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On a mixed boundary values problem for Lavrent'ev type equations

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Abstract. The main purpose of this note is to study the Riemann–Hilbert and Hilbert type boundary value problem for the general non-linear system of two first-order real equations

$$\begin{aligned}&\varphi(x, y, u, v, u_x, u_y, v_x, v_y) = 0,\\ &\psi(x, y, u, v, u_x, u_y, v_x, v_y) = 0\end{aligned} \tag{1.1}$$

for the unknown functions $u(x, y)$ and $v(x, y)$ of two independent variables x and y in the multiply connected domains.

1. Preliminaries. Let $L' = L_0+L_1+\ldots+L_m$ be the boundary contours of an $m+1$-connected Liapounoff region D(¹) and let L'' be another system of finite non-intersecting oriented contours in the domain D. Suppose that L' and L'' have no common points $(L'\cap L'') = \emptyset$; then the system L'' decomposes the domain D into a finite number of connected substes. The union of all these regions of the domain D will briefly be called the domain G. In particular, if L'' is empty, G is a multiply connected domain.

We shall consider a class of equations (1.1) which can be written in the following complex form:

$$\begin{aligned} w_{\bar z} &= q_1(z, w, w_z)w_z+q_2(z, w, w_z)\bar w_{\bar z}+A(z)w+B(z)\bar w+F(z)\\ &= q(z, w, w_z)w_z+A(z)w+B(z)\bar w+F(z)\\ &= h(z, w, w_z)+Aw+B\bar w+F\end{aligned} \tag{1.2}$$

(see [1], [2], [6]), $z = x+iy$, $w = w(z) = u(x, y)+iv(x, y)$, $w_{\bar z} = \frac{1}{2}(w_x+iw_y)$, $w_z = \frac{1}{2}(w_x-iw_y)$. Moreover, we assume that the function $h(\xi) = h(z, w, \xi)$ satisfies the Lipschitz condition

$$\begin{aligned}&|h(z, w, \xi_1)-h(z, w, \xi_2)| \leqslant q_0|\xi_1-\xi_2|\\ &|q(z, w, \xi)| \leqslant q_0 < 1\end{aligned} \qquad (q_0 < 1) \tag{1.3}$$

(see also [3], [8]). Equation (1.2) contains the complex form of the non-linear

(¹) L_0 contains all contours L_j $(1 \leqslant j \leqslant m)$.

systems of equations, elliptic in the sense of Lavrent'ev. The functions A, B and F belong to the class $L_p(G)$, $p > 2$. In addition, the solution w will be sought in the class of sectionally continuous functions in G, which have continuous extensions up to the boundary and belong to the class W_p^1, $p > 2$.

This class of equations (1.2) contains the general complex form

$$w_{\bar{z}} - q_1(z) w_z - q_2(z) \bar{w}_{\bar{z}} + A(z) w + B(z) \bar{w} = F(z) \tag{1.4}$$

of the systems of linear elliptic equations with generalized derivatives in the sense of Sobolev or Pompeiu (for instance, see [14]), and also the well-known complex form of the Beltrami system of equations $w_{\bar{z}} = q(z) w_z$ and the complex form of the Cauchy–Riemann system of equations $w_{\bar{z}} = 0$.

In this paper we describe some properties of the elliptic non-linear equation (1.2) in the domain G, for the unknown function w satisfying the boundary conditions

$$\operatorname{Re}[a(t) w(t)] = \gamma(t) \quad (t \in L'), \tag{1.5}$$

$$w^+(t) = g(t) w^-(t) + h(t) \quad (t \in L''). \tag{1.6}$$

The symbols w^+ and w^- are understood in the usual sense of the theory of the Hilbert boundary values problem; a, γ and g, h are given functions on L' and L'', respectively.

If $q = 0$ in (1.2), the boundary values problem (1.2)–(1.5) or, in a more general case, (1.4)–(1.5), was studied by L. Bers, B. Bojarski, T. Iwaniec, I. N. Vekua, V. S. Vinogradov, W. Wendland, author and many others(2). In the latter case, the boundary values problem (1.2)–(1.5)–(1.6) has been solved (see [9], [10]). When D is a simply-connected domain ($m = 0$) and L'' is empty, Vinogradov [15] solved the boundary condition (1.5) for the general linear case (1.4), by the use of a conformal mapping onto the unit disc.

Making use of complex variable methods, the general case (1.1) was investigated by Bojarski and Iwaniec (see [1]–[3], [7]). By application of the theory of quasi-conformal mapping, a method was elaborated by Iwaniec [6] for the multiply connected cases.

In this work, the effort to fill the gap between the boundary values problem of type (1.5) for the linear and general non-linear case has been continued.

