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The homology of tensor products

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CONTENTS

1. Introduction

Let (C, ∂), (C', ∂') be free abelian chain complexes and let G be an abelian group. We study the homology of $C\otimes G\otimes C'$ with the standard tensor product differential in order to investigate the circumstances in which there exists a natural projection

$$\theta_q\colon H_{p+q}\,(C\otimes G\otimes C') \to H_p\big(C, H_q(G\otimes C')\big).$$

In the case where C, C' are singular chain complexes of topological spaces X, Y respectively, $H_*(C\otimes G\otimes C')$ is naturally isomorphic to $H_*(X\times Y, G)$ and the projection θ_q gives rise to a natural map from $H_{p+q}(X\times Y, G)$ to $H_p\big(X, H_q(Y, G)\big)$. The interpretation of the term 'natural' is made clear in the text.

It is proved that when the groups Tor $\big(G, H_{q-1}(C')\big)$ and Tor $\big(H_p(C)\otimes G, H_{q-1}(C')\big)$ vanish, there exists such a θ_q which is described explicitly. A counterexample is given which shows that in general there does not exist a natural θ_q such that $\theta_q\eta = \varrho$, where $\eta\colon H_p(C)\otimes H_q(G\otimes C') \to H_{p+q}(C\otimes G\otimes C')$ and $\varrho\colon H_p(C)\otimes H_q(G\otimes C') \to H_p\big(C, H_q(G\otimes C')\big)$ are the well-known canonical monomorphisms.

However, there always exists a natural homomorphism

$$\mathcal{Q}\colon H_{p+q}(C\otimes G\otimes C') \to \frac{H_p\big(C, H_q(G\otimes C')\big)\oplus H_q\big(H_p(C)\otimes G\otimes C'\big)}{M},$$

where the group M is defined in the text. Then $\mathcal{Q}$ gives rise to a definition of a product of a class of $H_{p+q}(C\otimes G\otimes C')$ with a cohomology class $\{f\}$ of $H^q\big(\mathrm{Hom}\,(C', F)\big)$, where F is an abelian coefficient group, to give an element of $H_p(C, G\otimes F)$. This is done by taking the Kronecker index of $H^q\big(\mathrm{Hom}\,(C', F)\big)$ with the coefficient group $H_q(G\otimes C')$ and with $H_q\big(H_p(C)\otimes G\otimes C'\big)$. The resulting group maps canonically to $H_p(C, G\otimes F)$.

When C and C' are singular chain complexes of spaces X, Y this product pairing yields the product

$$H^q(Y, F)\otimes H_{p+q}(X\times Y, G) \to H_p(X, F\otimes G).$$

Let $X = Y$. Then combining the pairing with $d_*\colon H_{p+q}(X, G) \to H_{p+q}(X\times X, G)$ induced by the diagonal map $d\colon X \to X\times X$, $\big(d(x) = (x, x)\big)$ gives a product $H^q(X, F)\otimes H_{p+q}(X, G) \to H_p(X, F\otimes G)$, which is shown to be the classical cap product. Thus the cap product can be defined entirely in terms of homology, dispensing with discussion on the chain level.

Section 2 introduces systems of filtering subgroups $D^{r,s}$ and $D'^{s,r}$ respectively for $H_n(C\otimes G\otimes C')$, $r+s=n$. Then $D^{r-1,s+1}\subseteq D^{r,s}$ for all $r+s=n$, and there is an isomorphism $D^{r,s}/D^{r-1,s+1}\to H_r\big(C, H_s(G\otimes C')\big)$. The $D'^{s,r}$ are obtained from $D^{r,s}$ by interchanging the roles of C and C'. There are also defined subgroups $\Delta^{r,s}\subseteq D^{r,s}$, $\Delta'^{s,r}\subseteq D'^{s,r}$. In particular, $D'^{s-1,r+1}\subseteq \Delta'^{s,r}$ and there is a monomorphism $\Delta'^{s,r}/D'^{s-1,r+1}\to H_s\big(H_r(C)\otimes G\otimes C'\big)$.

In Section 3 the main theorem is proved, viz. that $H_{p+q}\,(C\otimes G\otimes C')$ is equal to $D^{p,q}+\Delta'^{q,p}$, and hence to $D^{p,q}+D'^{q,p}$, where $+$ is the group addition in $H_{p+q}(C\otimes G\otimes C')$. A full description is given of the cycles of $C\otimes G\otimes C'$.

Section 4 is devoted to establishing the properties of the groups D, D', Δ and Δ'. The main result is the following exact sequence which may be thought of as a restriction of the classical Künneth exact sequence:

$$\sum_{\substack{r\leqslant p\\ r+s=p+q}} H_r(C)\otimes H_s(G\otimes C')\rightarrowtail D^{p,q}\twoheadrightarrow \sum_{\substack{r\leqslant p\\ r+s=p+q}} \operatorname{Tor}\big(H_{r-1}(C), H_s(G\otimes C')\big).$$

There are also exact sequences for $D'^{q,p}$, $\Delta^{p,q}$, $\Delta'^{q,p}$ and the intersection group $D^{p,q}\cap D'^{q,p}$.

The conditions for the existence of the natural projection θ_q are established in Section 5. The main theorem and the homomorphisms $D^{p,q}\to H_p\big(C, H_q(G\otimes C')\big)$, $\Delta'^{q,p}\to H_q\big(H_p(C)\otimes G\otimes C'\big)$ give rise to the homomorphism

$$\mathscr{Q}\colon\ H_{p+q}(C\otimes G\otimes C')\to\frac{H_p\big(C, H_q(G\otimes C')\big)\oplus H_q\big(H_p(C)\otimes G\otimes C'\big)}{M}.$$

The required conditions are deduced from $\mathscr{Q}$. There is a related projection $\mathscr{P}$ obtained by using $D'^{q,p}$ in place of $\Delta'^{q,p}$ in the main theorem. The section ends with the counter-example already mentioned.

Finally, Sections 6 and 7 use the projection $\mathscr{P}$ and the homomorphism $\mathscr{Q}$ to obtain the product pairing of $H^q(C', F)$ and $H_{p+q}(C\otimes G\otimes C')$ to $H_p(C, G\otimes F)$, and it is proved that when $C=C'=C(X)$ this leads to the classical cap product.

Throughout this paper $\sum$ refers to the direct sum $\oplus$, the tensor product is written $\otimes$, and Tor denotes the torsion product. The image of a homomorphism f is written $\operatorname{Im} f$ and its kernel $\operatorname{Ker} f$. Let (C, ∂) be a chain complex. Then Z_n or $Z_n(C)$ denotes the group of n-cycles of C, and B_n or $B_n(C)$ is the subgroup of Z_n consisting of boundaries. The homology class of a cycle z is written $\{z\}$, and when there is no possibility of confusion these brackets are also used to denote cohomology classes.

