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NOTES ON ISOMETRIES

BY

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0. Introduction. Let M be an n-dimensional Riemannian manifold of differentiability class C^∞ covered by a system of coordinate neighborhoods $\{U; x^h\}$, where here and in the sequel the indices $h, i, j, \ldots$ run over the range $\{1, 2, \ldots, n\}$. We denote by g_{ji}, ∇_i, $K_{kji}{}^h$ and K_{ji} components of the metric tensor, the operator of covariant differentiation with respect to Christoffel symbols $\{{}_j{}^h{}_i\}$ formed with g_{ji}, components of the curvature tensor, and those of the Ricci tensor, respectively.

An infinitesimal transformation v^h is called an *isometry* if it satisfies

$$\pounds_v g_{ji} = \nabla_j v_i + \nabla_i v_j = 0, \tag{0.1}$$

where $\pounds_v$ denotes the operator of Lie differentiation with respect to v^h and $v_i = g_{ih} v^h$.

The present author proved (cf. [5])

THEOREM A. *In a compact Riemannian manifold, in order for an infinitesimal transformation v^h to be an isometry, it is necessary and sufficient that*

$$g^{ji} \nabla_j \nabla_i v^h + K_i{}^h v^i = 0, \qquad \nabla_i v^i = 0, \tag{0.2}$$

where g^{ji} are contravariant components of the metric tensor and $K_i{}^h = K_{it} g^{th}$.

Since the Lie derivative of Christoffel symbols with respect to v^h is given by (cf. [4] and [6])

$$\pounds_v \{{}_j{}^h{}_i\} = \nabla_j \nabla_i v^h + K_{kji}{}^h v^k, \tag{0.3}$$

the first equation of (0.2) can be written as

$$g^{ji} (\pounds_v \{{}_j{}^h{}_i\}) = 0. \tag{0.4}$$

An infinitesimal transformation v^h satisfying (0.4) is called an *almost isometry* (cf. [1] and [8]).

Since the Lie derivative of the volume element $\sqrt{g}$, g being the determinant formed with g_{ji}, is given by (cf. [6])

$$\pounds_v \sqrt{g} = \sqrt{g}\nabla_i v^i, \tag{0.5}$$

the second equation of (0.2) can be written as

$$\pounds_v \sqrt{g} = 0. \tag{0.6}$$

An infinitesimal transformation v^h satisfying (0.6) is said to be *volume-preserving.*

Consequently, we can restate Theorem A as follows:

THEOREM B. *In a compact Riemannian manifold, in order for an infinitesimal transformation to be an isometry, it is necessary and sufficient that it is an almost isometry and is volume-preserving.*

An infinitesimal transformation v^h is called an *affine transformation* if it satisfies (cf. [4] and [6])

$$\pounds_v \left\{ {h \atop j\ i} \right\} = \nabla_j \nabla_i v^h + K_{kji}{}^h v^k = 0. \tag{0.7}$$

From the general formula (cf. [6])

$$\pounds_v \left\{ {h \atop j\ i} \right\} = \tfrac{1}{2}(\nabla_j \pounds_v g_{it} + \nabla_i \pounds_v g_{jt} - \nabla_t \pounds_v g_{ji}) g^{th}, \tag{0.8}$$

it is easily seen that an infinitesimal isometry is an affine transformation. Conversely, if v^h is an infinitesimal affine transformation, it satisfies (0.7) from which we can deduce (0.4) and (0.6) in a compact case. Consequently, using Theorem B, we have (cf. [5])

THEOREM C. *An infinitesimal affine transformation in a compact Riemannian manifold is an isometry.*

The main purpose of the present paper is to discuss some topics related to the above-mentioned and to obtain some theorems not only on infinitesimal transformations but also on infinitesimal changes of metric and on finite changes of metric in a Riemannian manifold.

1. Infinitesimal isometries. In this section, we prove

THEOREM 1.1. *An infinitesimal transformation v^h in a compact Riemannian manifold such that*

$$U_{jih} = \left(\pounds_v \left\{ {t \atop j\ i} \right\}\right) g_{th} \tag{1.1}$$

is symmetric in all the indices and which is volume-preserving is an isometry.

