \quad b_k' = \sum_{j=n_k+1}^{n_{k+1}} b_j \quad (k = 1, 2, \ldots, \nu),$$

where $0 = n_1 < n_2 < \ldots < n_{\nu+1} = n$. Denote by $\|A'\|_{M_1}$ the upper bound of the sums

$$\sum_{k=1}^{\nu} |a_k'|\,u_k$$

taken over all non-negative sequences $U = (u_1, u_2, \ldots, u_\nu)$ such that

$$\sum_{k=1}^{\nu} N_1(u_k) \leqslant 1.$$

Write $\Gamma = 1/N_1^{-1}(1)N_2^{-1}(1)$ and

$$S_{M_1,M_2}(A, B) = \max_{A',B'} \{\|A'\|_{M_1} \cdot \|B'\|_{M_2}\}.$$

Applying inequality (3) of [3] and reasoning as in § 5 of [6], we easily get the estimate

$$\left|\sum_{j=1}^{n} \sum_{i=1}^{j} a_i b_j\right| \leqslant \left\{\Gamma + \sum_{\mu=1}^{n-1} M_1^{-1}\left(\frac{1}{\mu}\right) M_2^{-1}\left(\frac{1}{\mu}\right)\right\} S_{M_1,M_2}(A, B) \tag{2}$$

needed in the sequel.

2. Fundamental lemmas. Start with the following

2.1. LEMMA. *If $\delta \in (0, \pi)$ and $f \in L$, then*

$$\lim_{n\geqslant\nu\to\infty} \left(\int_{x-\pi}^{x-\delta} + \int_{x+\delta}^{x+\pi}\right) f(s) K_\nu^0(s-x)\, d\omega_n(s) = 0$$

uniformly in $x \in (-\infty, \infty)$.

Proof. Consider the integral

$$J_{n,\nu}^0(x) = \int_{x+\delta}^{x+\pi} f(s) K_\nu^0(s-x)\, d\omega_n(s)$$

which is 2π-periodic in x. Arguing as in § 2 of [5], we conclude that for any f Riemann-integrable over $\langle -\pi, \pi\rangle$,

$$\lim_{n\geqslant\nu\to\infty} J_{n,\nu}^0(x) = 0 \quad \text{uniformly in } x. \tag{3}$$

Now let $f \in L$, $x \in \langle -\pi, \pi\rangle$. Suppose that the majorant f^* is infinitely discontinuous at the points $x_1, x_2, \ldots, x_r$ belonging to $\langle -\pi, 2\pi\rangle$. Choose an arbitrary $\varepsilon > 0$ and the intervals $I_k = (x_k - \sigma, x_k + \sigma)$ such that

$$\sum_{k=1}^{r} \int_{x_k-\sigma}^{x_k+\sigma} f^*(s)\, d\omega_n(s) < \varepsilon \quad (n = 0, 1, 2, \ldots).$$

Write

$$J_{n,\nu}^0(x) = \int_{x+\delta}^{x+\pi} \{f(s) - g(s)\} K_\nu^0(s-x)\, d\omega_n(s) + \int_{x+\delta}^{x+\pi} g(s) K_\nu^0(s-x)\, d\omega_n(s),$$

where

$$g(s) = \begin{cases} 0 & \text{for } s \in I_k\ (k = 1, 2, \ldots, r), \\ f(s) & \text{for remaining } s \in \langle -\pi, 2\pi\rangle. \end{cases}$$

The first integral does not exceed (in absolute value)

$$\frac{1}{2\sin\frac{1}{2}\delta}\sum_{k=1}^{r}\int_{x_k-\sigma}^{x_k+\sigma} f^*(s)\,d\omega_n(s) < \frac{\varepsilon}{2\sin\frac{1}{2}\delta}.$$

Since $g(s)$ is Riemann-integrable, the second integral tends uniformly to zero. Thus the relation (3) holds for any f of class L.

Evidently, the integral

$$\bar{J}^0_{n,\nu}(x) = \int_{x-\pi}^{x-\delta} f(s)K^0_\nu(x-s)\,d\omega_n(s)$$

behaves analogously. Hence the result follows.