In diagrams of groups and homomorphisms $\rightarrowtail$ denotes a monomorphism and $\twoheadrightarrow$ an epimorphism.

If A, B are subgroups of C, we denote the subgroup of C consisting of elements common to A and B by $A \frown B$.

2. Definitions and preliminaries

Consider the more general case than that described above, namely the complex $C \otimes C''$, where (C, ∂), (C'', ∂'') are abelian chain complexes not for the moment assumed free. Write $A = C \otimes C''$. Define the differential in A to be the tensor product differential given by $\partial^{\otimes}(x \otimes y) = \partial x \otimes y + (-1)^{\dim x} x \otimes \partial'' y$, $x \in C$, $y \in C''$. The n-th chain group of the complex $(A, \partial^{\otimes})$ is $\sum_{r+s=n} C_r \otimes C''_s$.

We may filter and graduate A by defining $A^{p,q}$, the subgroup of A consisting of elements of filtration $\leqslant p$ and total degree $p+q$, to be $\sum_{r \leqslant p} C_r \otimes C''_s$, $r+s = p+q$. For a fixed p, $(A^p, \partial^{\otimes})$ is a subcomplex of $(A, \partial^{\otimes})$, where A^p has $A^{p,q}$ as its $(p+q)$-th chain group. Then $H_{p+q}(A^p)$ is well-defined.

The inclusion i: $A^p \to A$ is a chain mapping and induces a homology homomorphism i_*: $H_{p+q}(A^p) \to H_{p+q}(A)$. Write $\operatorname{Im} i_* = D^{p,q}$. Then $D^{p,q} \subseteq H_{p+q}(A)$ and clearly $D^{p-1,q+1} \subseteq D^{p,q}$ for all p, q. The groups $D^{p,q}$ form a system of filtering subgroups of $H_{p+q}(A)$.

Now consider the group $C_p \otimes Z''_q + \sum_{r<p} C_r \otimes C''_s$, $r+s = p+q$, which we denote by $\mathscr{A}^{p,q}$. Its image in A is a subgroup of $A^{p,q}$. If either C or C'' is free the injection i: $\mathscr{A}^{p,q} \to A$ is monomorphic. Assume the freedom of at least one of these two complexes. We define a homomorphism from $Z(\mathscr{A}^{p,q})/B(\mathscr{A}^{p,q})$ to $H_p(C, H_q(C''))$, where $Z(\mathscr{A}^{p,q}) = Z_{p+q}(A) \frown A^{p+q}$ and $B(\mathscr{A}^{p,q}) = B_{p+q}(A) \frown A^{p+q}$. Note that when C is free, $Z(\mathscr{A}^{p,q}) = Z_{p+q}(A^p)$ and $Z(\mathscr{A}^{p,q})/B(\mathscr{A}^{p,q}) = D^{p,q}$. However, if only C'' is free, it is only possible to say in general that $Z(\mathscr{A}^{p,q})/B(\mathscr{A}^{p,q}) \subseteq D^{p,q}$. In this case we write $Z(\mathscr{A}^{p,q})/B(\mathscr{A}^{p,q}) = \Delta^{p,q}$. Then clearly $D^{p-1,q+1} \subseteq \Delta^{p,q} \subseteq D^{p,q}$.

An element x of $\mathscr{A}^{p,q}$ is of the form $x = \sum_i c_p^{(i)} \otimes z_q''^{(i)} + \sum_i \sum_{r<p} c_r^{(i)} \otimes c_s''^{(i)}$. If $x \in Z(\mathscr{A}^{p,q})$ the following conditions are satisfied:

$$
(1) \qquad
\begin{aligned}
&\sum_i \partial c_p^{(i)} \otimes z_q''^{(i)} + \varepsilon \sum_i c_{p-1}^{(i)} \otimes \partial'' c_{q+1}''^{(i)} = 0, \\
&\sum_i \partial c_r^{(i)} \otimes c_s''^{(i)} + \varepsilon \sum_i c_{r-1}^{(i)} \otimes \partial'' c_{s+1}''^{(i)} = 0, \qquad \text{all } r < p.
\end{aligned}
$$

Here, and throughout this paper, $c_r^{(i)} \in C_r$, $c_s''^{(i)} \in C''_s$, $z_q''^{(i)} \in Z''_q$, and ε is the group automorphism in C_r defined by $\varepsilon c_r = (-1)^r c_r$.

Define Γ: $\mathscr{A}^{p,q} \to C_p \otimes H_q(C'')$ by $\Gamma(x) = \sum_i c_p^{(i)} \otimes \{z_q''^{(i)}\}$. Let $C \otimes H_q(C'')$ have boundary $\partial \otimes 1$. Then it is a chain complex whose p-th chain group is $C_p \otimes H_q(C'')$.

LEMMA 1.

(a) *If C or C'' is free, Γ induces a homomorphism*

$$\Gamma_*: Z(\mathscr{A}^{p,q})/B(\mathscr{A}^{p,q}) \to H_p(C, H_q(C'')).$$

(b) $\operatorname{Ker} \Gamma_* = D^{p-1,q+1}$.

(c) *If C is free, Γ_* induces an isomorphism*

$$l: D^{p,q}/D^{p-1,q+1} \to H_p(C, H_q(C'')).$$

(d) *If C'' is free, Γ_* induces a monomorphism*

$$\lambda: \Delta^{p,q}/D^{p-1,q+1} \to H_p(C, H_q(C'')).$$

Proof.

(a) Let x be a cycle of $\mathscr{A}^{p,q}$, satisfying relations (1).

Then $\sum_i \partial c_p^{(i)} \otimes z_q''^{(i)}$ lies in the image of $C_{p-1} \otimes B_q''$ in $C_{p-1} \otimes Z_q''$. Hence, by the exactness of the sequence $C_{p-1} \otimes B_q'' \to C_{p-1} \otimes Z_q'' \twoheadrightarrow C_{p-1} \otimes H_q''$ (where $H_q'' = H_q(C'')$) it is clear that $\sum_i \partial c_p^{(i)} \otimes \{z_q''^{(i)}\} = 0$, i.e. $\Gamma(x)$ is a cycle of $C \otimes H_q(C'')$.

Now let x be a boundary in A. Then $\sum_i c_p^{(i)} \otimes z_q''^{(i)} = \sum_j e_p^{(j)} \otimes \partial'' e_{q+1}''^{(j)} + \sum_k \partial e_{p+1}^{(k)} \otimes e_q''^{(k)}$ for some $e_r \in C_r$, $e_r'' \in C_r''$. The first sum $\sum_j$ is mapped to zero by Γ.