Proof. Since

$$U_{jih} = \left(\pounds_v \left\{ {t \atop j\ i} \right\}\right) g_{th} = \nabla_j \nabla_i v_h + K_{kjih} v^k \tag{1.2}$$

is symmetric in i and h, we have, by taking the skew-symmetric part of U_{jih} with respect to i and h,

$$\nabla_j(\nabla_i v_h - \nabla_h v_i) + 2K_{kjih}v^k = 0,$$

from which, transvecting with g^{ji},

$$g^{ji}\nabla_j\nabla_i v_h - g^{ji}\nabla_j\nabla_h v_i + 2K_{kh}v^k = 0. \tag{1.3}$$

On the other hand, from the Ricci formula

$$\nabla_j\nabla_h v_i - \nabla_h\nabla_j v_i = -K_{jhi}{}^t v_t,$$

we find, transvecting with g^{ji},

$$g^{ji}\nabla_j\nabla_h v_i = \nabla_h\nabla_i v^i + K_h{}^t v_t,$$

or

$$g^{ji}\nabla_j\nabla_h v_i = K_{kh}v^k, \tag{1.4}$$

since the infinitesimal transformation v^h is volume-preserving, and hence

$$\nabla_i v^i = 0. \tag{1.5}$$

Substituting (1.4) into (1.3), we find

$$g^{ji}\nabla_j\nabla_i v^h + K_k{}^h v^k = 0. \tag{1.6}$$

Thus, according to Theorem A, equations (1.5) and (1.6) show that the infinitesimal transformation v^h is an isometry. Thus the theorem is proved.

2. Infinitesimal changes of metric. Suppose that there is given a one-parameter family $\overset{*}{g}_{ji}(x, t)$ of Riemannian metrics such that

$$\overset{*}{g}_{ji}(x, 0) = g_{ji}(x) \tag{2.1}$$

in M. We call in this case the change of metric $g_{ji} \to \overset{*}{g}_{ji}$ an *infinitesimal change of metric.* We put

$$[\partial \overset{*}{g}_{ji}(x, t)/\partial t]_{t=0} = a_{ji} \tag{2.2}$$

and call a_{ji} the *variation tensor.* If $a_{ji} = 0$, we call the change of metric $g_{ji} \to \overset{*}{g}_{ji}$ an *infinitesimal isometric change* of metric.

Denoting by g^* the determinant formed with $\overset{*}{g}_{ji}$ and computing $[\partial\sqrt{g^*}/\partial t]_{t=0}$, we find

$$[\partial\sqrt{g^*}/\partial t]_{t=0} = \tfrac{1}{2}\sqrt{g}\,a_{ji}g^{ji}. \tag{2.3}$$

If this expression vanishes, we call the change of metric $g_{ji} \to \overset{*}{g}_{ji}$ an *infinitesimal volume-preserving change* of metric.

Denoting by $\left\{{h \atop j\,i}\right\}^*$ the Christoffel symbols formed with g^*_{ji} and computing

$$U_{ji}{}^h = [\partial \left\{{h \atop j\,i}\right\}^* / \partial t]_{t=0},$$

we find

$$U_{ji}{}^h = \tfrac{1}{2}(\nabla_j a_{it} + \nabla_i a_{jt} - \nabla_t a_{ji}) g^{th} \tag{2.4}$$

and, consequently, putting

$$U_{jih} = U_{ji}{}^t g_{th}, \tag{2.5}$$

we have

$$U_{jih} = \tfrac{1}{2}(\nabla_j a_{ih} + \nabla_i a_{jh} - \nabla_h a_{ji}). \tag{2.6}$$

If the tensor $U_{ji}{}^h$ vanishes identically, we say that the infinitesimal change of metric is *affine*. If the tensor $U_{ji}{}^h$ satisfies

$$g^{ji} U_{ji}{}^h = 0, \tag{2.7}$$

we say that the infinitesimal change of metric is *almost isometric.*

Suppose now that an infinitesimal change of metric $g_{ji} \to g^*_{ji}$ is affine. Then, from (2.6), we have

$$\nabla_j a_{ih} + \nabla_i a_{jh} - \nabla_h a_{ji} = 0,$$

from which

$$\nabla_j a_{ih} = 0. \tag{2.8}$$

Conversely, if (2.8) holds, then we have $U_{ji}{}^h = 0$ and, consequently, we obtain

THEOREM 2.1. *In order for an infinitesimal change of metric to be affine it is necessary and sufficient that the covariant derivative of the variation tensor vanishes.*

