COLLOQUIUM MATHEMATICUM

VOL. LIV 1987 FASC. 1

VECTOR FIELDS GENERATING CERTAIN SPECIAL TRANSFORMATIONS

BY

ZBIGNIEW OLSZAK (WROCŁAW)

1. Introduction. Let (M, g) be a pseudo-Riemannian manifold with an indefinite metric g.

Suppose X is a vector field on M. Denote by f_X the infinitesimal point transformation given in an (arbitrary) local coordinate system (u^i) $(1 \leqslant i \leqslant n = \dim M)$ of M by

$$(u^i) \mapsto (u^i + \varepsilon X^i(u^1, \ldots, u^n)),$$

where ε is an infinitesimal and $X = (X^i)$ (i.e., X^i are the local components of X).

Recall that X is said to be a *projective collineation* (PC) in (M, g) if any f_X maps geodesics into geodesics without necessarily preserving the geodesic parameter. X is called a *null-geodesic collineation* (NC) in (M, g) if any f_X maps null-geodesics into geodesics preserving the geodesic parameter (cf. [3]). Moreover, if X is a conformal collineation (Conf C) in (M, g), then any f_X maps null-geodesics into geodesics with general change in parameter (cf. [3]).

In the present paper we are concerned with vector fields X for which f_X maps null-geodesics into geodesics of pseudo-Riemannian manifold with general change in parameter. These collineations seem to be new and we shall call them *general null-geodesic collineations* (GNC's). For the definitions of all other motions and collineations used in this paper we refer to [3] and [6]. As we have just seen, any PC, ConfC and NC is a GNC. We shall see that any $\boldsymbol{R}$-linear combination of a PC, an NC and a ConfC is also a GNC. However, we shall prove that the converse statement fails in general.

In Section 2 we find the conditions for a vector field to be a GNC in terms of Lie derivatives of the Christoffel symbols and the metric of the manifold. Section 3 establishes the conditions for a GNC to be a special curvature collineation (SCC) or a curvature collineation (CC). In Section 4 we study GNC's in pseudo-Euclidean spaces. We get, among others, examples of GNC's which are not decomposable into a PC, an NC and a ConfC. Other examples of non-decomposable GNC's are given in Section 6. However, as we shall prove in Section 5, in a pseudo-Riemannian manifold of non-zero constant curvature and of dimension $\geqslant 3$, any GNC decomposes into a PC and a conformal motion (Conf M). Finally, we show in Section 7 that a GNC gives rise to a quadratic first integral of null-geodesics.

2. General null-geodesic collineations. Let (M, g) be an n-dimensional ($n \geqslant 2$) pseudo-Riemannian manifold with an indefinite metric tensor $g = (g_{ij})$.

Definition. Let X be a vector field on M. We call X a *general null-geodesic collineation* (shortly, GNC) in (M, g) if every infinitesimal point transformation f_X maps null-geodesics of (M, g) into geodesics of (M, g) without necessarily preserving the geodesic parameter.

THEOREM 2.1. *A vector field X is a GNC in (M, g) if and only if*

$$L_X \Gamma^i_{jk} = \varphi_j \delta^i_k + \varphi_k \delta^i_j + \psi^i g_{jk}, \tag{2.1}$$

where Γ^i_{jk} are the Christoffel symbols, L_X denotes the Lie differentiation with respect to X, and $\varphi = (\varphi_i)$ and $\psi = (\psi_i)$ are, respectively, certain covariant vector fields on M and $\psi^i = g^{is}\psi_s$.

In the proof of this theorem we need the following algebraic lemma:

LEMMA 2.1. *Let W be a real n-dimensional ($n \geqslant 2$) vector space and let h be an indefinite scalar product in W. If A is a symmetric trilinear form in W such that $A(X, X, X) = 0$ for any isotropic vector $X \in W$ (i.e., $h(X, X) = 0$), then*

$$A(X, Y, Z) = u(X)h(Y, Z) + u(Y)h(Z, X) + u(Z)h(X, Y) \tag{2.2}$$

for any $X, Y, Z \in W$, where u is certain linear form in W.