Let C'' be free. In the following commutative diagram the horizontal sequences are exact and the vertical arrows are monomorphisms since Z'' is a direct factor in C''. Then, since

$$\sum_k \partial e_{p+1}^{(k)} \otimes e_q''^{(k)} \in C \otimes Z'' \cap B \otimes C'',$$

it must lie in $B \otimes Z''$ and hence maps to a boundary in $C \otimes H_q(C'')$:

$$\begin{array}{ccccc} B \otimes Z'' & \rightarrowtail & C \otimes Z'' & \twoheadrightarrow & C/B \otimes Z'' \\ \downarrow & & \downarrow & & \downarrow \\ B \otimes C'' & \rightarrowtail & C \otimes C'' & \twoheadrightarrow & C/B \otimes C'' \end{array}$$

In the case C is free but not C'', the same result is obtained by noting that $\sum_k e_{p+1}^{(k)} \otimes \partial'' e_q''^{(k)} = 0$, and hence $e_q''^{(k)} \in Z_q''$.

This completes the proof of the first part of the lemma.

(b) Clearly, $\Gamma_*(D^{p-1,q+1}) = 0$.

Let x be a cycle of $\mathscr{A}^{p,q}$ such that $\Gamma(x)$ is a boundary of $C\otimes H_q(C'')$. Then the following commutative diagram shows that x must be of the form $x = \sum\limits_i \partial c_{p+1}^{(i)}\otimes z_q''^{(i)} + \sum\limits_i c_p^{(i)}\otimes \partial'' e_{q+1}''^{(i)} + \sum\limits_i \sum\limits_{r<p} c_r^{(i)}\otimes c_s''^{(i)}$:

$$\begin{array}{ccc} B\otimes Z_q'' & \twoheadrightarrow & B\otimes H_q'' \\ \downarrow & & \downarrow \\ C\otimes Z_q'' & \twoheadrightarrow & C\otimes H_q'' \end{array}$$

Thus $x = \partial^{\otimes}\big(\sum\limits_i c_{p+1}^{(i)}\otimes z_q''^{(i)} + \varepsilon\sum\limits_i c_p^{(i)}\otimes e_{q+1}''^{(i)}\big) + y$, where $y \in A^{p-1,q+1}$ and is a cycle.

Hence $\operatorname{Ker}\Gamma_* = D^{p-1,q+1}$.

(c) Let C be free and let $\big\{\sum\limits_i c_p^{(i)}\otimes\{z_q''^{(i)}\}\big\} \in H_p\big(C, H_q(C'')\big)$. Then $\sum\limits_i \partial c_p^{(i)}\otimes\{z_q''^{(i)}\} = 0$ in $C_{p-1}\otimes H_q(C'')$ and hence in $Z_{p-1}\otimes H_q(C'')$. (This is where the argument breaks down for C'' free but not C.) Then the exact sequence $Z_{p-1}\otimes B_q'' \rightarrowtail Z_{p-1}\otimes Z_q'' \twoheadrightarrow Z_{p-1}\otimes H_q''$ implies that $\sum\limits_i \partial c_p^{(i)}\otimes z_q''^{(i)} \in Z_{p-1}\otimes B_q''$.

Let $\sum\limits_i \partial c_p^{(i)}\otimes z_q''^{(i)} = \sum\limits_i z_{p-1}^{(i)}\otimes \partial'' c_{q+1}''^{(i)}$, and let $v = \sum\limits_i c_p^{(i)}\otimes z_q''^{(i)} - \varepsilon\sum\limits_i z_{p-1}^{(i)}\otimes c_{q+1}''^{(i)}$. Clearly v is a cycle of A^p, and $\Gamma_*\{v\} = \big\{\sum\limits_i c_p^{(i)}\otimes\{z_q''^{(i)}\}\big\}$. Hence Γ_* is an epimorphism and l is an isomorphism.

(d) This follows immediately from (b).

COROLLARY 1.1. *In the spectral sequence E_r associated with the filtration A^p of A, if C is free, then $E_2 = E_\infty$.*

Proof. It is well known that $D^{p,q}/D^{p-1,q+1} = E_\infty$ and that $E_2^{p,q} \cong H_p\big(C, H_q(C'')\big)$. The isomorphism l completes the proof.

The roles of C and C'' may be interchanged to give a second filtration of $A = C\otimes C''$. Define the subgroup of A with filtration $\leqslant q$ to be $\sum\limits_{s\leqslant q} C_r\otimes C_s''$, and denote it by A'^q. Its $(p+q)$-th chain group $A'^{q,p}$ is $\sum\limits_{s\leqslant q} C_r\otimes C_s''$, $r+s = p+q$. Then there are defined $D'^{s,r}$, $\mathscr{A}'^{s,r}$ and $\Delta'^{s,r}$ exactly analogously to $D^{r,s}$, $\mathscr{A}^{r,s}$ and $\Delta^{r,s}$. Application of Lemma 1 immediately yields

COROLLARY 1.2. *When C is free, there is a monomorphism*

$$\lambda'\colon\ \Delta'^{q,p}/D'^{q-1,p+1} \to H_q\big(H_p(C)\otimes C''\big)$$

and when C'' is free, there is an isomorphism $l'\colon D'^{q,p}/D'^{q-1,p+1} \to H_q\big(H_p(C)\otimes C''\big)$.

Now consider the complex $C\otimes G\otimes C'$, where (C, ∂) and (C', ∂') are free and G is an abelian coefficient group. Applying the above discussion to $C\otimes(G\otimes C')$ defines groups $D_1^{p,q}$, $\Delta_1^{p,q}$, $D_1'^{q,p}$ and $\Delta_1'^{q,p}$ while $(C\otimes G)\otimes C'$

yields $D_2^{p,q}$, $\Delta_2^{p,q}$, $D_2'^{q,p}$ and $\Delta_2'^{q,p}$. These groups satisfy the following relations:

$$D_1^{p,q} = \Delta_1^{p,q}, \quad \Delta_1'^{q,p} \subseteq D_1'^{q,p}, \quad \Delta_2^{p,q} \subseteq D_2^{p,q},$$

$$\Delta_2'^{q,p} = D_2'^{q,p}, \quad D_1^{p,q} = D_2^{p,q}, \quad D_1'^{q,p} = D_2'^{q,p}.$$