Suppose next that the Riemannian manifold M is irreducible and an infinitesimal change of metric $g_{ji} \to g^*_{ji}$ in M is affine and volume-preserving. Then we have $\nabla_j a_{ih} = 0$, from which, the Riemannian manifold being irreducible, we have, by a theorem of Schur, $a_{ih} = c g_{ih}$, c being a constant. On the other hand, the change being volume-preserving, we have $a_{ih} g^{ih} = 0$ and, consequently, $nc = 0$, from which $a_{ih} = 0$. Thus the change of metric is isometric. Hence we have

THEOREM 2.2. *An infinitesimal affine and volume-preserving change of metric in an irreducible Riemannian manifold is isometric.*

We suppose now that, for an infinitesimal volume-preserving change of metric, the tensor U_{jih} is symmetric in all the indices j, i and h. Then, from (2.6), we have

$$\nabla_j a_{ih} + \nabla_i a_{jh} - \nabla_h a_{ji} = \nabla_j a_{hi} + \nabla_h a_{ji} - \nabla_i a_{jh},$$

from which

$$\nabla_i a_{jh} = \nabla_h a_{ji}, \tag{2.9}$$

that is to say, $\nabla_j a_{ih}$ is also symmetric in all the indices. Thus (2.4) reduces to

$$U_{ji}{}^h = \tfrac{1}{2}\nabla_j a_i{}^h, \tag{2.10}$$

where $a_i{}^h = a_{it}g^{th}$, from which

$$g^{ji}U_{ji}{}^h = 0, \tag{2.11}$$

since the infinitesimal change of metric is volume-preserving. Thus we have

THEOREM 2.3. *An infinitesimal volume-preserving change of metric for which the tensor U_{jih} is symmetric in all the indices is almost isometric.*

Denoting by $K^*_{kji}{}^h$ the curvature tensor formed with the Christoffel symbols $\{{}_j{}^h{}_i\}^*$, and computing $[\partial K^*_{kji}{}^h/\partial t]_{t=0}$, we find

$$[\partial K^*_{kji}{}^h/\partial t]_{t=0} = \nabla_k U_{ji}{}^h - \nabla_j U_{ki}{}^h. \tag{2.12}$$

Thus, supposing that U_{jih} is symmetric in all the indices and substituting (2.10) into (2.12), we find

$$[\partial K^*_{kji}{}^h/\partial t]_{t=0} = \tfrac{1}{2}(\nabla_k\nabla_j a_i{}^h - \nabla_j\nabla_k a_i{}^h),$$

that is,

$$[\partial K^*_{kji}{}^h/\partial t]_{t=0} = \tfrac{1}{2}(K_{kjt}{}^h a_i{}^t - K_{kji}{}^t a_t{}^h). \tag{2.13}$$

Thus we have

THEOREM 2.4. *If an infinitesimal change of metric such that U_{jih} is symmetric in all the indices preserves the curvature tensor, then the curvature transformation and the transformation $a_i{}^h$ commute.*

3. Finite changes of metric. Let g_{ji} and g^*_{ji} be both Riemannian metrics in an n-dimensional differentiable manifold M. We call the change $g_{ji} \to g^*_{ji}$ a *finite change* of metric. We denote by $\{{}_j{}^h{}_i\}$ and $\{{}_j{}^h{}_i\}^*$ the Christoffel symbols formed with g_{ji} and those formed with g^*_{ji}, respectively.

We put

$$a_{ji} = g^*_{ji} - g_{ji} \tag{3.1}$$

and call a_{ji} the *difference tensor*. If $a_{ji} = 0$, the change is an *isometry*.

We denote by g^* the determinant formed with g^*_{ji}. If

$$\sqrt{g^*} = \sqrt{g}, \tag{3.2}$$

the finite change of metric is said to be *volume-preserving.*

We put

$$U_{ji}{}^h = \left\{{h \atop j\,i}\right\}^* - \left\{{h \atop j\,i}\right\}. \tag{3.3}$$

If the tensor $U_{ji}{}^h$ vanishes identically, then the finite change of metric is said to be *affine.*

