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Absolute Cesàro summability of infinite series

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Abstract. Let σ_n^a denote the n-th Cesàro mean of order a, $a > -1$ of a given series Σa_n. The series Σa_n is said to be summable $|C, a|_k$, $k \geq 1$ if $\Sigma n^{k-1}|\sigma_n^a - \sigma_{n-1}^a|^k < \infty$. In this note the following theorem has been proved.

THEOREM. *A series Σa_n is summable $|C, a+1|_k$, $a > -1$, $k \geq 1$ if and only if Σb_n is summable $|C, a|_k$, where*

$$b_n = \sum_{v=n}^{\infty} \frac{a_v}{v+1} (C, a).$$

It may be remarked that the above theorem is an analogue of a theorem of Hardy and Littlewood (Math. Z. 19 (1924), p. 67-96) for ordinary Cesàro summability and that of Chow (JLMS 14 (1939), p. 101-112) for summability $|C, a|$, $a \geq 0$.

1. Let $\sum a_n$ be a given infinite series with s_n as its nth partial sum and let $\{\sigma_n^a\}$ and $\{t_n^a\}$ denote the nth Cesàro means of order a $(a > -1)$ of the sequences $\{s_n\}$ and $\{na_n\}$ respectively. A series $\sum a_n$ is said to be (C, a)*-summable to* s if $\sigma_n^a \to s$, as $n \to \infty$. It is said to be $|C, a|$*-summable*, if $\sum |\sigma_n^a - \sigma_{n-1}^a| < \infty$, and $|C, a|_k$*-summable*, $k \geq 1$, if $\sum n^{k-1} |\sigma_n^a - \sigma_{n-1}^a|_k < \infty$. By virtue of the well-known identity $t_n^a = n(\sigma_n^a - \sigma_{n-1}^a)$ the last condition can be written as $\sum_1^{\infty} \frac{|t_n^a|^k}{n} < \infty$.

2. The following theorem concerning absolute Cesàro summability of an infinite series was established by Chow [4].

THEOREM A. *In order that the series $\sum a_n$ be $|C, a+1|$-summable $(a \geq 0)$ it is necessary and sufficient that the series $\sum b_n$, where*

$$b_n = \sum_{v=n}^{\infty} \frac{a_v}{v+1} (C, a)$$

should be $|C, a|$-summable.

The above theorem is an analogue of a theorem of Hardy and Littlewood [6] for ordinary Cesàro summability. The object of this note is twofold: firstly, to prove that there exists an analogue of Theorem A for generalized absolute summability, namely $|C, a|_k$-summability; secondly, to extend the range of a from $a \geq 0$ to $a > -1$.

Our theorem is as follows:

THEOREM. *A series $\sum a_n$ is $|C, \alpha+1|_k$-summable $\alpha > -1$, $k \geqslant 1$ if and only if $\sum b_n$ is $|C, \alpha|_k$-summable, where*

$$b_n = \sum_{v=n}^{\infty} \frac{a_v}{v+1} (C, \alpha). \tag{2.1}$$

3. The following lemmas will be required for the proof of our theorem.

LEMMA 1. *A series $\sum a_n$ is $|C, \alpha+1|_k$-summable, $\alpha > -1$ if and only if $\sum \frac{t_n^\alpha}{n}$ is $|C, 1|_k$-summable, $k \geqslant 1$.*

Proof. Suppose that $\sum a_n$ is $|C, \alpha+1|_k$-summable, $\alpha > -1$, $k \geqslant 1$. Then it is easy to see that

$$\sum_{k=0}^{n} t_k^\alpha = \frac{\alpha}{\alpha+1} \sum_{k=0}^{n-1} t_k^{\alpha+1} + \frac{(\alpha+n+1)}{\alpha+1} t_n^{\alpha+1},$$

and therefore

$$\begin{aligned}\sum_1^{\infty} \frac{1}{n} \left| \frac{1}{n} \sum_{v=1}^{n} t_v^\alpha \right|^k &\leqslant C \sum_{n=1}^{\infty} \frac{1}{n^{k+1}} \left| \sum_{v=1}^{n-1} t_v^{\alpha+1} \right|^k + C \sum_{n=1}^{\infty} \frac{1}{n} |t_n^{\alpha+1}|^k \text{ (}^1\text{)} \\ &= O\left(\sum_{n=1}^{\infty} \frac{1}{n^2} \sum_{v=1}^{n} |t_v^{\alpha+1}|^k \right) + O(1) \\ &= O\left(\sum_{v=1}^{\infty} \frac{|t_v^{\alpha+1}|^k}{v} \right) + O(1) = O(1),\end{aligned}$$

by virtue of the hypothesis. Thus $\sum \frac{t_n^\alpha}{n}$ is $|C, 1|_k$-summable. Similarly, we can prove the converse part.