Thus there are only four different groups involved, which will be written $D^{p,q}$, $\Delta^{p,q}$, $D'^{q,p}$, $\Delta'^{q,p}$, where $D^{p,q} = D_1^{p,q} = \Delta_1^{p,q} = D_2^{p,q}$, $\Delta'^{q,p} = \Delta_1'^{q,p}$, etc. Then $\Delta^{p,q} \subseteq D^{p,q}$ and $\Delta'^{q,p} \subseteq D'^{q,p}$. There are defined l: $D^{p,q}/D^{p-1,q+1} \xrightarrow{\cong} H_p(C, H_q(G\otimes C'))$ and λ: $\Delta^{p,q}/D^{p-1,q+1} \rightarrowtail H_p(C, G\otimes H_q(C'))$, also corresponding homomorphisms l', λ'. Let ϱ'_*: $H_p(C, G\otimes H_q(C')) \to H_p(C, H_q(G\otimes C'))$ be induced by the canonical monomorphism ϱ': $G\otimes H_q(C') \to H_q(G\otimes C')$, and let τ: $\Delta^{p,q} \twoheadrightarrow \Delta^{p,q}/D^{p-1,q+1}$, t: $D^{p,q} \twoheadrightarrow D^{p,q}/D^{p-1,q+1}$ be the canonical projections, while ι: $\Delta^{p,q} \rightarrowtail D^{p,q}$, ι': $\Delta^{p,q}/D^{p-1,q+1} \rightarrowtail D^{p,q}/D^{p-1,q+1}$ are the canonical embeddings.

COROLLARY 1.3. *The following diagram commutes*:

$$\begin{array}{ccccc}
\Delta^{p,q} & \overset{\tau}{\twoheadrightarrow} & \Delta^{p,q}/D^{p-1,q+1} & \overset{\lambda}{\rightarrowtail} & H_p(C, G\otimes H_q(C')) \\
\downarrow \iota & & \downarrow \iota' & & \downarrow \varrho'_* \\
D^{p,q} & \underset{t}{\twoheadrightarrow} & D^{p,q}/D^{p-1,q+1} & \underset{l}{\overset{\cong}{\rightarrowtail}} & H_p(C, H_q(G\otimes C'))
\end{array}$$

Proof. This is immediate from the above definitions since $\lambda\tau = \Gamma_*$ for $(C\otimes G)\otimes C'$ while $lt = \Gamma_*$ for $C\otimes(G\otimes C')$.

There is an obvious analogue to this corollary for $D'^{q,p}$ and $\Delta'^{q,p}$.

3. Main Theorem

We recall the proof of the Künneth formula for the tensor product of chain complexes C and C'', where C is free.

Let $Z\otimes C''$ be the tensor product complex obtained by regarding Z as a chain complex with zero boundary. Then the embedding of Z in C induces a chain monomorphism $\bar{\iota}$: $Z\otimes C'' \to C\otimes C''$. Let $B\otimes C''$ be the chain complex with the usual graduation and with boundary $-\varepsilon\otimes\partial''$. Then the epimorphism $\bar{\partial} = \partial\otimes 1$: $C\otimes C'' \to B\otimes C''$ is a chain map, and there is an exact sequence of chain complexes and chain maps, $Z\otimes C'' \rightarrowtail C\otimes C'' \twoheadrightarrow B\otimes C''$.

By a well-known theorem ([4], p. 196) this gives rise to an exact homology sequence

$$\ldots \xrightarrow{\lambda_*} H_n(Z\otimes C'') \xrightarrow{\bar{\iota}_*} H_n(C\otimes C'') \xrightarrow{\bar{\partial}_*} H_{n-1}(B\otimes C'') \xrightarrow{\lambda_*} H_{n-1}(Z\otimes C'') \to \ldots$$

Here λ_* is induced by the embedding $B \subseteq Z$ (see [4], p. 212), and $\bar{\iota}_*$, $\bar{\partial}_*$ are induced by $\bar{\iota}$, $\bar{\partial}$ respectively. Since C is free then $H_n(Z\otimes C'') = \sum_{r+s=n} Z_r\otimes H_s(C'')$, and $H_{n-1}(B\otimes C'') = \sum_{r+s=n} B_{r-1}\otimes H_s(C'')$. The cokernel of λ_*: $\sum_{r+s=n} B_r\otimes H_s(C'') \to \sum_{r+s=n} Z_r\otimes H_s(C'')$ is $\sum_{r+s=n} H_r(C)\otimes H_s(C'')$, and λ_* induces η: $\sum_{r+s=n} H_r(C)\otimes H_s(C'') \rightarrowtail H_n(C\otimes C'')$, the classical monomorphism induced by $\iota\otimes\iota''$: $Z\otimes Z'' \to C\otimes C''$. Thus there is an exact sequence

$$(2)\qquad \sum_{r+s=n} H_r(C)\otimes H_s(C'') \overset{\eta}{\rightarrowtail} H_n(C\otimes C'') \xrightarrow{\bar{\partial}_*} \sum_{r+s=n} B_{r-1}\otimes H_s(C'') \xrightarrow{\lambda_*} \sum_{r+s=n} Z_{r-1}\otimes H_s(C'') \to \dots$$

Since $\operatorname{Ker}\lambda_* = \operatorname{Im}\bar{\partial}_*$, the image of $\bar{\partial}_*$ can be identified with $\sum_{r+s=n} \operatorname{Tor}\big(H_{r-1}(C), H_s(C'')\big)$, and denoting by ξ the homomorphism defined by $\bar{\partial}_*$, we have the Künneth exact sequence

$$(3)\qquad \sum_{r+s=n} H_r(C)\otimes H_s(C'') \overset{\eta}{\rightarrowtail} H_n(C\otimes C'') \overset{\xi}{\twoheadrightarrow} \sum_{r+s=n} \operatorname{Tor}\big(H_{r-1}(C), H_s(C'')\big).$$

We now prove the main theorem upon which the subsequent discussion is based. Let C and C'' be the usual chain complexes and let C be free. There is defined a subgroup $D^{p,q}+\Delta'^{q,p}$ of $H_{p+q}(C\otimes C'')$, where $+$ is the group addition in $H_{p+q}\,(C\otimes C'')$. The main theorem is as follows:

THEOREM 2. *Let C, C'' be chain complexes and let C be free. Then, with the notation for D, Δ' established in section 2, $D^{p,q}+\Delta'^{q,p} = H_{p+q}(C\otimes C'')$ for any integers p, q.*