Proof. Fix $X \in W$ such that $h(X, X) = 1$. Let Y be a vector in W such that $h(Y, Y) < 0$. The equation $h(Y+\lambda X, Y+\lambda X) = 0$ has two solutions λ_1 and λ_2 ($\lambda_1 \neq \lambda_2$) such that $\lambda_1 \lambda_2 = h(Y, Y)$ and $\lambda_1 + \lambda_2 = -2h(X, Y)$. By the assumption on A we have

$$A(Y+\lambda X, Y+\lambda X, Y+\lambda X) = 0 \quad \text{for } \lambda = \lambda_1, \lambda_2,$$

that is,

$$\lambda_i^3 A(X, X, X) + 3\lambda_i^2 A(X, X, Y) + 3\lambda_i A(X, Y, Y) + A(Y, Y, Y) = 0 \tag{2.3}$$

for $i = 1, 2$. Multiplying the first equation of (2.3) by λ_2 and the second one by λ_1, and subtracting the resulting equations we find

$$A(Y, Y, Y) = 3\lambda_1 \lambda_2 A(X, X, Y) + \lambda_1 \lambda_2 (\lambda_1 + \lambda_2) A(X, X, X)$$

or

$$A(Y, Y, Y) = 3h(Y, Y)[A(X, X, Y) - \tfrac{2}{3}h(X, Y)A(X, X, X)]. \tag{2.4}$$

Now, define the linear form u in W by

$$u(Z) = A(X, X, Z) - \tfrac{2}{3}h(X, Z)A(X, X, X), \qquad Z \in W.$$

The relation (2.4) shows that $A(Y, Y, Y) = 3u(Y)h(Y, Y)$ for any $Y \in W$ such that $h(Y, Y) < 0$. This clearly implies (2.2), completing the proof.

Proof of Theorem 2.1. The parameter-independent equation of a geodesic in (M, g) is

$$p^j \frac{Dp^i}{dt} - p^i \frac{Dp^j}{dt} = 0,$$

where $p^i = du^i/dt$, t is an arbitrary parameter of the geodesic and D/dt denotes intrinsic differentiation along the geodesic. Hence X is a GNC if and only if

$$L_X\left(p^j \frac{Dp^i}{dt} - p^i \frac{Dp^j}{dt}\right) = 0 \tag{2.5}$$

for any null-geodesic, i.e., for any geodesic such that $g_{ij} p^i p^j = 0$. By the same method as, e.g., in [3], p. 22, we show that (2.5) takes the form

$$(\delta^j_m L_X \Gamma^i_{kl} - \delta^i_m L_X \Gamma^j_{kl}) p^m p^k p^l = 0 \tag{2.6}$$

for any null-geodesic. But, by Lemma 2.1, from (2.6) we find

$$\begin{aligned}\delta^j_m L_X \Gamma^i_{kl} - \delta^i_m L_X \Gamma^j_{kl} + \delta^j_k L_X \Gamma^i_{lm} - \delta^i_k L_X \Gamma^j_{lm} + \delta^j_l L_X \Gamma^i_{mk}& \\ - \delta^i_l L_X \Gamma^j_{mk} = u^{ji}_m g_{kl} + u^{ji}_k g_{lm} + u^{ji}_l g_{mk}&\end{aligned} \tag{2.7}$$

for some u^{ji}_m $(= -u^{ij}_m)$. Write

$$A_{ijk} = g_{ia} L_X \Gamma^a_{jk} \qquad \text{and} \qquad u_{jim} = g_{ja} g_{ib} u^{ab}_m.$$

Lowering the indices j and i in (2.7), we rewrite it in the form

$$\begin{aligned}g_{jm} A_{ikl} - g_{im} A_{jkl} + g_{jk} A_{ilm} - g_{ik} A_{jlm} + g_{jl} A_{imk} - g_{il} A_{jmk}& \\ = u_{jim} g_{kl} + u_{jik} g_{lm} + u_{jil} g_{mk}.&\end{aligned} \tag{2.8}$$