LEMMA 2 [4]. *If $\alpha > -1$ and $a_n = (n+1)(b_n - b_{n+1})$, then*

$$\frac{t_n^{\alpha+1}}{n} = (\alpha+2) \frac{T_n^{\alpha+1}}{n} - (\alpha+1) \frac{T_{n+1}^\alpha}{n+1},$$

where T_n^α is the n-th (C, α) mean of $\{nb_n\}$.

LEMMA 3 ([1], [10]). *Let*

$$\delta_v = \sum_{n=v}^{\infty} \frac{A_{n-v}^\alpha U_n}{A_n^\alpha}, \quad \alpha \geqslant 0,$$

where $\sum |U_n| < \infty$. Then $\delta_n = O(1)$, $\Delta^\alpha \delta_n = O(n^{-\alpha})$ and

$$\sum_1^{\infty} n^\alpha |\Delta^{\alpha+1} \delta_n| < \infty.$$

(1) C is a constant not necessarily the same at each occurrence.

LEMMA 4 [3]. *Let* $k > 0$, λ_n *be non-increasing and positive and* $\varepsilon_n = O(\lambda_n)$. *Then* $\Delta^k \varepsilon_n = O(n^{-k}\lambda_n)$ *if and only if* $\Delta^k\left(\frac{\varepsilon_n}{n}\right) = O(n^{-k-1}\lambda_n)$.

LEMMA 5 [3]. *If* $\lambda > 0$ *and* $\varepsilon_n = O(1)$, *then*

$$\Delta^\lambda \varepsilon_n = (n+\lambda)\Delta^\lambda\left(\frac{\varepsilon_n}{n}\right) - \lambda\Delta^{\lambda-1}\left(\frac{\varepsilon_n}{n}\right).$$

LEMMA 6. *If* $\sum a_n$ *is* $|C, a+1|_k$*-summable,* $a > -1$, $k \geqslant 1$, *then* $\sum \frac{a_n}{n+1}$ *is* (C, a)*-summable.*

For $a \geqslant 0$ this a special case of a more general result due to Mehdi [9]. However, he states and proves his result for integral values of a only. Since a proof of the above lemma for non-integral values of a is slightly more difficult, it seems desirable to include one here. A proof seems to be all the more appropriate in view of the fact that we take $a > -1$ in place of $a \geqslant 0$.

It may be observed that for $k = 1$, $a \geqslant 0$ the following stronger result is known [7]

LEMMA 7. *If* $a \geqslant 0$ *and* $\sum a_n$ *is* $|C, a+1|$*-summable, then* $\sum \frac{a_n}{n+1}$ *is* $|C, a|$*-summable.*

However, for $k > 1$ it is not possible to obtain such a result for the $|C, a+1|_k$-summability. In other words, the statement "If $\sum a_n$ is $|C, a+1|_k$-summable, $a \geqslant 0$, $k > 1$, then $\sum \frac{a_n}{n+1}$ is $|C, a|$-summable" is false, for we know [9] that a necessary condition for the series $\sum a_n \varepsilon_n$ to be $|C, a|$-summable whenever $\sum a_n$ is $|C, a+1|_k$-summable is $\sum n^{k'-1}|\varepsilon_n|^{k'} < \infty$, $\frac{1}{k} + \frac{1}{k'} = 1$. Thus in Lemma 6 the (C, a)-summability cannot be replaced by the $|C, a|$-summability.