Proof. Let $\bar{z} = \sum_{r,s}\sum_i c_r^{(i)}\otimes c_s''^{(i)}$ be a $(p+q)$-cycle of $C\otimes C''$. Then $\bar{\partial}(\bar{z}) = \sum_{r,s}\sum_i \partial c_r^{(i)}\otimes c_s''^{(i)}$ is an $(n-1)$-cycle, $n = p+q$, of $B\otimes C''$, since $\bar{\partial} = \partial\otimes 1$ is a chain map. It follows, since C is free, that $\sum_{r,s}\sum_i \partial c_r^{(i)}\otimes c_s''^{(i)} \in \sum_{r+s=n} B_{r-1}\otimes Z_s''$, i.e. $\sum_{r,s}\sum_i \partial c_r^{(i)}\otimes c_s''^{(i)} = \sum_{r,s}\sum_j \partial e_r^{(j)}\otimes z_s''^{(j)}$ for some $e_r\in C_r$ and $z_s''\in Z_s''$. This equality holds in both $B\otimes C''$ and $C\otimes C''$. Hence $\bar{\partial}\big(\bar{z}-\sum_{r,s} e_r^{(j)}\otimes z_s''^{(j)}\big) = 0$. It follows, since $Z\otimes C'' \overset{\bar{\iota}}{\rightarrowtail} C\otimes C'' \overset{\bar{\partial}}{\twoheadrightarrow} B\otimes C''$ is exact, that there exist $z_r^{(k)}\in Z_r$, $e_s''^{(k)}\in C_s''$ such that

$$\bar{z} = \sum_{r,s}\sum_j e_r^{(j)}\otimes z_s''^{(j)} + \sum_{r,s}\sum_k z_r^{(k)}\otimes e_s''^{(k)}.$$

Since $\bar{z}$ is a cycle, then in each dimension r,

$$(4)\qquad \sum_j \partial e_{r+1}^{(j)}\otimes z_{s-1}''^{(j)} + (-1)^r \sum_k z_r^{(k)}\otimes \partial'' e_s''^{(k)} = 0.$$

Write

$$\bar{u} = \sum_{r \leqslant p} \sum_{j} e_r^{(j)} \otimes z_s''^{(j)} + \sum_{r<p} \sum_{k} z_r^{(k)} \otimes e_s''^{(k)},$$

$$\bar{v} = \sum_{r>p} \sum_{j} e_r^{(j)} \otimes z_s''^{(j)} + \sum_{r \geqslant p} \sum_{k} z_r^{(k)} \otimes e_s''^{(k)}.$$

Then $\bar{u} \in Z_{p+q}(A^p)$, $\bar{v} \in Z(\mathscr{A}'^{q,p})$, and $\bar{u}+\bar{v} = \bar{z}$. Thus $\{\bar{z}\} = \{\bar{u}\}+\{\bar{v}\}$, $\{\bar{u}\} \in D^{p,q}$, and $\{\bar{v}\} \in \Delta'^{q,p}$, and the theorem is proved.

COROLLARY 2.1. *If C, C'' are chain complexes such that $C_r = 0 = C_r''$ for $r < 0$, and C is free, then each n-cycle $\bar{z}$ of $C \otimes C''$ is equal to a sum $\sum_i \bar{z}_i$, $0 \leqslant i < n$, where $\bar{z}_r$ is a cycle of the group $C_r \otimes C_{n-r}'' \oplus C_{r+1} \otimes C_{n-r-1}''$.*

Proof. Using the notation of the theorem, define $\bar{z}_r = \sum_j e_{r+1}^{(j)} \otimes z_{s-1}''^{(j)} + \sum_k z_r^{(k)} \otimes e_s''^{(k)}$. Then from (4), $\partial^{\otimes} \bar{z}_r = 0$, $r = 0, 1, \ldots, n-1$. There is no difficulty in dimensions 0 and $n-1$. Thus $\bar{z} = \sum_i \bar{z}_i$, $0 \leqslant i < n$.

COROLLARY 2.2. *If C or C'' is free, $H_{p+q}(C \otimes C'') = D^{p,q} + D'^{p,q}$.*

Proof. Let C be free. Since $\Delta'^{q,p} \subseteq D'^{q,p}$, the result follows immediately from the theorem.

Since $D^{p,q} + D'^{q,p}$ is symmetric in C and C'', the corollary is also true for C'' free.

Let X, Y be topological spaces, and consider a singular homology theory such that there is a chain equivalence ψ: $C(X) \otimes C(Y) \to C(X \times Y)$, where $C(T)$ is the singular chain complex of T ($T = X, Y, X \times Y$), and $X \times Y$ is the cartesian product space with the usual product topology. Then ψ induces an isomorphism ψ_*: $H_{p+q}(C(X) \otimes G \otimes C(Y)) \cong H_{p+q}(X \times Y, G)$, where G is as usual an abelian coefficient group. Using the notation of Section 2 for the groups D, Δ' associated with a complex $C \otimes G \otimes C'$, with $C(X) = C$, $C(Y) = C'$, let $\psi_*(D^{p,q}) = \mathscr{D}^{p,q}$, $\psi_*(\Delta'^{q,p}) = \mathbf{\Delta}'^{q,p}$. The following corollary is then an immediate consequence of the theorem:

COROLLARY 2.3.

$$H_{p+q}(X \times Y, G) = \mathscr{D}^{p,q} + \mathbf{\Delta}'^{q,p}.$$

4. The groups D, D', Δ and Δ'

All the necessary information about these groups may be deduced from their Künneth-type exact sequences.

THEOREM 3 (Restricted Künneth formula). *Let (C, ∂), (C'', ∂'') be chain complexes and let C be free.*

Let η_p be the restriction of the monomorphism η of the Künneth sequence to $\sum_{r \leqslant p} H_r(C) \otimes H_s(C'')$, and let ξ_p be the restriction of ξ to $D^{p,q}$. Then

(a) $\operatorname{Im}\eta_p \subseteq D^{p,q}$, (b) $\operatorname{Im}\xi_p = \sum_{r\leqslant p} \operatorname{Tor}\big(H_{r-1}(C), H_s(C'')\big)$, *and* (c) *there is a short exact sequence*

$$\sum_{r\leqslant p} H_r(C)\otimes H_s(C'') \overset{\eta_p}{\rightarrowtail} D^{p,q} \overset{\xi_p}{\twoheadrightarrow} \sum_{r\leqslant p}\operatorname{Tor}\big(H_{r-1}(C), H_s(C'')\big). \tag{5}$$

Proof.

(a) It is immediately clear from the definition of $D^{p,q}$ that $\operatorname{Im}\eta_p \subseteq D^{p,q}$.

(b) ξ_p is induced by $\bar\partial$ on $A^{p,q}$, so $\xi_p(D^{p,q}) \subseteq \sum_{r\leqslant p}\operatorname{Tor}\big(H_{r-1}(C), H_s(C'')\big)$.