2.2. LEMMA. *If $\delta\in(0,\pi)$ and $f\in V^*_M$, where*

$$M(u) = u^{-1/a}\left(\log\frac{1}{u}\right)^\beta \qquad (-1<a<0,\ \beta>-1/a)$$

for small $u>0$, then

$$\lim_{n\geqslant\nu\to\infty}\left(\int_{x-\pi}^{x-\delta}+\int_{x+\delta}^{x+\pi}\right)f(s)K^a_\nu(s-x)\,d\omega_n(s) = 0$$

uniformly in $x\in(-\infty,\infty)$.

Proof. Confine our attention to

$$J^a_{n,\nu}(x) = \int_{x+\delta}^{x+\pi} f(s)K^a_\nu(s-x)\,d\omega_n(s)\ (-\pi\leqslant x\leqslant\pi).$$

As it is easily seen,

$$\int_{x+\delta}^{x+\pi}\frac{|f(s)|}{\nu\{2\sin\frac{1}{2}(s-x)\}^2}\,d\omega_n(s) \leqslant \frac{\pi^2}{4\nu}\int_{x+\delta}^{x+\pi}\frac{|f(s)|}{(s-x)^2}\,d\omega_n(s)$$

$$\leqslant \frac{\pi^2 H}{4\nu\delta^2}\left(\pi-\delta+\frac{2\pi}{2n+1}\right) \qquad (0<\nu\leqslant n),$$

where $H = \sup_{-\pi\leqslant s\leqslant\pi}|f(s)|<\infty$. Hence

$$J^a_{n,\nu}(x) = \int_{x+\delta}^{x+\pi} f(s)\frac{\sin\{(\nu+\frac{1}{2}+\frac{1}{2}a)(s-x)-\frac{1}{2}\pi a\}}{A^a_\nu\{2\sin\frac{1}{2}(s-x)\}^{a+1}}\,d\omega_n(s)+o(1)$$

uniformly in x. Therefore, setting

$$F_1(s) = f(s)\frac{\cos\{\frac{1}{2}a(s-x)-\frac{1}{2}\pi a\}}{\{2\sin\frac{1}{2}(s-x)\}^{a+1}},\qquad F_2(s) = f(s)\frac{\sin\{\frac{1}{2}a(s-x)-\frac{1}{2}\pi a\}}{\{2\sin\frac{1}{2}(s-x)\}^{a+1}},$$

we obtain

$$J^{a}_{n,\nu}(x) = \frac{1}{A^{a}_{\nu}} \int\limits_{x+\delta}^{x+\pi} F_1(s)\sin(\nu+\tfrac{1}{2})(s-x)\,d\omega_n(s)+$$

$$+\frac{1}{A^{a}_{\nu}} \int\limits_{x+\delta}^{x+\pi} F_2(s)\cos(\nu+\tfrac{1}{2})(s-x)\,d\omega_n(s)+o(1)$$

$$= \frac{1}{A^{a}_{\nu}} J^{a,1}_{n,\nu}(x)+\frac{1}{A^{a}_{\nu}} J^{a,2}_{n,\nu}(x)+o(1)$$

as $n \geqslant \nu \to \infty$.

Suppose that

$$s^{(n)}_{k-1} < x+\delta \leqslant s^{(n)}_{k} < s^{(n)}_{k+1} < \ldots < s^{(n)}_{m} < x+\pi \leqslant s^{(n)}_{m+1}.$$

Then

$$J^{a,1}_{n,\nu}(x) = \frac{2\pi}{2n+1} \sum_{j=k}^{m} F_1(s^{(n)}_{j})\sin(\nu+\tfrac{1}{2})(s^{(n)}_{j}-x)$$

and, by the Abel transformation,

$$J^{a,1}_{n,\nu}(x) = \frac{2\pi}{2n+1} \sum_{j=k}^{m-1} \sum_{l=k}^{j} \{F_1(s^{(n)}_{j})-F_1(s^{(n)}_{j+1})\}\sin(\nu+\tfrac{1}{2})(s^{(n)}_{l}-x)+$$

$$+\frac{2\pi}{2n+1} F_1(s^{(n)}_{m}) \sum_{l=k}^{m} \sin(\nu+\tfrac{1}{2})(s^{(n)}_{l}-x) = R^{a}_{n,\nu}(x)+T^{a}_{n,\nu}(x).$$