Proof. We take $a_0 = x_0 = 0$. Let

$$x_n = \frac{1}{n^{1/k}} \frac{1}{A_n^{a+1}} \sum_{v=1}^{n} A_{n-v}^{a} v a_v, \qquad n \geqslant 1,$$

and

$$y_n = \frac{1}{A_n^a} \sum_{m=1}^{n} A_{n-m}^{a} \frac{a_m}{m+1}.$$

Expressing y_n in terms of x_n we have

$$y_n = \frac{1}{A_n^\alpha}\sum_{v=1}^{n} v^{1/k}A_v^{\alpha+1}x_v \sum_{m=v}^{n}\frac{A_{m-v}^{-\alpha-2}A_{n-m}^{\alpha}}{m(m+1)} = \sum_{v=1}^{\infty} C_{n,v}x_v,$$

where

$$C_{n,v} = \begin{cases} \dfrac{v^{1/k}A_v^{\alpha+1}}{A_n^\alpha}\displaystyle\sum_{m=v}^{n}\frac{A_{m-v}^{-\alpha-2}A_{n-m}^{\alpha}}{m(m+1)}, & 1\leqslant v\leqslant n,\\ 0 & v>n.\end{cases}$$

Now the sufficient conditions for the $\lim\limits_{n\to\infty} y_n$ to exist whenever $\sum|x_n|^k<\infty$, $1<k<\infty$ are [9]:

(3.1) $$\lim_{n\to\infty} C_{n,v} \text{ exists for } v\geqslant 1,$$

(3.2) $$\sum_{n=v}^{\infty} C_{n,v}U_n \text{ converges for all } U_n \text{ such that } \sum|U_n|<\infty,\ v\geqslant 1,$$

(3.3) $$\sum_{v=1}^{\infty}\Big|\sum_{n=v}^{\infty} C_{n,v}U_n\Big|^{k'}<\infty \text{ whenever } \sum|U_n|<\infty.$$

Proof of (3.1). Since $\sum\limits_{m=v}^{\infty}|A_{m-v}^{-\alpha-2}|<\infty$, $v\geqslant 1$, $\alpha>-1$, it follows that $\sum\limits_{m=v}^{\infty}\dfrac{A_{m-v}^{-\alpha-2}}{m(m+1)}$ is (C,α)-summable, $\alpha>-1$, for every $v\geqslant 1$. Thus

$$\lim_{n\to\infty} C_{n,v} = \lim_{n\to\infty}\frac{v^{1/k}A_v^{\alpha+1}}{A_n^\alpha}\sum_{m=v}^{n}\frac{A_{m-v}^{-\alpha-2}A_{n-m}^{\alpha}}{m(m+1)} = v^{1/k}A_v^{\alpha+1}\sum_{m=v}^{\infty}\frac{A_{m-v}^{-\alpha-2}}{m(m+1)}$$
$$= v^{1/k}A_v^{\alpha+1}\Delta^{\alpha+1}\left(\frac{1}{v(v+1)}\right), \qquad v\geqslant 1.$$

Proof of (3.2).

$$\sum_{n=v}^{\infty}|C_{n,v}U_n| \leqslant \sum_{n=v}^{\infty}\frac{v^{1/k}A_v^{\alpha+1}}{A_n^\alpha}|U_n|\sum_{m=v}^{n}\frac{|A_{m-v}^{-\alpha-2}|A_{n-m}^{\alpha}}{m(m+1)}$$
$$\leqslant v^{1/k}A_v^{\alpha+1}\sum_{m=v}^{\infty}\frac{|A_{m-v}^{-\alpha-2}|}{m^2}\sum_{n=m}^{\infty}\frac{A_{n-m}^{\alpha}|U_n|}{A_n^\alpha}.$$

It can easily be seen that $\dfrac{A_{n-m}^{\alpha}}{A_n^\alpha}$ is decreasing for $-1<\alpha<0$. Hence for $\alpha>-1$

$$\sum_{n=v}^{\infty}|C_{n,v}U_n| \leqslant C\,v^{1/k}A_v^{\alpha+1}\sum_{m=1}^{\infty}\frac{1}{m^2A_m^\alpha}\sum_{n=1}^{\infty}|U_n| \;(^2)$$
$$\leqslant Cv^{1/k}A_v^{\alpha+1}, \qquad v\geqslant 1,\ \alpha>-1.$$

Hence (3.2) is satisfied.

(2) When $\alpha\geqslant 0$, the factor $1/A_m^\alpha$ is omitted.

