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# A METRIZATION THEOREM

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If $X$ is a space, then $\mathscr{F}X$ will denote the space of closed subsets of $X$ with the exponential topology, and $C^+(X)$ will denote the space of continuous functions from $X$ into the non-negative reals. In [1], Borges shows that the space $X$ is stratifiable if and only if there is a function $T$ taking $\mathscr{F}X$ into $C^+(X)$ such that (1) if $H \in \mathscr{F}X$, then $H = \{x: [T(H)](x) = 0\}$, and (2) if $K \subset H$ are in $\mathscr{F}X$, then $[T(K)](x) \geqslant [T(H)](x)$ for each $x$.

It is the purpose of this note to prove the following

THEOREM. *The following conditions on a $T_1$-space $X$ are equivalent:*

1. *$X$ is metrizable.*
2. *There is a continuous function $T$ taking $\mathscr{F}X$ into $C^+(X)$ such that*

a. *if $H \in \mathscr{F}X$, then $H = \{x: [T(H)](x) = 0\}$;*

b. *if $H$ is a finite set and $p \in H$, then $[T(\{p\})](x) \geqslant [T(H)](x)$ for all $x$.*

If $\{U_1, U_2, \ldots, U_n\}$ is a finite collection of subsets of $X$, then $\langle U_1, U_2, \ldots, U_n\rangle$ denotes the set to which the member $H$ of $\mathscr{F}X$ belongs if and only if $H$ intersects each $U_i$ and $H \subset \bigcup_{i=1}^{n} U_i$. A set $B$ will be called a *basic open set* in $\mathscr{F}X$ if and only if there is a finite set $\{U_1, U_2, \ldots, U_n\}$ of open subsets of $X$ such that $B = \langle U_1, U_2, \ldots, U_n\rangle$.

If $K$ is a subset of $X$ and $U$ is a subset of $[0, \infty)$, then $(K, U)$ denotes the set $\{f \in C^+(X): f(K) \subset U\}$. The statement that $B$ is a *subbasic open set* in $C^+(X)$ means that there is a compact subset $K$ of $X$ and an open subset $U$ of $[0, \infty)$ such that $B = (K, U)$.

We will make use of the following result:

LEMMA (Wilson [2]). *A $T_3$-space $X$ is metrizable if and only if there is a semi-metric $d$ for $X$ such that, whenever,*

$$\lim_{n\to\infty} d(x_n, p) = \lim_{n\to\infty} d(x_n, y_n) = 0,$$

*then*

$$\lim_{n\to\infty} d(y_n, p) = 0.$$

Proof of the theorem. First, suppose that $X$ admits a metric $d$. For each $H$ in $\mathcal{F}X$, define $T(H)$ by $[T(H)](x) = \text{g.l.b.}\ \{d(x, y): y \in H\}$. It is well known that $T(H)$ is in $C^+(X)$. We must show that $T$ is continuous. To that end, let $(K, U)$ be a basic open set in $C^+(X)$ that contains $T(H)$; then $U$ is an open set in $[0, \infty)$ and $K$ is a compact set in $X$ such that $[T(H)](K) \subset U$. Since $T(H)$ is continuous, there is an $\varepsilon > 0$ such that if $r \in [T(H)](K)$ and if $r'$ is in $[0, \infty)$ such that $|r-r'| < \varepsilon$, then $r'$ is in $U$. Let $W = \{S_\varepsilon(y): y \in H\}^*$. (If $H$ is a set collection, then $H^*$ denotes the union of the members of $H$.) For each $k$ in $K$, let $h(k)$ be a point of $H$ such that $|d(h(k), k) - [T(H)](k)| < \varepsilon/3$. Since $K$ is compact, there is a finite subset $\{k_1, k_2, \ldots, k_n\}$ of $K$ such that $\{S_{\varepsilon/3}(k_i): i = 1, 2, \ldots, n\}$ covers $K$. Then

$$V = \langle W, S_{\varepsilon/3}(h(k_1)), S_{\varepsilon/3}(h(k_2)), \ldots, S_{\varepsilon/3}(h(k_n))\rangle$$

is a basic open set in $\mathcal{F}X$ that contains $H$. Let $H'$ be a member of $\mathcal{F}X$ in $V$ and let $y$ denote a point of $K$. We would like to show that $[T(H')](y)$ is in $U$. Since $H' \subset W$, $[T(H')](y) > [T(H)](y) - \varepsilon$. Let $j$ be an integer such that $y$ is in $S_{\varepsilon/3}(k_j)$ and let $p$ be a point of $H'$ in $S_{\varepsilon/3}(h(k_j))$. Then

$$\begin{aligned} T(H')(y) &\leqslant d(y, p) \leqslant d(y, k_j) + d(k_j, h(k_j)) + d(h(k_j), p) \\ &< \varepsilon/3 + T(H)(y) + \varepsilon/3 + \varepsilon/3. \end{aligned}$$

Now, to show that condition (2) implies that $X$ is metrizable, let $d$ be the function taking $X \times X$ into $[0, \infty)$ defined by

$$d(x, y) = [T(\{x\})](y) + [T(\{y\})](x).$$

Claim 1. $d$ is a semi-metric on $X$.