Since

$$|F_1(s^{(n)}_{m})| \leqslant \frac{H}{(2\sin\frac{1}{2}\delta)^{a+1}}, \qquad \Big|\sum_{l=k}^{m} \sin(\nu+\tfrac{1}{2})(s^{(n)}_{l}-x)\Big| \leqslant \frac{2n+1}{2\nu+1},$$

we have

$$T^{a}_{n,\nu}(x) = O\left(\frac{1}{\nu}\right) \quad \text{as } n \geqslant \nu \to \infty,$$

uniformly in x.

Further, let

$$p = -1/a, \quad 1/p+1/q = 1, \quad 0 < \gamma < (\beta/p)-1.$$

Write

$$\bar{M}(u) = u^{q} \Big/ \left(\log\frac{c}{u}\right)^{q\gamma} \quad (c = 6\pi e)$$

for $u \in (0, 4\pi\rangle$, and

$$Q = \frac{1}{N^{-1}(1)\bar{N}^{-1}(1)} + \sum_{\mu=1}^{\infty} M^{-1}\left(\frac{1}{\mu}\right)\bar{M}^{-1}\left(\frac{1}{\mu}\right).$$

The last series converges, because

$$M^{-1}(v) \leqslant v^{1/p}\left(\frac{p}{\log\frac{1}{v}}\right)^{\beta/p}, \qquad \overline{M}^{-1}(v) \leqslant v^{1/q}\left(\frac{1}{q}\log\frac{c^q}{v}\right)^{\gamma}$$

if v is small enough. Thus, inequality (2) yields

$$|R^{a}_{n,\nu}(x)| \leqslant Q V^{*}_{M}(F_1; x+\delta, x+\pi)\, V^{*}_{\overline{M}}(G^{x}_{n,\nu}; x+\delta, x+\pi),$$

where

$$G^{x}_{n,\nu}(t) = \int_{-\pi}^{t} \sin\left(\nu+\frac{1}{2}\right)(s-x)\, d\omega_n(s).$$

Denote by $V(g; a, b)$ the ordinary variation of $g(s)$ in $\langle a, b\rangle$, and set

$$\varphi_x(s) = \left\{2\sin\frac{1}{2}(s-x)\right\}^{-a-1}, \qquad \psi_x(s) = \cos\left\{\frac{1}{2}a(s-x)-\frac{1}{2}\pi a\right\}.$$

Then,

$$V(\varphi_x; x+\delta, x+\pi) = \left(2\sin\frac{1}{2}\delta\right)^{-a-1} - 2^{-a-1}$$

and

$$V(\psi_x; x+\delta, x+\pi) \leqslant \frac{1}{2}a(\pi-\delta).$$

Consequently, for all x,

$$\begin{aligned} &V^{*}_{M}(F_1; x+\delta, x+\pi)\\ &\quad\leqslant \frac{V^{*}_{M}(f; x+\delta, x+\pi)}{(2\sin\frac{1}{2}\delta)^{a+1}} + HN^{-1}(1)\, V(\varphi_x\psi_x; x+\delta, x+\pi)\\ &\quad\leqslant (2\sin\tfrac{1}{2}\delta)^{-a-1}\{2V^{*}_{M}(f; 0, 2\pi) + HN^{-1}(1)(1+\tfrac{1}{2}a\pi)\}. \end{aligned}$$

To evaluate the $\overline{M}$-variation of $G^{x}_{n,\nu}(t)$, let us choose

$$z_\lambda = x + \frac{2\pi}{2\nu+1}\lambda \ (\lambda = 0, 1, 2, \ldots, \nu), \qquad z_{\nu+1} = x+\pi.$$