Proof of (3.3). We have

$$\sum_{n=v}^{\infty} C_{n,v} U_n = \sum_{n=v}^{\infty} \frac{v^{1/k} A_v^{a+1} U_n}{A_n^a} \sum_{m=v}^{n} \frac{A_{m-v}^{-a-2} A_{n-m}^a}{m(m+1)}.$$

Case (i). Let $-1 < a < 0$. Then by Abel's transformation

$$\sum_{m=v}^{n} \frac{A_{m-v}^{-a-2} A_{n-m}^a}{m(m+1)} = \frac{1}{(n+1)(n+2)} \sum_{r=v}^{n} A_{r-v}^{-a-2} A_{n-r}^a + \\ + \sum_{m=v}^{n} \Delta\left(\frac{1}{m(m+1)}\right) \sum_{r=v}^{m} A_{r-v}^{-a-2} A_{n-r}^a = I_1 + I_2, \text{ say}.$$

Now

$$I_1 = \frac{1}{(n+1)(n+2)} A_{n-v}^{-1}$$

and

$$I_2 \leqslant C \sum_{m=v}^{n} \frac{1}{m^3} \left| \sum_{r=v}^{m} A_{r-v}^{-a-2} A_{n-r}^a \right| \leqslant C \sum_{m=v}^{n} \frac{1}{m^3} A_{n-v}^a A_{m-v}^{-a-1}$$

(see [2]).

Hence

$$\left| \sum_{n=v}^{\infty} C_{n,v} U_n \right| \leqslant C v^{1/k+a+1} \sum_{n=v}^{\infty} \frac{|U_n|}{A_n^a} \frac{1}{n^2} A_{n-v}^{-1} + \\ + C v^{1/k+a+1} \sum_{n=v}^{\infty} \frac{|U_n|}{A_n^a} \sum_{m=v}^{n} \frac{1}{m^3} A_{n-v}^a A_{m-v}^{-a-1}$$

$$\leqslant C v^{1/k-1} |U_v| + C v^{1/k+a+1} \sum_{m=v}^{\infty} \frac{1}{m^3} A_{m-v}^{-a-1} \sum_{n=m}^{\infty} \frac{A_{n-v}^a |U_n|}{A_n^a}$$

$$= O(v^{-1/k'} |U_v|) + O\left(v^{1/k+a+1} \sum_{m=v}^{\infty} \frac{1}{m^3} A_{m-v}^{-a-1} \sum_{n=m}^{\infty} \frac{A_{n-m}^a |U_n|}{A_n^a}\right)$$

$$= O(v^{-1/k'} |U_v|) + O\left(v^{1/k+a+1} \sum_{m=v}^{\infty} \frac{A_{m-v}^{-a-1}}{m^3 A_m^a}\right)$$

$$= O(v^{-1/k'} |U_v|) + O(v^{1/k-a-2}),$$

so that

$$\sum_{v=1}^{\infty} \left| \sum_{n=v}^{\infty} C_{n,v} U_n \right|^{k'} = O\left(\sum_{v=1}^{\infty} |U_v|^{k'}\right) + O\left(\sum_{v=1}^{\infty} v^{-ak'-k'-1}\right) = O(1),$$

$$-1 < a < 0.$$

Case (ii). $\alpha = 0$. It is easy to see that

$$\sum_{n=v}^{\infty} C_{n,v} U_n = O(v^{1/k-2})$$

and hence the result follows.

Case (iii). $\alpha > 0$. In this case

$$\sum_{n=v}^{\infty} C_{n,v} U_n = v^{1/k} A_v^{\alpha+1} \sum_{m=v}^{\infty} \frac{A_{m-v}^{-\alpha-2}}{m(m+1)} \sum_{n=m}^{\infty} \frac{A_{n-m}^{\alpha} U_n}{A_n^{\alpha}}$$

$$= v^{1/k} A_v^{\alpha+1} \sum_{m=v}^{\infty} \frac{A_{m-v}^{-\alpha-2} \delta_m}{m(m+1)} = v^{1/k} A_v^{\alpha+1} \Delta^{\alpha+1}\left(\frac{\delta_v}{v(v+1)}\right).$$