Note that $A_{ijk} = A_{ikj}$ and $u_{jim} = -u_{ijm}$. Set, moreover, $A_i = g^{ab} A_{aib}$ and $B_i = g^{ab} A_{iab}$. Transvecting (2.8) with g^{jm} we get

$$(n+1) A_{ikl} - g_{ik} A_l - g_{il} A_k = u_{lik} + u_{kil} + g^{ab} u_{aib} g_{kl}. \tag{2.9}$$

Hence, by the transvection with g^{kl}, we have

$$g^{ab} u_{aib} = \frac{n+1}{n+2} B_i - \frac{2}{n+2} A_i. \tag{2.10}$$

Permuting (2.9) cyclically with respect to all indices, adding the resulting equalities, and using (2.10) we find

$$\begin{aligned}A_{ikl} + A_{kli} + A_{lik} = \frac{1}{n+2}[(B_i + 2A_i) g_{kl} + (B_k + 2A_k) g_{li}& \\ + (B_l + 2A_l) g_{ik}].&\end{aligned} \tag{2.11}$$

On the other hand, transvecting (2.8) with g^{kl} we obtain

$$u_{jim} = \frac{1}{n+2}(2A_{imj} - 2A_{jmi} + g_{jm} B_i - g_{im} B_j).$$

This together with (2.10) substituted into (2.9) gives

$$(2.12)\qquad (n-1)(n+4) A_{ikl} + 2(A_{ikl} + A_{kli} + A_{lik}) = [(n+3) B_i - 2A_i] g_{kl} + [(n+2) A_k - B_k] g_{li} + [(n+2) A_l - B_l] g_{ik}.$$

Finally, substituting (2.11) into (2.12) and writing

$$\varphi_i = \frac{nA_i - B_i}{(n-1)(n+2)}, \qquad \psi_i = \frac{(n+1) B_i - 2A_i}{(n-1)(n+2)},$$

we get $A_{ijk} = \psi_i g_{jk} + \varphi_j g_{ki} + \varphi_k g_{ij}$, which gives (2.1). Conversely, one can easily see that (2.1) implies (2.6). This completes the proof.

PROPOSITION 2.1. *A vector field X is a GNC in (M, g) if and only if*

$$(2.13)\qquad a_{ij,k} = (\varphi_i + \psi_i) g_{jk} + (\varphi_j + \psi_j) g_{ik} + 2\varphi_k g_{ij}$$

for certain covariant vector fields φ and ψ, where the comma denotes covariant differentiation with respect to g and $a = L_X g$.

This proposition is an immediate consequence of our Theorem 2.1 and the following relation (cf. [6]):

$$L_X \Gamma^i_{jk} = \tfrac{1}{2} g^{is}(a_{js,k} + a_{ks,j} - a_{jk,s}).$$

COROLLARY 2.1. *In Theorem 2.1 and Proposition 2.1 the covariant vector field $(n+1)\varphi + \psi$ is a gradient.*

Indeed, by (2.13) we have

$$(n+1)\varphi_i + \psi_i = \tfrac{1}{2}(g^{sr} a_{sr})_{,i}.$$

From Theorem 2.1 it is clear that PC's, Conf C's, NC's and any their $\boldsymbol{R}$-linear combinations are GNC's. In the sequel a GNC will be called *essential* if it is not such a combination. Note that if a GNC is not essential, then φ (and therefore ψ) is a gradient. For an essential GNC, φ and ψ may be non-gradients (even local), as we shall see in Section 6. Examples of essential GNC's will be given in Sections 4 and 6.

3. GNC's and CC's. In this section we determine necessary and sufficient conditions for a GNC to be an SCC or a CC. Here and in the sequel, considering a GNC X, we shall denote by φ and ψ the covector fields that it determines as in (2.1).

PROPOSITION 3.1. *A GNC X is an SCC if and only if*

$$\varphi_{i,j} = \psi_{i,j} = 0.$$

Proof. The condition defining an SCC is $(L_X \Gamma^i_{jk})_{,l} = 0$. Thus, by Theorem 2.1, a GNC is an SCC if and only if

$$\varphi_{j,l}\delta^i_k + \varphi_{k,l}\delta^i_j + \psi^i_{,l} g_{jk} = 0,$$

which is equivalent to $\varphi_{i,j} = \psi_{i,j} = 0$. The proof is complete.