Taking non-negative u_λ such that

$$\sum_{\lambda=0}^{\nu} \overline{N}(u_\lambda) \leqslant 1,$$

we have

$$\begin{aligned} \sum_{\lambda=0}^{\nu}\left|\int_{z_\lambda}^{z_{\lambda+1}} \sin\left(\nu+\frac{1}{2}\right)(s-x)\, d\omega_n(s)\right| u_\lambda &\leqslant \frac{2\pi}{2\nu+1}\sum_{\lambda=0}^{\nu} u_\lambda\\ &\leqslant \frac{2\pi}{2\nu+1}(\nu+1)\overline{N}^{-1}\left(\frac{1}{\nu+1}\right) \leqslant C\frac{(\nu+1)^{a}}{\{\log(\nu+1)\}^{\gamma}}, \qquad C = \text{const} \end{aligned}$$

(see [3], p. 453 and [1], p. 25). Analogously, for any subsequence $\{z_{\lambda_j}\}_{\varrho}^{\tau+1}$ of $\{z_\lambda\}_0^{\nu+1}$ and for all $\tilde{u}_j \geqslant 0$ such that

$$\sum_{j=\varrho}^{\tau} \overline{N}(\tilde{u}_j) \leqslant 1,$$

we have

$$\sum_{j=\varrho}^{\tau} \left| \int_{z_{\lambda_j}}^{z_{\lambda_j+1}} \sin\left(\nu+\frac{1}{2}\right)(s-x)\,d\omega_n(s) \right| \tilde{u}_j \leqslant \frac{2\pi}{2\nu+1} \sum_{j=\varrho}^{\tau} \tilde{u}_j \leqslant C \frac{(\nu+1)^a}{\{\log(\nu+1)\}^\gamma}.$$

Consider now an arbitrary partition

$$x = t_0 < t_1 < \ldots < t_i < t_{i+1} < \ldots < t_r < t_{r+1} = x+\pi$$

together with non-negative w_i satisfying the condition $\sum_{i=0}^{r} \overline{N}(w_i) \leqslant 1$.

Let us write

$$\sum_{i=0}^{r} \left| \int_{t_i}^{t_{i+1}} \sin\left(\nu+\frac{1}{2}\right)(s-x)\,d\omega_n(s) \right| w_i$$
$$= \Big(\sum_i{}' + \sum_i{}''\Big) \left| \int_{t_i}^{t_{i+1}} \sin\left(\nu+\frac{1}{2}\right)(s-x)\,d\omega_n(s) \right| w_i,$$

where $\sum'$ denotes summation over all these i for which $\langle t_i, t_{i+1})$ contain no z_λ $(\lambda = 0, 1, \ldots, r)$. The integrand is of constant sign in $\langle z_\lambda, z_{\lambda+1}\rangle$. Hence

$$\sum_i{}' \left| \int_{t_i}^{t_{i+1}} \sin(\nu+\tfrac{1}{2})(s-x)\,d\omega_n(s) \right| w_i \leqslant \sum_j \left| \int_{z_{\lambda_j}}^{z_{\lambda_j+1}} \sin(\nu+\tfrac{1}{2})(s-x)\,d\omega_n(s) \right| w_{\lambda_j}$$

for some subsequences $\{z_{\lambda_j}\}$ and $\{w_{\lambda_j}\}$. Further,

$$\sum_i{}'' \left| \int_{t_i}^{t_{i+1}} \sin(\nu+\tfrac{1}{2})(s-x)\,d\omega_n(s) \right| w_i \leqslant \frac{2\pi}{2\nu+1} \sum_i{}'' w_i.$$

Therefore, by the above estimates,

$$V_M^*(G_{n,\nu}^x;\, x+\delta,\, x+\pi) \leqslant V_M^*(G_{n,\nu}^x;\, x,\, x+\pi) \leqslant 2C \frac{(\nu+1)^a}{\{\log(\nu+1)\}^\gamma}$$

uniformly in x $(0 \leqslant \nu \leqslant n,\ n = 0, 1, 2, \ldots)$.

Collecting the results, we obtain

$$J_{n,\nu}^{a,1}(x) = O\left(\frac{(\nu+1)^a}{\{\log(\nu+1)\}^\gamma}\right) \quad \text{as } n \geqslant \nu \to \infty,$$

uniformly in x. Similar calculations show that this relation remains true for $J_{n,\nu}^{a,2}(x)$, too. Thus the proof is completed.