Now let $\partial c_r\otimes\{z_s''\}\in\operatorname{Ker}\big(\lambda_*\colon B_{r-1}\otimes H_s(C'')\to Z_{r-1}\otimes H_s(C'')\big)$. Then there exists an integer $k\geqslant 0$ such that $\partial c_r = kz_{r-1}$, $kz_s'' = \partial c_{s+1}''$ for some $z_{r-1}\in Z_{r-1}$, $c_{s+1}''\in C_{s+1}''$ (see [4], p. 226). Hence $c_r\otimes z_s'' + (-1)^{r-1}z_{r-1}\otimes c_{s+1}''$ is a cycle of $A^{r,s}$ whose homology class is mapped by $\bar\partial_*$ to $\partial c_r\otimes\{z_s''\}$. Thus ξ_p is onto.

(c) η_p is a monomorphism, ξ_p is an epimorphism, and since $\xi\eta = 0$; then $\xi_p\eta_p = 0$. It remains to show that $\xi_p^{-1}(0)\subseteq\operatorname{Im}\eta_p$. Let $\xi_p\{x\} = 0$, $x\in Z_{p+q}(A^p)$. Then from the exactness of the Künneth sequence, $\{x\}\in\operatorname{Im}\eta$. So $x = \sum_i\sum_{r,s} z_r^{(i)}\otimes z_s''^{(i)} - \partial^\otimes\sum_i\sum_{r,s} c_r^{(i)}\otimes c_{s+1}''^{(i)}$. In particular, $\sum_i \partial c_{p+2}^{(i)}\otimes c_{q-1}''^{(i)} + \varepsilon\sum_i c_{p+1}^{(i)}\otimes\partial'' c_q''^{(i)} = \sum_i z_{p+1}^{(i)}\otimes z_{q-1}''^{(i)}$, i.e. $\sum_i c_{p+1}^{(i)}\otimes\partial'' c_q''^{(i)}$ lies in the intersection of the images of $C_{p+1}\otimes B_{q-1}''$ and $Z_{p+1}\otimes C_{q-1}''$ in $C_{p+1}\otimes C_{q-1}''$. It follows from the diagram that $\sum_i c_{p+1}^{(i)}\otimes\partial'' c_q''^{(i)}\in\operatorname{Im} Z_{p+1}\otimes B_{q-1}''$:

$$\begin{tikzcd}
Z_{p+1}\otimes B_{q-1}'' \arrow[r, tail] \arrow[d] & Z_{p+1}\otimes C_{q-1}'' \arrow[r, two heads] \arrow[d] & Z_{p+1}\otimes C_{q-1}''/B_{q-1}'' \arrow[d] \\
C_{p+1}\otimes B_{q-1}'' \arrow[r, tail] & C_{p+1}\otimes C_{q-1}'' \arrow[r, two heads] & C_{p+1}\otimes C_{q-1}''/B_{q-1}''
\end{tikzcd}$$

Thus $\sum_i c_{p+1}^{(i)}\otimes\partial'' c_q''^{(i)}$ is a boundary under $\partial^\otimes$, so $\sum_i\sum_{r>p} z_r^{(i)}\otimes z_s''^{(i)}$ is also a boundary under $\partial^\otimes$, and $\{x\}\in\operatorname{Im}\eta_p$.

Corollary 3.1. *The following diagram, in which $\zeta_\otimes$, ζ, ζ_* are the canonical inclusions, is commutative for all positive integers p'*

$$\begin{tikzcd}
\sum_{r\leqslant p} H_r(C)\otimes H_s(C'') \arrow[r, tail, "\eta_p"] \arrow[d, "\zeta_\otimes"] & D^{p,q} \arrow[r, two heads, "\xi_p"] \arrow[d, "\zeta"] & \sum_{r\leqslant p}\operatorname{Tor}\big(H_{r-1}(C), H_s(C'')\big) \arrow[d, "\zeta_*"] \\
\sum_{r\leqslant p+p'} H_r(C)\otimes H_s(C'') \arrow[r, tail, "\eta_{p+p'}"'] & D^{p+p',q-p'} \arrow[r, two heads, "\xi_{p+p'}"'] & \sum_{r\leqslant p+p'}\operatorname{Tor}\big(H_{r-1}(C), H_s(C'')\big)
\end{tikzcd}$$

Proof. This follows immediately from the definitions of ξ, η_r, $r = p$, $p+p'$.

The inclusion $\zeta\colon D^{p,q}\to H_{p+q}(C\otimes C'')$ also gives rise to a commutative diagram similar to that of the above corollary. When C'' is a positive complex this is simply a special case of the corollary since $D^{p+q,0} = H_{p+q}(C\otimes C'')$.

COROLLARY 3.2. *Let C be a free complex.*

(a) *There is a short exact sequence*

$$H_p(C)\otimes H_q(C'') \xrightarrow{[\eta_p]} D^{p,q}/D^{p-1,q+1} \xrightarrow{[\xi_p]} \operatorname{Tor}\big(H_{p-1}(C), H_q(C'')\big),$$

where $[\eta_p]$, $[\xi_p]$ are induced by η_p, ξ_p respectively on the appropriate factor groups.

(b) *The following diagram commutes. The upper sequence is that defined in* (a), *the lower sequence is the classical universal coefficient sequence, and l is the isomorphism of Section* 2

$$\begin{tikzcd} H_p(C)\otimes H_q(C'') \arrow[r, tail, "{[\eta_p]}"] \arrow[d, "1", "\cong"'] & D^{p,q}/D^{p-1,q+1} \arrow[r, two heads, "{[\xi_p]}"] \arrow[d, "l", "\cong"'] & \operatorname{Tor}\big(H_{p-1}(C), H_q(C'')\big) \arrow[d, "1", "\cong"'] \\ H_p(C)\otimes H_q(C'') \arrow[r, tail, "\varrho"'] & H_p\big(C, H_q(C'')\big) \arrow[r, two heads, "\sigma"'] & \operatorname{Tor}\big(H_{p-1}(C), H_q(C'')\big) \end{tikzcd}$$

Proof.

(a) This follows by a well-known theorem on exact sequences ([4], p. 207) from Corollary 3.1 with $p' = 1$.

(b) Let $t\colon D^{p,q} \to D^{p,q}/D^{p-1,q+1}$ be the canonical projection. Then $[\eta_p] = t\eta_p$ restricted to $H_p(C)\otimes H_q(C'')$. Let $z \in Z_p$, $z'' \in Z''_q$ and let [ ] denote equivalence class modulo $D^{p-1,q+1}$. Then $l[\eta_p](\{z\}\otimes\{z''\}) = l[\{z\otimes z''\}] = \{z\otimes\{z''\}\} = \varrho(\{z\}\otimes\{z''\})$, i.e. the left-hand square commutes.