PROPOSITION 3.2. *Let X be a GNC in (M, g). If X is a CC and $n \geqslant 3$, then $\varphi_{i,j} = \psi_{i,j} = \lambda g_{ij}$ for a certain scalar function λ on M. The converse is true in any dimension.*

Proof. The condition defining a CC is $L_X R^i_{jkl} = 0$, where R indicates the curvature tensor of the manifold. Since (cf. [6])

$$L_X R^i_{jkl} = (L_X \Gamma^i_{jk})_{,l} - (L_X \Gamma^i_{jl})_{,k},$$

we have for a GNC, by Theorem 2.1,

$$L_X R^i_{jkl} = \varphi_{j,l}\delta^i_k - \varphi_{j,k}\delta^i_l + (\varphi_{k,l} - \varphi_{l,k})\delta^i_j + \psi^i_{,l} g_{jk} - \psi^i_{,k} g_{jl}. \tag{3.1}$$

From this we see that to obtain the assertion we need only to prove that the vanishing of the right-hand side of (3.1) implies $\varphi_{i,j} = \psi_{i,j} = \lambda g_{ij}$. Thus, let us suppose

$$\varphi_{j,l}\delta^i_k - \varphi_{j,k}\delta^i_l + (\varphi_{k,l} - \varphi_{l,k})\delta^i_j + \psi^i_{,l} g_{jk} - \psi^i_{,k} g_{jl} = 0. \tag{3.2}$$

From (3.2), by contraction with respect to i and k, we find

$$n\varphi_{j,l} - \varphi_{l,j} + \psi_{j,l} = \mu g_{jl}, \tag{3.3}$$

where $\mu = \psi^s_{,s}$. Hence, by contraction with g^{jl}, we obtain $\mu = \varphi^s_{,s}$. On the other hand, contracting (3.2) with g^{jk}, we get

$$2\varphi_{j,l} - \varphi_{l,j} + (n-1)\psi_{j,l} = \mu g_{jl}.$$

Subtracting the last equality from (3.3) we have $\psi_{j,l} = \varphi_{j,l}$. This applied to (3.3) gives $(n+1)\varphi_{j,l} - \varphi_{l,j} = \mu g_{jl}$. Therefore

$$\varphi_{j,l} = \frac{\mu}{n} g_{jl}.$$

This completes the proof.

4. GNC's in pseudo-Euclidean spaces. First we recall that for any vector field X in any pseudo-Riemannian manifold we have (cf. [6])

$$L_X \Gamma^i_{jk} = X^i_{,jk} + X^r R^i_{jkr}. \tag{4.1}$$

Let (E^n, g) be a pseudo-Euclidean space of dimension $n \geqslant 2$ with the Cartesian coordinates $(u^1, \ldots, u^n)$. In this case $\Gamma^i_{jk} = 0$ and $R^i_{jkl} = 0$. Thus, by (4.1) and Theorem 2.1, we see that a vector field X is a GNC in (E^n, g) if and only if it satisfies the system of differential equations

$$\partial_k \partial_j X^i = \varphi_j \delta^i_k + \varphi_k \delta^i_j + \psi^i g_{jk}, \tag{4.2}$$

where we have put $\partial_i = \partial/\partial u^i$.

THEOREM 4.1. *For $n \geq 3$, take Cartesian coordinates $(u^1, \ldots, u^n)$ in E^n so that*

$$ds^2 = \sum_a e_a (du^a)^2,$$

where $e_a = \pm 1$ $(a = 1, \ldots, n)$, is the fundamental form of (E^n, g). Then a vector field X is a GNC in (E^n, g) if and only if it is of the form

$$X^i = \tfrac{1}{2}(Bu^i + D_i e_i) \sum_a e_a (u^a)^2 + \sum_a (A^i_a + C_a u^i) u^a + B^i, \tag{4.3}$$

where B, B^i, C_i, D_i and A^i_j $(i, j = 1, \ldots, n)$ are arbitrary constants. Moreover, X is essential if and only if $B \neq 0$.