Remark 1. In the case $n = O(\nu)$ it may be supposed $\beta > 0$. Indeed, observing that

$$|R_{n,\nu}^{a}(x)| \leqslant \frac{2\pi}{2\nu+1} \sum_{j=k}^{m-1} |F_1(s_j^{(n)}) - F_1(s_{j+1}^{(n)})|$$

and applying the inequality of Hölder's type ([3], p. 453 or [8], p. 175), we get

$$|R_{n,\nu}^{a}(x)| \leqslant \frac{2\pi}{2\nu+1} V_M^*(F_1; x+\delta, x+\pi)\cdot(2n+1)M^{-1}\left(\frac{1}{2n+1}\right);$$

whence

$$R_{n,\nu}^{a}(x) = O\big(\nu^{a}(\log n)^{a\beta}\big) \quad \text{as } n \geqslant \nu \to \infty.$$

The conclusion is now evident.

Remark 2. Also we can prove that the Fourier-Lagrange coefficients $a_k^{(n)}$, $b_k^{(n)}$ of $f \in V_M^*$ are of the order $O\big(k^{a}/(\log k)^{\gamma}\big)$ as $n \geqslant k \to \infty$, where γ is an arbitrary positive number less than $-a\beta - 1$ (cf. 3.3 of [5]).

Finally, we shall give an auxiliary result proved in [4].

2.3. LEMMA. *Let $f(s)$ be continuous at every point of the interval $\langle a, b\rangle$, and let*

$$V_{M_0}^*(f; a-\eta, b+\eta) < \infty$$

for some $\eta > 0$. Suppose that, for any positive integer k and all integers r large enough ($r \geqslant k$),

$$rN_0(u) \leqslant N_1(ru/k)$$

whenever $0 < ru/k \leqslant l$ (l being fixed). Then,

$$\lim_{\delta\to 0} V_{M_1}^*(f; x-\delta, x+\delta) = 0$$

uniformly in $x \in \langle a, b\rangle$.

3. Main results. Now we shall present two theorems similar to that of [6], p. 275, [7], p. 610, and [5], § 3.

3.1. THEOREM. *Suppose that $f \in L$ is continuous at every point x of an interval $\langle a, b\rangle$, $a \leqslant b$, and that*

$$V_{M_0}^*(f; a-\eta, b+\eta) < \infty$$

for some positive η, where

$$M_0(u) = \exp(-u^{-1/\beta_0}) \qquad (\beta_0 > 2)$$

for sufficiently small $u > 0$. Then

$$\lim_{n\geqslant\nu\to\infty} \sigma_{n,\nu}^{0}(x; f) = f(x)$$

uniformly in $x \in \langle a, b\rangle$.

3.2. THEOREM. *Given any* $\alpha \in (-1, 0)$, *let* $M(u)$ *be as in* 2.2 *and let*

$$M_0(u) = u^{-1/\alpha}\left(\log\frac{1}{u}\right)^{\beta_0} \qquad \left(\beta_0 > -\frac{2+\alpha}{\alpha}\right)$$

for small $u > 0$. *Suppose that* $f \in V_M^*$ *is continuous at every* x *of* $\langle a, b\rangle$, $a \leqslant b$, *and that*

$$V_{M_0}^*(f; a-\eta, b+\eta) < \infty$$

for some $\eta > 0$. *Then*

$$\lim_{n \geqslant \nu \to \infty} \sigma_{n,\nu}^{\alpha}(x; f) = f(x)$$

uniformly in $x \in \langle a, b\rangle$.