To establish this, first, let $x$ be a limit point of the set $H$ and let $\varepsilon > 0$. Since $T(\{x\})$ is continuous, there is an open set $U$ containing $x$ such that if $y \in U$, then $[T(\{x\})](y) < \varepsilon/2$. Since $T$ is continuous, there is a basic open set $B$ in $\mathcal{F}X$ such that if $H \in B$, then $[T(H)](x) < \varepsilon/2$. Since $\{x\}$ is degenerate, there is an open set $U'$ in $X$ such that $\{x\} \in \langle U'\rangle \subset B$. Thus, if $y \in U'$, then $[T(\{y\})](x) < \varepsilon/2$; and so, if $y \in U \cap U'$, $d(x, y) < \varepsilon/2 + \varepsilon/2$.

Suppose now that $x$ is not a limit point of the set $H$. Let $2\varepsilon = [T(H^-)](x)$. We will show that if $y \in H$, then $[T(\{y\})](x) > \varepsilon$; and so, if $y \in H$, then $d(x, y) > \varepsilon$. To this end, let $\langle U_1, U_2, \ldots, U_n\rangle$ be a basic open set in $\mathcal{F}X$ that contains $H^-$ such that if $K$ is in $\langle U_1, U_2, \ldots, U_n\rangle$, then $|[T(K)](x) - [T(H^-)](x)| < \varepsilon$. Let $\{x_1, x_2, \ldots, x_n\}$ be a finite subset of $X$ such that $x_i \in U_i$ for each $i$. Then, if $y$ is a point of $H$, it must be true that $\{y, x_1, x_2, \ldots, x_n\}$ is in $\langle U_1, U_2, \ldots, U_n\rangle$. Hence

$$[T(\{y\})](x) \geqslant [T(\{y, x_1, x_2, \ldots, x_n\})](x) > \varepsilon.$$

This establishes the truth of claim 1.

Claim 2. $d$ satisfies the conditions of our Lemma.

Suppose that $\{x_i\}$ and $\{y_i\}$ are sequences of points of $X$, and $x$ is a point of $X$ such that

$$\lim_{i\to\infty} d(x_i, y_i) = \lim_{i\to\infty} d(x, x_i) = 0.$$

We wish to show that $\{y_i\}$ converges to $x$. To see that this is so, suppose otherwise; i.e., suppose that there are an $\varepsilon > 0$ and an infinite subset $\mathcal{N}$ of integers such that $d(x, y_j) > \varepsilon$ for each $j \in \mathcal{N}$. Let $H = \mathrm{cl}\{y_j\colon j \in \mathcal{N}\}$. Let $M$ denote an integer such that if $j \in \mathcal{N}$ and $j > M$, then $x_j$ is not in $H$. Then, since $K = \{x\} \cup \{x_j\colon j \in \mathcal{N}, j > M\}$ is compact, there is a $\delta > 0$ such that if $p \in K$, then $[T(H)](p) > \delta$. Let $\varepsilon = \delta/2$. Then there is a basic open set $\langle U_1, U_2, \ldots, U_n\rangle$ in $\mathcal{F}X$ containing $H$ such that if $H' \in \langle U_1, U_2, \ldots, U_n\rangle$ and if $y \in K$, then

$$|[T(H)](y) - [T(H')](y)| < \varepsilon.$$

For each $i \leqslant n$, let $z_i$ be a point of $U_i$ so that if $j \in \mathcal{N}$, then

$$\{y_j, z_1, z_2, \ldots, z_n\} \subset \langle U_1, U_2, \ldots, U_n\rangle.$$

Thus, for each $j \in \mathcal{N}$ such that $j > M$,

$$\begin{aligned} d(x_j, y_j) &\geqslant [T(\{y_j\})](x_j) \geqslant [T(\{y_j, z_1, z_2, \ldots, z_n\})](x_j) \\ &> [T(H)](x_j) - \delta/2 \end{aligned}$$

which contradicts the assumption that $\lim_{n\to\infty} d(x_n, y_n) = 0$.

## *REFERENCES*

[1] C. J. R. Borges, *On stratifiable spaces*, Pacific Journal of Mathematics 17 (1966), p. 1-16.

[2] W. A. Wilson, *On semi-metric spaces*, American Journal of Mathematics 53 (1931), p. 361-373.

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