Now ξ_p is induced by $\bar\partial_*$ in sequence (2), and σ is induced by $\partial\otimes 1$ on $C\otimes H_q(C'')$. Let $x = \sum_{r\leqslant p}\sum_j e_r^{(j)}\otimes z_s''^{(j)} + \sum_{r<p}\sum_k z_r^{(k)}\otimes e_s''^{(k)}$ be a cycle of $A^{p,q}$ (see proof of Theorem 2), and let $[\bar\partial_*]$ denote the homomorphism induced by $\bar\partial_*$ on $D^{p,q}/D^{p-1,q+1}$. Then $[\bar\partial_*][\{x\}] = \sum_j \partial e_p^{(j)}\otimes\{z_q''^{(j)}\}$. But

$$(\partial\otimes 1)_* l[\{x\}] = (\partial\otimes 1)_*\Big\{\sum_j e_p^{(j)}\otimes\{z_q''^{(j)}\}\Big\} = \sum_j \partial e_p^{(j)}\otimes\{z_q''^{(j)}\}.$$

Identifying $\operatorname{Im}[\bar\partial_*]$ and $\operatorname{Im}(\partial\otimes 1)_*$ with $\operatorname{Tor}\big(H_{p-1}(C), H_q(C'')\big)$ completes the proof.

The above corollary furnishes an alternative demonstration that l is an isomorphism.

There is a sequence similar to that of Theorem 3 for the group $\Delta'^{q,p}$.

THEOREM 3. *Let C, C'' be chain complexes such that C is free. Let η'_q be the restriction of η to $\sum_{s\leqslant q} H_r(C)\otimes H_r(C'')$, and ξ'_q, the restriction of ξ to $\Delta'^{q,p}$. Then $\operatorname{Im}\eta'_q \subseteq \Delta'^{q,p}$, $\operatorname{Im}\xi'_q = \sum_{s\leqslant q}\operatorname{Tor}\big(H_r(C), H_{s-1}(C'')\big)$, and there is a short exact sequence,*

$$(6)\qquad \sum_{s\leqslant q} H_r(C)\otimes H_s(C'') \xrightarrow{\eta'_q} \Delta'^{q,p} \xrightarrow{\xi'_q} \sum_{s\leqslant q}\operatorname{Tor}\big(H_r(C), H_{s-1}(C'')\big).$$

Proof. Clearly η_q' is a monomorphism, and $\operatorname{Im}\eta_q' \subseteq \Delta'^{q,p}$. Let $x = \sum_{s<q} e_r^{(j)} \otimes z_s''^{(j)} + \sum_{s\leqslant q} z_r^{(k)} \otimes e_s''^{(k)}$ be a cycle of $\mathscr{A}'^{q,p}$ (see proof of Theorem 2). Then $\bar{\partial} x = \sum_{s<q} \partial e_r^{(j)} \otimes z_s''^{(j)}$. Thus $\xi_q'(\Delta'^{q,p}) \subseteq \sum_{s\leqslant q} \operatorname{Tor}\big(H_r(C), H_{s-1}(C'')\big)$. The proofs that ξ_q' maps onto $\sum_{s\leqslant q} \operatorname{Tor}\big(H_r(C), H_{s-1}(C'')\big)$, and that the sequence is exact follow exactly the argument of the corresponding parts of Theorem 3.

There are obvious analogues to Corollary 3.1 and to Corollary 3.2 (a), but not to Corollary 3.2 (b) since the universal coefficient theorem cannot be applied to $H_p(C) \otimes C''$ when C'' is not free.

In the next section it will be necessary to be able to describe the intersections of the groups D with the Δ'.

Lemma 4. *Let C be free, and let $\eta\colon H_p(C)\otimes H_q(C'') \to H_{p+q}(C\otimes C'')$ be the canonical monomorphism. Then $D^{p,q} \cap \Delta'^{q,p} = \eta\big(H_p(C)\otimes H_q(C'')\big)$.*

Proof. Let $x \in D^{p,q} \cap \Delta'^{q,p}$. Since $x \in D^{p,q}$, then $\xi(x) \in \sum_{r\leqslant q} \operatorname{Tor}\big(H_{r-1}(C), H_s(C'')\big)$, by Theorem 3.

But $x \in \Delta'^{q,p}$ so by Theorem 3', $\xi(x) \in \sum_{s\leqslant p} \operatorname{Tor}\big(H_r(C), H_{s-1}(C'')\big)$. Hence $\xi(x) = 0$ and so, by the exactness of sequences (5) and (6), $x \in \eta_p\big(\sum_{r\leqslant p} H_r(C)\otimes H_s(C'')\big) \cap \eta_q'\big(\sum_{s\leqslant q} H_r(C)\otimes H_s(C'')\big) = \eta\big(H_p(C)\otimes H_q(C'')\big)$.

Clearly $\eta\big(H_p(C)\otimes H_q(C'')\big) \subseteq D^{p,q} \cap \Delta'^{q,p}$, and the lemma is proved.

Lemma 5. *When C is free, $D^{p,q} \cap \Delta'^{q-1,p+1} = 0 = D^{p-1,q+1} \cap \Delta'^{q,p}$.*

Proof. Let $x \in D^{p,q} \cap \Delta'^{q-1,p+1}$. Then again $\xi(x) = 0$, and

$$x \in \eta_p\Big(\sum_{r\leqslant p} H_r(C)\otimes H_s(C'')\Big) \cap \eta_{q-1}'\Big(\sum_{s\leqslant q-1} H_r(C)\otimes H_s(C'')\Big) = 0.$$

Similarly $D^{p-1,q+1} \cap \Delta'^{q,p} = 0$.

Lemma 6. *For all positive integers t there is an exact sequence*

$$\sum_{\substack{p\leqslant r\leqslant p+t\\ r+s=p+q}} H_r(C)\otimes H_s(C'') \xrightarrow{{}_p\eta_{p+t}} D^{p+t,q-t} \cap \Delta'^{q,p} \xrightarrow{{}_p\xi_{p+t}} \sum_{\substack{p\leqslant r\leqslant p+t-1\\ r+s=p+q-1}} \operatorname{Tor}\big(H_r(C), H_s(C'')\big).$$

${}_p\eta_{p+t}$, ${}_p\xi_{p+t}$ are the restrictions of the Künneth sequence maps.

Proof. This is an immediate consequence of Theorems 3 and 3'.