In respect of the data of the boundary values problem (1.2)–(1.5)–(1.6), we shall make the following usual assumptions on the coefficients:

HYPOTHESIS 1. (1) $L' \in C_\alpha^1$, $0 < \alpha \leqslant 1$, $L'' \in C^1$;

(2) $a(t)$, $\gamma(t) \in C_\beta(L)$, $0 < \beta \leqslant 1$ $(t \in L',\ a(t) \neq 0)$;

(3) $g(t)$, $h(t) \in C_\nu(L'')$, $\frac{1}{2} < \nu < 1$ $(t \in L'',\ g \neq 0)$;

(4) $A(z)$, $B(z)$, $F(z) \in L_p$, $p > 2$.

(2) For many special references, see [5], [14].

If $A = B = F = 0$, equation (1.2) is elliptic in the sense of Lavrent'ev ([2], [6]), called by him *strong ellipticity in the geometric sense.*

Notation. Let $n_1 = \frac{1}{2}\pi\Delta_{L'} \arg \overline{a(t)}$ and $n_2 = \frac{1}{2}\pi\Delta_{L''} \arg g(t)$; then the number $n = n_1 + n_2$ will be called the *total winding number* corresponding to the boundary condition (1.5)–(1.6).

2. General transmission and boundary value problems. We state the following

PROPOSITION 1. *Under Hypothesis* 1, *the general boundary values problem* (1.2)–(1.5)–(1.6) *in G is equivalent to a boundary values problem of the type* (1.2)–(1.5) *in D.*

We shall prove the proposition by the use of the following

LEMMA 1. *Let Γ be a finite number of smooth non-intersecting oriented contours $\Gamma_1, \Gamma_2, \ldots, \Gamma_p$ in the finite part of the plane, for instance in the domain D, such that the domain bounded by the contours Γ with respect to the all plane is connected, and let $g(t)$ be a non-vanishing function, continuous in the Hölder sense on Γ. Then there exists* (for example, see [13]) *a function, holomorphic inside and outside the connected domain ($f^+(z), f^-(z)$, respectively), Hölder continuous from left and right on Γ, and vanishing nowhere in the finite part of the plane, including the boundary values $f^+(t), f^-(t)$, $t \in \Gamma$ and satisfying the boundary condition*

$$\frac{f^+(t)}{f^-(t)} = g(t), \qquad t \in \Gamma. \tag{2.1}$$

Now, we prove Proposition 1. For the proof, we shall restrict ourselves to the case where the domain bounded by the contours L'' with respect to the whole plane is connected. This condition was merely chosen for the sake convenience. As a matter of fact, this method permits us to treat the general case of the domain G. We make a substitution of the form

$$w(z) = \frac{W(z)}{f(z)} - \frac{Q(z)}{f(z)}, \tag{2.2}$$

where $f(z)$ is a sectionally holomorphic function inside and outside of the connected domain, vanishing nowhere in the finite part of the plane, satisfying the boundary condition (2.1), $g(t)$ being the coefficient of the boundary condition (1.6), and

$$Q(z) = -\frac{1}{2\pi i} \int\limits_{L''} \frac{f^-(t)h(t)}{t-z}\, dt; \tag{2.3}$$

$h(t)$ is the free term of the boundary condition (1.6). In view of the properties of the Cauchy type integral (2.3) and Hypothesis 1, it can be verified that the boundary values problem (1.2)–(1.5)–(1.6) is reduced to a boundary values problem of type (1.2)–(1.5) in the domain D for the unknown function W. We observe that W is continuous on the closed domain $D+L$ and belongs to the class $W_p^1(D)$, $p > 2$. The index corresponding to this boundary values problem is $n = n_1 + n_2$. Substitutions of the form (2.2) leave the form of equation (1.2) invariant. Furthermore, they do not violate the conditions in Hypothesis 1. By simple computations, we find that the Lipschitz condition (1.3) holds in the domain D. In accordance with the above consideration, Proposition 1 is also valid for the case where the domain bounded by L'' is not connected (in the sense described in Section 1). This completes the proof.

3. As in the linear case, the solutions of the general mixed boundary values problem (1.2)–(1.5)–(1.6) are intimately connected with the solutions of the Riemann–Hilbert type boundary values problem (1.2)–(1.5). Hence, in this section, we shall study the latter problem.

Suppose that the function h in (1.2) does not involve w, in other words, consider the equation

$$w_{\bar{z}} = h(z, w_z) + A(z)w + B(z)\bar{w} + F(z) \quad \text{in } D \tag{3.1}$$

satisfying the boundary condition

$$\operatorname{Re}[w(t)] = c, \tag{3.2}$$

where $c = c_k$ on L_k $(k = 0, 1, \ldots, m)$ are real constants; $c_0 = 0$.