Proof. Let us suppose that X is a GNC in (E^n, g), $n \geq 3$. Since any vector field on E^n is a CC in (E^n, g), by Proposition 3.2 we must have $\partial_j \varphi_i = \partial_j \psi_i = Bg_{ij}$. However, in our case $B = \text{const}$. Indeed, $\partial_j \varphi_i = Bg_{ij}$ implies $(\partial_k B) g_{ji} - (\partial_j B) g_{ki} = 0$, that is, $\partial_k B = 0$. Therefore

$$\varphi_i = Be_i u^i + C_i \quad \text{and} \quad \psi_i = Be_i u^i + D_i$$

(no summation over i), where C_i and D_i are certain constants. Note that, in view of Proposition 3.1, X is an SCC in (E^n, g) if and only if $B = 0$.

Thus, to find all GNC's in (E^n, g), we need to solve the following system of differential equations (cf. (4.2)):

$$\partial_k \partial_j X^i = (Be_j u^j + C_j) \delta^i_k + (Be_k u^k + C_k) \delta^i_j + (Bu^i + e_i D_i) e_j \delta_{jk}$$

(no summation over j and k). Formula (4.3) gives the all solutions of this system. Conversely, one can easily verify that any vector field given by (4.3) is a GNC in (E^n, g).

In the diagram of Katzin and Levine [3] one can see that any PC, ConfC and NC is an SCC in (E^n, g). It is not hard to verify that a GNC in (E^n, g) is not essential if and only if it is an SCC. So, X is not essential if and only if $B = 0$ in (4.3). This completes our proof.

THEOREM 4.2. *Take coordinates (u^1, u^2) in E^2 so that $ds^2 = 2du^1 du^2$ is the fundamental form of (E^2, g). Then a vector field X on E^2 is a GNC in (E^2, g) if and only if it is of the form*

$$X^1 = Hu^2 + Q, \qquad X^2 = Fu^1 + P, \tag{4.4}$$

where H, Q, F, P are functions on E^2 such that H, Q depend only on u^1, and F, P depend only on u^2. Moreover, X is not essential if and only if

$$\partial_1^2 H = 0, \quad \partial_1^3 Q = 0, \quad \partial_2^2 F = 0, \quad \partial_2^3 P = 0. \tag{4.5}$$

Proof. Suppose that X is a GNC in (E^2, g) and $X_i = X^a g_{ia}$. With the help of (4.2) we obtain

$$0 = \partial_l \partial_k \partial_j X_i - \partial_k \partial_l \partial_j X_i = (\partial_l \varphi_j) g_{ki} - (\partial_k \varphi_j) g_{li} + (\partial_l \varphi_k - \partial_k \varphi_l) g_{ji} + (\partial_l \psi_i) g_{jk} - (\partial_k \psi_i) g_{jl}.$$

Hence, by the contraction with g^{ki} and the use of $g^{sr}(\partial_s \psi_r) = \partial_1 \psi_2 + \partial_2 \psi_1$ we find

$$2\partial_l \varphi_j - \partial_j \varphi_l + \partial_l \psi_j - (\partial_1 \psi_2 + \partial_2 \psi_1) g_{lj} = 0.$$

Therefore, we must have

$$\begin{aligned} \varphi_1 &= \tfrac{1}{2}(u^2 \partial_1 h + q), & \varphi_2 &= \tfrac{1}{2}(u^1 \partial_2 f + p), \\ \psi_1 &= f - \tfrac{1}{2}(u^2 \partial_1 h + q), & \psi_2 &= h - \tfrac{1}{2}(u^1 \partial_2 f + p), \end{aligned} \tag{4.6}$$

h, q, f, p being certain functions, where h and q depend only on u^1, and f, p depend only on u^2. Now, solve the following system:

$$\partial_k \partial_j X_i = \varphi_j g_{ki} + \varphi_k g_{ji} + \psi_i g_{jk}.$$

In view of (4.6) we obtain $X_1 = Fu^1 + P$, where F and P are functions depending only on u^2 and such that $\partial_2 F = f$ and $\partial_2^2 P = p$. Consequently, $X^2 = Fu^1 + P$. Similarly, one can find $X^1 (= X_2) = Hu^2 + Q$, where H and Q are functions depending only on u^1 and such that $\partial_1 H = h$, $\partial_1^2 Q = q$. Thus, any GNC in (E^2, g) is of the form (4.4). The converse is easy to verify.