From Lemmas 3 and 4 we deduce that

$$\Delta^{\alpha}\left(\frac{\delta_n}{n}\right) = O(n^{-\alpha-1}) \quad \text{and} \quad \Delta^{\alpha}\left(\frac{\delta_n}{n(n+1)}\right) = O(n^{-\alpha-2}).$$

From Lemma 5 we have

$$\Delta^{\alpha+1}\left(\frac{\delta_n}{n(n+1)}\right) = O(n^{-2}|\Delta^{\alpha+1}\delta_n|) + O(n^{-\alpha-3})$$

and therefore

$$\sum_{v=1}^{\infty}\Big|\sum_{n=v}^{\infty} C_{n,v} U_n\Big|^{k'} \leqslant C \sum_{v=1}^{\infty} v^{(\alpha+2)k'-1}\left|\Delta^{\alpha+1}\frac{\delta_v}{v(v+1)}\right|^{k'}$$

$$\leqslant C \sum_{v=1}^{\infty} v^{\alpha k'-1} |\Delta^{\alpha+1}\delta_v|^{k'} + C\sum_{v=1}^{\infty} v^{-k'-1}$$

$$= O\Big\{\Big(\sum_{v=1}^{\infty} v^{\alpha-1/k'} |\Delta^{\alpha+1}\delta_v|\Big)^{k'}\Big\} + O(1) = O(1),$$

by virtue of Lemma 3.

This completes the proof of Lemma 7.

LEMMA 8. *If* $\sum a_n$ *is* $|C, 1|_k$*-summable and*

$$d_n = \sum_{v=n}^{\infty} \frac{a_v}{(v+1+\alpha)}, \qquad \alpha > -1,\ k \geqslant 1,$$

then $\sum d_n$ *is* $|C, 0|_k$*-summable.*

For $k = 1$ we get a result of Chow [4].

Proof. By virtue of the hypothesis and Lemma 6, $\sum \frac{a_n}{n+1+\alpha}$ is convergent. Thus d_n exists for every n. Also $\sum a_n$ is $|C, 1|_k$-summable if and only if $\sum \frac{|t_n^1|^k}{n} < \infty$. Using the identity

$$a_n = \frac{1}{n}\{(n+1)t_n^1 - nt_{n-1}^1\},$$

we get

$$\begin{aligned}
d_n = \sum_{v=n}^{\infty} \frac{a_v}{v+1+a} &= \lim_{m\to\infty} \sum_{v=n}^{m} \frac{1}{v(v+1+a)} [v(t_v^1 - t_{v-1}^1) + t_v^1] \\
&= \sum_{v=n}^{\infty} \frac{t_v^1}{(v+1+a)(v+2+a)} + \sum_{v=n}^{\infty} \frac{t_v^1}{v(v+1+a)} - \frac{t_{n-1}^1}{n+1+a}.
\end{aligned}$$

Therefore

$$\begin{aligned}
\sum_{1}^{\infty} n^{k-1}|d_n|^k &= O(1)\sum_{n=1}^{\infty} n^{k-1}\left(\sum_{v=n}^{\infty}\frac{|t_v^1|}{v^2}\right)^k + O(1)\sum_{n=1}^{\infty}\frac{|t_{n-1}^1|^k}{n} \\
&= O(1)\sum_{n=1}^{\infty} n^{k-1}\sum_{v=n}^{\infty}\frac{|t_v^1|^k}{v^2}\left(\sum_{v=n}^{\infty}\frac{1}{v^2}\right)^{k-1} + O(1) \\
&= O(1)\sum_{n=1}^{\infty}\sum_{v=n}^{\infty}\frac{|t_v^1|^k}{v^2} + O(1) \\
&= O(1)\sum_{v=1}^{\infty}\frac{|t_v^1|^k}{v} + O(1) = O(1),
\end{aligned}$$

by virtue of the hypothesis. Thus $\sum d_n$ is $|C, 0|_k$-summable.