To sum up, we study the boundary values problem (1.2)–(1.5)–(1.6) in the case where the total winding number corresponding to this problem is zero.

PROPOSITION 2. *The boundary values problem* (3.1)–(3.2) *is equivalent to an integral equation which is of some kind of the Fredholm equation with compact operator* (*for this equation the Fredholm alternatives hold*).

As a matter of fact, the unknown constants $c_1, c_2, \ldots, c_m$ *on* L_1, $L_2, \ldots, L_m$, *respectively*, $c_0 = 0$ *on* L_0, *can be determined such that the proposition is dervied.*

Proof. We shall apply a suitable integral representation $w = T(\omega)_{(z)}$, $\omega \in L_p(D)$, $p > 2$, for the solution w of problem (3.1)–(3.2) with the following properties (see [6]):

There exist an operator $T(\omega)_{(z)}$ $(\omega \in L_p,\ p > 2)$ and a constant c such that

(a) $\operatorname{Re}[w(t)] = \operatorname{Re}[T(\omega)_{(t)}] = c$, $t \in L$, where c is the collection of the determined constants introduced in (3.2).

(b) $\partial w/\partial\bar{z} = \partial T(\omega)_{(z)}/\partial\bar{z} = \omega$.

(c) Writing $S(\omega) = \partial w/\partial z$, we have

$$\|S(\omega)\|_{L_2(D)} = \|\omega\|_{L_2(D)}$$

and $\lim_{p\to 2} \Lambda_p = \Lambda_2 = 1$, Λ_p is the norm of S in L_p, $p > 2$. Then the boundary values problem (3.1)–(3.2) is reduced to the following integral equation:

$$\omega = h(z, S(\omega)) + AT(\omega) + \overline{BT(\omega)} + F \tag{3.3}$$

or, in other words,

$$\omega - h(z, S(\omega)) + \dot{T}(\omega) = F(z), \tag{3.4}$$

where $\dot{T}(\omega) = -AT(\omega) - \overline{BT(\omega)}$. According to condition (1.3), we find out that equation (3.3) is reducible to the following integral equation with compact operator

$$\omega - (I-S)^{-1}\dot{T}(\omega) = (I-S)^{-1}F, \tag{3.5}$$

which is a kind of Fredholm equations with compact operator(3).

The results above have been performed by the use of the theory of quasiconformal mapping which was elaborated by Iwaniec [5]. If one uses the properties of Green's function of the Dirichlet problem, we propose rather different method, and one which has been used as well as for the case where the total winding number corresponding to the boundary values problem (1.2)–(1.5)–(1.6) is zero.

Consider equation (3.1) in the multiply-connected domain D, satisfying the boundary condition

$$\mathrm{Re}[w(t)] = 0 \quad \text{on } L; \tag{3.6}$$

L is the boundary of D.

To begin, let us at first state the following:

Hypothesis 2. Suppose $D \in C^3$, the remaining assumptions concerning Hypothesis 1 being preserved.

Proposition 3. *Under Hypothesis 2, the boundary values problem* (3.1)–(3.6) *is equivalent to an integral equation of Fredholm type with compact operator; for instance, Fredholm alternatives hold.*

Proof. If w is a non-trivial solution of the boundary values problem (3.1)–(3.6), then it can be represented in the form

$$w(z) = T_1(\omega) = \frac{1}{\pi}\iint_D \left\{\left[2\frac{\partial g(\xi, z)}{\partial \xi} + \overline{A(\xi, z)}\right]\omega(\xi) - A(\xi, z)\overline{\omega(\xi)}\right\} d\sigma_\xi + ip(z), \tag{3.7}$$

(3) It is an analogue of Fredholm equations.

where $z \in D+L$, $\xi \in D$, and $g(\xi, z)$ is Green's function for the domain D of the Dirichlet problem (see [4]). The function A is introduced as

$$A(\xi, z) = \frac{1}{2\pi i} \int_L \frac{dt}{(\bar{t}-\bar{\xi})(t-z)};$$

L is the boundary of the domain D, ω is a function belonging to the class $L_p(D)$, $p > 2$, and $p(z)$ is a continuous constant function of the point z, in other words, it is constant on each connected component of the boundary.

We observe that w is continuous in the closed domain $D+L$ and belongs to the class W_p^1, $p > 2$. Furthermore, the real part of w defined by (3.7) is equal to zero on L.