As we know, a GNC X in (E^2, g) is not essential if and only if it is an SCC or, by Proposition 3.1, $\partial_j \varphi_i = \partial_j \psi_i = 0$. But in virtue of (4.6), this holds if and only if equalities (4.5) are satisfied. This completes the proof.

For the description of PC's and Conf C's in (E^n, g) see [1] and [4].

5. GNC's in manifolds of constant curvature. First we prove the following lemma:

LEMMA 5.1. *Suppose we are given a GNC X in a pseudo-Riemannian manifold (M, g), $n \geqslant 3$, so that condition* (2.1) *is satisfied. Then φ and ψ are local gradients if and only if the Ricci tensor fulfils the identity $a_{is} R_j^{\ s} = a_{js} R_i^{\ s}$, where $a_{ij} = L_X g_{ij}$ and $R_i^{\ j} = R_{is} g^{sj}$, $R_{ij} = R^s_{\ ijs}$.*

Proof. By Proposition 2.1, equality (2.13) is fulfilled. The integrability condition of (2.13) is

$$a_{is} R^s_{\ jkl} + a_{js} R^s_{\ ikl} = -(\varphi_{i,l} + \psi_{i,l}) g_{jk} + (\varphi_{i,k} + \psi_{i,k}) g_{jl} - (\varphi_{j,l} + \psi_{j,l}) g_{ik} + (\varphi_{j,k} + \psi_{j,k}) g_{il} - 2(\varphi_{k,l} - \varphi_{l,k}) g_{ij}. \tag{5.1}$$

Contracting (5.1) with g^{jk} we get

$$a_{is} R_l^{\ s} - a_{sr} R^s_{\ il}{}^r = -n(\varphi_{i,l} + \psi_{i,l}) + (\varphi^a_{\ ,a} + \psi^a_{\ ,a}) g_{il} - 2(\varphi_{i,l} - \varphi_{l,i}). \tag{5.2}$$

Hence, by the antisymmetrization, we find

$$a_{is}R_l^{\ s}-a_{ls}R_i^{\ s} = -(n+4)(\varphi_{i,l}-\varphi_{l,i})-n(\psi_{i,l}-\psi_{l,i}).$$

But, by Corollary 2.1, we have $\psi_{i,l}-\psi_{l,i} = -(n+1)(\varphi_{i,l}-\varphi_{l,i})$. Therefore, we obtain

$$a_{is}R_l^{\ s}-a_{ls}R_i^{\ s} = (n^2-4)(\varphi_{i,l}-\varphi_{l,i}),$$

which completes the proof.

Let us now assume that (M, g), $n \geqslant 3$, is a pseudo-Riemannian manifold of non-zero constant curvature, i.e.,

$$R_{ijkl} = \mu(g_{il}g_{jk}-g_{ik}g_{jl}), \qquad \mu = \text{const} \neq 0.$$

Let X be a GNC in (M, g). By Lemma 5.1, φ and ψ are local gradients. Therefore (5.2) gives us

$$\mu a_{il} = \lambda g_{il}-\varphi_{i,l}-\psi_{i,l} \tag{5.3}$$

for some scalar function λ. On the other hand, using (3.1), we have

$$L_X R_{jk} = -(n-1)\varphi_{j,k}-\psi_{j,k}+\psi^s{}_{,s}g_{jk}.$$

In our case this gives

$$(n-1)\mu a_{jk} = -(n-1)\varphi_{j,k}-\psi_{j,k}+\psi^s{}_{,s}g_{jk}, \tag{5.4}$$

which compared with (5.3) yields

$$\text{(a)}\ \mu a_{ij} = (\lambda-\nu)g_{ij}-\varphi_{i,j}, \qquad \text{(b)}\ \psi_{i,j} = \nu g_{ij} \tag{5.5}$$

for a certain scalar function ν.