For a finite volume-preserving change of metric, we have

$$U_{jt}{}^t = \left\{{t \atop j\,t}\right\}^* - \left\{{t \atop j\,t}\right\} = \partial_j \log\sqrt{g^*} - \partial_j \log\sqrt{g} \qquad (\partial_j = \partial/\partial x^j),$$

that is,

$$U_{jt}{}^t = 0. \tag{3.4}$$

Now, if the finite change of metric is an isometry, then we have $U_{ji}{}^h = 0$. Conversely suppose that we have $U_{ji}{}^h = 0$. Then we have

$$\left\{{h \atop j\,i}\right\}^* = \left\{{h \atop j\,i}\right\}$$

and, consequently,

$$0 = \nabla_j^* g_{ih}^* = \nabla_j g_{ih}^*,$$

where ∇_j^* denotes the operator of covariant differentiation with respect to the Christoffel symbols $\left\{{h \atop j\,i}\right\}^*$. Thus, if (M, g) is irreducible, we have, by a theorem of Schur, $g_{ih}^* = cg_{ih}$, c being a positive constant and, consequently, if the change is moreover volume-preserving, then we must have $g_{ih}^* = g_{ih}$. Thus we have (cf. [2] and [3])

THEOREM 3.1. *If (M, g) is irreducible and a finite change of metric $g_{ji} \to g_{ji}^*$ is volume-preserving and affine, then the change is isometric.*

We denote by

$$K_{kji}^{*\ \ h} = \partial_k \left\{{h \atop j\,i}\right\}^* - \partial_j \left\{{h \atop k\,i}\right\}^* + \left\{{h \atop k\,t}\right\}^* \left\{{t \atop j\,i}\right\}^* - \left\{{h \atop j\,t}\right\}^* \left\{{t \atop k\,i}\right\}^* \tag{3.5}$$

the curvature tensor of g_{ji}^* and by

$$K_{kji}{}^h = \partial_k \left\{{h \atop j\,i}\right\} - \partial_j \left\{{h \atop k\,i}\right\} + \left\{{h \atop k\,t}\right\} \left\{{t \atop j\,i}\right\} - \left\{{h \atop j\,t}\right\} \left\{{t \atop k\,i}\right\} \tag{3.6}$$

that of g_{ji}. Since

$$\left\{{h \atop j\,i}\right\}^* = \left\{{h \atop j\,i}\right\} + U_{ji}{}^h, \tag{3.7}$$

we substitute (3.7) into (3.5) and find

$$K_{kji}^{*\ \ h} = K_{kji}{}^h + \nabla_k U_{ji}{}^h - \nabla_j U_{ki}{}^h + U_{kt}{}^h U_{ji}{}^t - U_{jt}{}^h U_{ki}{}^t, \tag{3.8}$$

from which, by contraction with respect to h and k,

$$K_{ji}^* = K_{ji} + \nabla_t U_{ji}{}^t - \nabla_j U_{ti}{}^t + U_{st}{}^s U_{ji}{}^t - U_{jt}{}^s U_{si}{}^t \tag{3.9}$$

and, by transvection with g^{ji},

$$g^{ji}K_{ji}^* = K + \nabla_t g^{ji} U_{ji}{}^t - g^{ji}\nabla_j U_{ti}{}^t + U_{st}{}^s g^{ji} U_{ji}{}^t - g^{ji} U_{jt}{}^s U_{si}{}^t, \tag{3.10}$$

where K_{ji}^* and K_{ji} are Ricci tensors of g_{ji}^* and g_{ji}, respectively, and K is the scalar curvature of g_{ji}.

Now, for a finite volume-preserving change of metric, equations (3.9) and (3.10) become, respectively,

(3.11) $$K^*_{ji} = K_{ji} + \nabla_t U_{ji}{}^t - U_{jt}{}^s U_{si}{}^t$$

and

(3.12) $$g^{ji}K^*_{ji} = K + \nabla_t g^{ji} U_{ji}{}^t - g^{ji} U_{jt}{}^s U_{si}{}^t .$$

If the tensor U_{jih} is symmetric in all the indices, we have

$$g^{ji} U_{jt}{}^s U_{si}{}^t = U_{jts} U^{jts} \geqslant 0, \qquad \text{where } U^{jts} = U^{jst} = U_{ir}{}^t g^{ij} g^{rs}.$$

Thus, from (3.12), we have

THEOREM 3.2. *If a finite volume-preserving almost isometric change of metric satisfies*

(3.13) $$g^{ji}K^*_{ji} - K \geqslant 0, \qquad U_{jih} = U_{jhi},$$

then the change is affine.

Moreover, if the manifold (M, g) *is irreducible, then the finite change is an isometry.*

In fact, if (3.13) are satisfied, then, from (3.12), we have $U_{jts} U^{jts} = 0$, from which $U_{jih} = 0$, which means that the change is affine. The latter part of the theorem follows from Theorem 3.1.

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