4. Proof of the theorem. The method of proof is similar to that of Chow.

Sufficiency. Since $\sum b_n$ is $|C, a|_k$-summable, it follows from a known consistency theorem ([5]) that it is also $|C, a+1|_k$-summable. Using condition (2.1) we deduce that $a_n = (n+1)(b_n - b_{n+1})$ and then, applying Lemma 2, we have

$$\frac{t_n^{a+1}}{n} = (a+2)\frac{T_n^{a+1}}{n} - (a+1)\frac{T_{n+1}^{a}}{n+1}.$$

Therefore

$$\sum \frac{|t_n^{a+1}|^k}{n} = O\left(\sum_{1}^{\infty}\frac{|T_n^{a+1}|^k}{n}\right) + O\left(\sum_{1}^{\infty}\frac{|T_{n+1}^{a}|^k}{n+1}\right) = O(1).$$

Thus $\sum a_n$ is $|C, a+1|_k$-summable.

Necessity. Using the well-known identities ([7]; [8])

$$t_n^{\alpha} = n(\sigma_n^{\alpha} - \sigma_{n-1}^{\alpha}),$$
$$t_n^{\alpha+1} = (\alpha+1)(\sigma_n^{\alpha} - \sigma_n^{\alpha+1}),$$

we have

$$\frac{\alpha T_{n+1}^{\alpha-1}}{n+1} = \frac{n+\alpha+1}{n+1} T_{n+1}^{\alpha} - T_n^{\alpha}, \qquad \alpha > -1,\ \alpha \neq 0. \tag{4.1}$$

By virtue of Lemma 6, $\sum \frac{a_n}{n+1}$ is (C, α)-summable. Hence b_n defined by (2.1) exist for all n and $a_n = (n+1)(b_n - b_{n+1})$. Using Lemma 2 with $(\alpha-1)$ in place of α, we have

$$\frac{t_n^{\alpha}}{n} = (\alpha+1)\frac{T_n^{\alpha}}{n} - \alpha\frac{T_{n+1}^{\alpha-1}}{n+1}, \qquad \alpha > -1,\ \alpha \neq 0. \tag{4.2}$$

Combining (4.1) and (4.2) we get

$$\frac{t_n^{\alpha}}{n} = (n+\alpha+1)\left(\frac{T_n^{\alpha}}{n} - \frac{T_{n+1}^{\alpha}}{n+1}\right) \quad \text{for} \quad \alpha > -1\ (^3). \tag{4.3}$$

Let

$$\gamma_n = \sum_{v=n}^{\infty} \frac{t_v^{\alpha}}{v(v+1+\alpha)}.$$

From Lemma 1, $\sum_1^{\infty} \frac{t_n^{\alpha}}{n}$ is $|C, 1|_k$-summable and applying Lemma 6 we observe that $\sum_1^{\infty} \frac{t_n^{\alpha}}{n(n+1+\alpha)}$ is convergent. Thus γ_n exist for all n and

$$\frac{t_n^{\alpha}}{n} = (n+1+\alpha)(\gamma_n - \gamma_{n+1}). \tag{4.4}$$

Comparing this result with (4.3) we infer that

$$\gamma_n = \frac{T_n^{\alpha}}{n} + d,$$

where d is some constant.

Now $\gamma_n \to 0$. By hypothesis $\{b_n\}$ is (C, α)-summable, $\alpha > -1$, to zero and therefore, by virtue of the consistency theorem it is also $(C, \alpha+1)$-summable to zero. Using the following identity

$$\frac{T_n^{\alpha}}{n} = -\frac{\alpha}{\alpha+n}\frac{1}{A_{n-1}^{\alpha}}\sum_{v=0}^{n-1} A_{n-1-v}^{\alpha} b_v + \frac{1}{A_n^{\alpha}}\sum_{v=0}^{n} A_{n-v}^{\alpha-1} b_v,$$

(3) It can easily be verified that this identity is also true for $\alpha = 0$.

it follows immediately that $\frac{T_n^\alpha}{n} \to 0$. Hence $d = 0$ so that $\gamma_n = \frac{T_n^\alpha}{n}$. Since by Lemma 1, $\sum_1^\infty \frac{t_n^\alpha}{n}$ is $|C, 1|_k$-summable we conclude in view of Lemma 8 that $\sum_1^\infty \gamma_n$ is $|C, 0|_k$-summable, that is to say $\sum_1^\infty n^{-1}|T_n^\alpha|^k < \infty$. Thus $\sum_1^\infty b_n$ is $|C, \alpha|_k$-summable, $\alpha > -1$.

This completes the proof of the theorem.

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