From the integrability conditions of (5.5) (b) one can find $\mu\psi_j = -\nu_{,j}$. So, defining $\omega = -\nu/\mu$, we have $\psi_i = \omega_{,i}$. Hence (5.5) (b) takes the following form:

$$\psi_{i,j} = -\mu\omega g_{ij}. \tag{5.6}$$

Assuming $a = g^{rs}a_{rs}$ and contracting (5.4) and (5.5) (a) with g^{ij} we get

$$\mu a+\varphi^s{}_{,s} = \psi^s{}_{,s} \qquad \text{and} \qquad \mu a+\varphi^s{}_{,s} = n(\lambda-\nu),$$

respectively. Hence $\lambda-\nu = \psi^s{}_{,s}/n$ and, finally, $\lambda-\nu = -\mu\omega$ because of (5.6). Substitution of the last relation into (5.5) (a), covariant differentiation of the resulting expression, and using (2.13) lead to

$$\varphi_{i,jk} = -\mu[(\varphi_i+\psi_i)g_{jk}+(\varphi_j+\psi_j)g_{ik}+(2\varphi_k+\psi_k)g_{ij}]. \tag{5.7}$$

Now define vector fields X_1 and X_2 by

$$\text{(a)}\ X_1^i = -\frac{1}{2\mu}(\varphi^i+\psi^i), \qquad \text{(b)}\ X_2^i = \frac{1}{\mu}\psi^i. \tag{5.8}$$

With the help of (4.1), (5.6), and (5.7) we can verify that X_1 and X_2 are a PC and a ConfM in (M, g), respectively, and moreover

$$L_X \Gamma^i_{jk} = L_{X_1} \Gamma^i_{jk} + L_{X_2} \Gamma^i_{jk}.$$

Setting $X_0 = X - X_1 - X_2$ we see that X_0 is an affine collineation in (M, g). Actually, X_0 is a motion in (M, g) (cf. [5]).

Thus we have proved the following theorem:

THEOREM 5.1. *Let (M, g), $n \geqslant 3$, be a pseudo-Riemannian manifold of non-zero constant curvature μ. If X is a GNC in (M, g), then we have a decomposition $X = X_0 + X_1 + X_2$, where X_0 is a motion, X_1 is a PC given by* (5.8) (a), *and X_2 is a Conf M given by* (5.8) (b). *So, up to adding a motion X_0, X is determined uniquely by φ and ψ via the relation*

$$X^i = X^i_0 + \frac{1}{2\mu}(\psi^i - \varphi^i).$$

Remark 5.1. Let (M, g) be as in Theorem 5.1. From this theorem we obtain the following:

(a) If X is an NC in (M, g), that is, if we have $\varphi = 0$, then from (5.7) it follows that $\psi = 0$. This means that X is a motion.

(b) If X is a GNC in (M, g) such that $\varphi = \psi$, then X is a motion.

(c) If X is a ConfC in (M, g) (in this case, $\varphi + \psi = 0$), then it must be a ConfM.

6. Two-dimensional examples. Let $\boldsymbol{R}^2$ be the plane with Cartesian coordinates (u^1, u^2). Suppose we have an open and connected subset V of $\boldsymbol{R}^2$ and a non-constant scalar function h defined on V, depending only on u^2 and such that $u^1 + h(u^2) > 0$ for any $(u^1, u^2) \in V$. We take (u^1, u^2) as coordinates on V. All geometric objects considered below will be defined with respect to these coordinates.