Simultaneous proof. To fix the ideas, let $-\pi \leqslant a < b \leqslant \pi$. Considering $\alpha \in (-1, 0\rangle$ and $\delta \in (0, \pi)$, we write

$$\sigma_{n,\nu}^{\alpha}(x; f) - f(x) = \frac{1}{\pi}\left(\int_{x-\pi}^{x-\delta} + \int_{x-\delta}^{x+\delta} + \int_{x+\delta}^{x+\pi}\right)\{f(s)-f(x)\}K_\nu^\alpha(s-x)\,d\omega_n(s).$$

In view of 2.1 and 2.2, the first and the third integrals tend to zero uniformly in x. Hence

$$\sigma_{n,\nu}^{\alpha}(x; f) - f(x) = \frac{1}{\pi} I_{n,\nu}^{\alpha}(x, \delta) + o(1) \qquad \text{as } n \geqslant \nu \to \infty,$$

where

$$I_{n,\nu}^{\alpha}(x, \delta) = \int_{x-\delta}^{x+\delta} \{f(s)-f(x)\}K_\nu^\alpha(s-x)\,d\omega_n(s)$$

$$= \frac{2\pi}{2n+1}\sum_{l=k}^{m}\{f(s_l^{(n)})-f(x)\}K_\nu^\alpha(s_l^{(n)}-x)$$

if

$$s_{k-1}^{(n)} < x-\delta \leqslant s_k^{(n)} < s_{k+1}^{(n)} < \ldots < s_m^{(n)} < x+\delta \leqslant s_{m+1}^{(n)}.$$

Next, the Abel transformation gives

$$I_{n,\nu}^{\alpha}(x, \delta) = \frac{2\pi}{2n+1}\sum_{j=k}^{m-1}\sum_{l=k}^{j}\{f(s_j^{(n)})-f(s_{j+1}^{(n)})\}K_\nu^\alpha(s_l^{(n)}-x) +$$

$$+\frac{2\pi}{2n+1}\{f(s_m^{(n)})-f(x)\}\sum_{l=k}^{m}K_\nu^\alpha(s_l^{(n)}-x) = T_1+T_2.$$

Choose, for small $u > 0$, the following pairs of positive convex functions:

$$M_1(u) = \exp\left(\frac{-1}{u^{1/\beta_1}}\right), \qquad M_2(u) = u\Big/\left(\log\frac{c}{u}\right)^{\beta_2} \qquad \text{if } \alpha = 0,$$

where $2 < \beta_1 < \beta_0$, $1 < \beta_2 < \beta_1 - 1$, $c \geqslant 1$, and

$$M_1(u) = u^{-1/a}\left(\log\frac{1}{u}\right)^{\beta_1}, \qquad M_2(u) = u^{1/(1+a)}\Big/\left(\log\frac{c}{u}\right)^{\beta_2} \qquad \text{if } a < 0,$$

assuming that $-(2+a)/a < \beta_1 \leqslant \beta_0$, $1 < \beta_2 < -(1+a\beta_1)/(1+a)$, $c \geqslant 1$.

Clearly, the series

$$\sum M_1^{-1}\left(\frac{1}{\mu}\right) M_2^{-1}\left(\frac{1}{\mu}\right)$$

converges. Writing

$$\Phi_{n,\nu}^{a,x}(t) = \int_{-2\pi}^{t} K_\nu^a(s-x)\,d\omega_n(s)$$

and reasoning as in [6] and [7] (see also [5], § 2), we observe that the first variations

$$V_{M_2}(\Phi_{n,\nu}^{a,x}; x-\delta, x), \qquad V_{M_2}(\Phi_{n,\nu}^{a,x}; x, x+\delta)$$

are bounded uniformly in x, n, ν. By (1) this extends to the second M_2-variations, too.

In view of (2),

$$|T_1| \leqslant \left\{\Gamma + \sum_{\mu=1}^{\infty} M_1^{-1}\left(\frac{1}{\mu}\right) M_2^{-1}\left(\frac{1}{\mu}\right)\right\} V_{M_1}^*(f; x-\delta, x+\delta)\, V_{M_2}^*(\Phi_{n,\nu}^{a,x}; x-\delta, x+\delta).$$

Hence, by continuity of f and 2.3,

$$|T_1| + |T_2| \to 0 \qquad \text{as } \delta \to 0+,$$

uniformly in x, n, ν, and the desired results follow.

Remark. If $n = O(\nu)$, the last theorem is valid for arbitrary $M(u)$ in which $\beta > 0$ (see Remark 1 to 2.2).

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