The formulae for the generalized derivatives of w (3.7) with respect to $\bar{z}$ and z take the forms

$$\frac{\partial w}{\partial \bar{z}} = \frac{\partial T_1(\omega)_{(z)}}{\partial \bar{z}} = \omega + \frac{1}{\pi} \iint_D \chi(\xi, z)\, \omega(\xi)\, d\sigma_\xi, \tag{3.8}$$

$$\begin{aligned} \frac{\partial w}{\partial z} &= \frac{\partial T_1(\omega)_{(z)}}{\partial z} \\ &= \frac{1}{\pi} \iint_D \left\{ \frac{-\omega(\xi)}{(\xi - z)^2} + M(\xi, z)\, \omega(\xi) - \frac{\partial A(\xi, z)}{\partial z} \overline{\omega(\xi)} \right\} d\sigma_\xi, \end{aligned} \tag{3.9}$$

where

$$M(z, \xi) = 2 \frac{\partial^2 g(z, \xi)}{\partial z \partial \xi} + \frac{1}{(z-\xi)^2},$$

$$\chi(\xi, z) = 2 \frac{\partial^2 g(z, \xi)}{\partial \bar{z} \partial \xi} + \overline{\frac{\partial A(\xi, z)}{\partial z}}.$$

$M(z, \xi)$ is a regular analytic function with respect to z and ξ inside the domain D. Furthermore, in view of Hypothesis 2, it can be verified that the function $M(z, \xi)$ is continuous in both variables in the closed domain $D+L$. Consequently by some simple computation we find out that $\chi(\xi, z)$ is continuous with respect to ξ and z in closed domain $D+L$ (see also [4]). On the other hand, writing

$$\frac{\partial w}{\partial z} = \frac{\partial T_1(\omega)_{(z)}}{\partial z} = \frac{\partial T_1(\omega)}{\partial z} = S_1(\omega) = S_1 \omega, \tag{3.10}$$

we obtain

$$\|S_1(\omega)\|_{L_2(D)} = \|\omega\|_{L_2(D)}, \tag{3.11}$$

and $\lim_{p\to 2} \lambda_p = \lambda_2 = 1$; λ_p is the norm of S_1 in L_p, $p > 2$. This is a consequence of the Riesz–Thorin convexity theorem (see also [14]).

By substitution of representation (3.7) into equation (3.1) and boundary condition (3.6) we find that the following equation holds:

$$\omega = h(z, S_1(\omega)) + AT_1(\omega) + B\overline{T_1(\omega)} - \frac{1}{\pi}\iint_D \chi(\xi, z)\omega(\xi)\, d\sigma_\xi + F, \tag{3.12}$$

or,

$$\omega - h(z, S_1(\omega)) + T_2(\omega) = F, \tag{3.13}$$

where

$$T_2(\omega) = -AT_1(\omega) - B\overline{T_1(\omega)} + \frac{1}{\pi}\iint_D \chi(\xi, z)\omega(\xi)\, d\sigma_\xi .$$

Bearing in mind the above results and taking into account the Lipschitz condition (1.3) ($q_0 < 1$), we observe that equation (3.13) is reducible to the integral equation with compact operator

$$\omega - (I - S_1)^{-1} T_2(\omega) = (I - S_1)^{-1} F, \tag{3.14}$$

which has the fundamental properties of Fredholm type equations.

In fact, according to condition (1.3), since $\lambda_2 = 1$, a number $p > 2$ can be found such that

$$q_0 \lambda_p < 1.$$

For a fixed p the operator $(I - S_1)$ has the inverse $(I - S_1)^{-1}$ in L_p. Consequently, equation (3.13), which is equivalent to the original boundary values problem (3.1)–(3.6), is reducible to a kind of Fredholm equation (3.14).

It is a remarkable though extremely simple fact that the above results can be applied to the non-homogeneous boundary values problem (1.2)–(1.5)–(1.6). We shall give a brief hint at this fact:

Indeed, the non-homogeneous problem (1.2)–(1.5)–(1.6) can be reduced to the corresponding homogeneous (with respect to the boundary) problem. Let the domain bounded by the contours L'' be connected; then the non-homogeneous problem (1.2)–(1.5)–(1.6) is reducible to the corresponding homogeneous problem if the following conditions hold:

$$\int_{L'} a(t)Q(t)\gamma(t)\,dt + i\,\mathrm{Im}\Big(\int_{L''} h(t)Q^+(t)\,dt\Big) = 0, \tag{3.15}$$

where Q is an arbitrary solution of the homogeneous problem, adjoint to the boundary values problem (1.5)–(1.6), with the unknown holomorphic function $\tilde{Q}(z)$ in G.

In view of identity (3.15), the problem can also be established for the general case of the domain G.

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