Let $g = (g_{ij})$ be a pseudo-Riemannian metric on V given by

$$g_{11} = g_{22} = 0, \quad g_{12} = g_{21} = C[u^1 + h(u^2)]^{-1/2},$$

where C is a non-zero constant. Then the non-zero Christoffel symbols are those related to

$$\Gamma^1_{11} = -\tfrac{1}{2}[u^1 + h(u^2)]^{-1}, \quad \Gamma^2_{22} = -\tfrac{1}{2} h'(u^2)[u^1 + h(u^2)]^{-1}.$$

Consider a covector field (X_i) defined by $X_1 = 4$, $X_2 = 0$ at every point of V. One can verify that for $a_{ij} = L_X g_{ij} = X_{i,j} + X_{j,i}$, where L_X is the Lie derivative with respect to $X = (X^i = g^{is} X_s)$, we have

$$a_{11,2} = -4h'(u^2)[u^1 + h(u^2)]^{-2}, \tag{6.1}$$

and $a_{ij,k} = 0$ in other cases. Define in V covector fields φ and ψ by

$$\varphi_1 = \frac{1}{C} h'(u^2)[u^1 + h(u^2)]^{-3/2}, \quad \varphi_2 = 0, \quad \psi_i = -3\varphi_i, \; i = 1, 2. \tag{6.2}$$

In view of (6.1) and (6.2) it is not hard to see that (a_{ij}) fulfils equality (2.13), that is, X is a GNC in (V, g). Since φ is not a gradient (even local), X is an essential GNC.

7. Quadratic first integrals of null-geodesics. Let (M, g) be a pseudo-Riemannian manifold, and A a symmetric tensor field of type (0, 2) on M. Consider the geodesic of (M, g) written in the parameter-dependent form

$$\frac{Dp^i}{ds} = 0 \tag{7.1}$$

with $p^i = du^i/ds$ and $(u^1, \ldots, u^n)$ as local coordinates in M.

The tensor field A is said to be a *quadratic first integral of geodesics* in (M, g) if $A_{ij} p^i p^j = \mathrm{const}$ along any geodesic (7.1) (cf. [2], pp. 128–129). A necessary and sufficient condition for A to be a quadratic first integral of geodesics in (M, g) is

$$A_{ij,k} + A_{jk,i} + A_{ki,j} = 0.$$

Define a quadratic first integral of null-geodesics in (M, g) as a symmetric tensor field A of type (0, 2) such that $A_{ij} p^i p^j = \mathrm{const}$ along any null-geodesic (7.1). In virtue of our Lemma 2.1, a necessary and sufficient condition for A to be a quadratic first integral of null-geodesics in (M, g) is

$$A_{ij,k} + A_{jk,i} + A_{ki,j} = u_i g_{jk} + u_j g_{ki} + u_k g_{ij}$$

for certain covector field u.

Suppose that X is a GNC in (M, g). Then, by means of Proposition 2.1, for $a = L_X g$ we have

$$a_{ij,k} + a_{jk,i} + a_{ki,j} = 2(2\varphi_i + \psi_i) g_{jk} + 2(2\varphi_j + \psi_j) g_{ki} + 2(2\varphi_k + \psi_k) g_{ij}.$$

This shows that a is a quadratic first integral of null-geodesics in (M, g). If, moreover, $2\varphi + \psi = 0$, then a is a quadratic first integral of geodesics in (M, g).

REFERENCES

[1] L. P. Eisenhart, *Non-Riemannian Geometry*, New York 1927.

[2] — *Riemannian Geometry*, Princeton 1966.

[3] G. H. Katzin and J. Levine, *Applications of Lie derivatives to symmetries, geodesic mappings, and first integrals in Riemannian spaces*, Colloq. Math. 26 (1972), pp. 21–38.

[4] — *Special curvature collineations in a flat space*, Tensor (N. S.) 22 (1971), pp. 64–68.

[5] — and W. R. Davies, *Curvature collineations: a fundamental symmetry property of the space-times of general relativity defined by the vanishing Lie derivative of the Riemann curvature tensor*, J. Math. Phys. 10 (1969), pp. 617–629.

[6] K. Yano, *The Theory of Lie Derivatives and its Applications*, Amsterdam 1957.

Reçu par la Rédaction le 7.6.1983