Now let C be free and $C'' = G\otimes C'$, where C' is free, and let $D^{p,q}$, $\Delta^{p,q}$, $D'^{q,p}$, $\Delta'^{q,p}$ be the subgroups of $H_{p+q}(C\otimes G\otimes C')$ defined in Section 2. Then Lemma 4 shows that $D^{p,q} \cap \Delta'^{q,p} \cong H_p(C)\otimes H_q(G\otimes C')$, while $\Delta^{p,q} \cap D'^{q,p} \cong H_p(C, G)\otimes H_q(C')$. Clearly $\Delta^{p,q} \cap \Delta'^{q,p} \cong H_p(C)\otimes G\otimes H_q(C')$. We complete the description of these (p, q)-intersections by obtaining an exact sequence for the remaining intersection, viz. $D^{p,q} \cap D'^{q,p}$.

LEMMA 7. *There is a short exact sequence*

$$(7)\qquad H_p(C)\otimes H_q(G\otimes C')+H_p(C,G)\otimes H_q(C')$$
$$\xrightarrow{\eta,\eta'} D^{p,q}\cap D'^{q,p} \xrightarrow{\xi_\cap} \operatorname{Tor}\big(H_{p-1}(C),\operatorname{Tor}(G,H_{q-1}(C'))\big).$$

Here η, η' are the canonical embeddings in $H_{p+q}(C\otimes G\otimes C')$ while $\xi_\cap$ is described in the course of the proof, where it will also be seen that the term $\operatorname{Tor}\big(H_{p-1}(C),\ \operatorname{Tor}(G,H_{q-1}(C'))\big)$ is not in fact asymmetrical in C and C' since it may be replaced by the canonically isomorphic group $\operatorname{Tor}\big(\operatorname{Tor}(H_{p-1}(C),G),H_{q-1}(C')\big)$.

Proof. Write $C\otimes G\otimes C' = A$, $\sum_{r\leqslant p} C_r\otimes G\otimes C'_s = A^p$, and $\sum_{s\leqslant q} C_r\otimes G\otimes C'_s = A'^q$. The filtrations A^p, A'^q give rise to $D^{p,q}$, $D'^{q,p}$ respectively.

Let $x\in Z_{p+q}(A'^q)$ be written $x = \sum_i\sum_{s\leqslant q} c_r^{(i)}\otimes g^{(i)}\otimes c_s'^{(i)}$. Then

$$(8)\qquad \sum_i \partial c_p^{(i)}\otimes g^{(i)}\otimes c_q'^{(i)} = 0.$$

Now let $\{x\}\in D^{p,q}$. Then, for some $e_r\in C_r$, $e'_s\in C'_s$, $x+\partial^{\otimes}\sum_i\sum_{r+s=p+q} e_r^{(i)}\otimes g^{(i)}\otimes e_{s+1}'^{(i)}\in Z_{p+q}(A^p)$. Hence $\sum_i\sum_{s<q} c_r^{(i)}\otimes g^{(i)}\otimes c_s'^{(i)}+\partial^{\otimes}\sum_i\sum_{s<q-1} e_r^{(i)}\otimes g^{(i)}\otimes e_{s+1}'^{(i)}+\varepsilon\sum_i e_{p+1}^{(i)}\otimes g^{(i)}\otimes\partial' e_q'^{(i)} = 0$. So $x = \sum_i c_p^{(i)}\otimes g^{(i)}\otimes c_q'^{(i)}-\varepsilon\sum_i e_{p+1}^{(i)}\otimes g^{(i)}\otimes\partial' e_q'^{(i)}+\partial^{\otimes} y$ for some $y\in A_{p+q+1}$, and, since $\partial^{\otimes}x = 0$, we have

$$(9)\qquad (-1)^p\sum_i c_p^{(i)}\otimes g^{(i)}\otimes\partial' c_q'^{(i)}-(-1)^{p+1}\sum_i \partial e_{p+1}^{(i)}\otimes g^{(i)}\otimes\partial' e_q'^{(i)} = 0.$$

Thus $\{x\} = \{\sum_i c_p^{(i)}\otimes g^{(i)}\otimes c_q'^{(i)}+\sum_i \partial e_{p+1}^{(i)}\otimes g^{(i)}\otimes e_q'^{(i)}\}$.

Relations (8) and (9) show that each class of the group $D^{p,q}\cap D'^{q,p}$ has a representative cycle of the form $\sum_i f_p^{(i)}\otimes g^{(i)}\otimes f_q'^{(i)}$, $f_p\in C_p$, $f'_q\in C'_q$, with the relations

$$(10)\qquad \sum_i \partial f_p^{(i)}\otimes g^{(i)}\otimes f_q'^{(i)} = 0 = \sum_i f_p^{(i)}\otimes g^{(i)}\otimes\partial' f_q'^{(i)}.$$

The class of each such cycle clearly belongs to $D^{p,q}\cap D'^{q,p}$.

We now obtain the required exact sequence (7). First, η and η' are monomorphisms, and clearly $\operatorname{Im}\eta\subseteq D^{p,q}\cap D'^{q,p}$, $\operatorname{Im}\eta'\subseteq D^{p,q}\cap D'^{q,p}$.

Define a homomorphism $\delta\colon C_p\otimes G\otimes C'_q\to B_{p-1}\otimes G\otimes B'_{q-1}$ by $\delta(c_p\otimes g\otimes c'_q) = \partial c_p\otimes g\otimes\partial' c'_q$. Then δ maps boundaries in $C_p\otimes G\otimes C'_q$ under $\partial^{\otimes}$ to zero and induces $\delta_*\colon D^{p,q}\cap D'^{q,p}\to B_{p-1}\otimes G\otimes B'_{q-1}$. Clearly $\delta_*\eta = 0 = \delta_*\eta'$. Let $\delta_*\{x\} = 0$, $x = \sum_i f_p^{(i)}\otimes g^{(i)}\otimes f_q'^{(i)}$, satisfying relations (10).

Then $\sum_i \partial f_p^{(i)}\otimes g^{(i)}\otimes \partial' f_q'^{(i)} = 0$ in $B_{p-1}\otimes G\otimes B'_{q-1}$. Hence $x\in Z_p\otimes G\otimes C'_q+C_p\otimes G\otimes Z'_q$, then $\{x\}\in \operatorname{Im}\eta, \eta'$.

It remains to describe $\operatorname{Im}\delta_*(D^{p,q}\frown D'^{q,p})$. Again let $x=\sum_i f_p^{(i)}\otimes g^{(i)}\otimes f_q'^{(i)}$ be a cycle such that $\{x\}\in D^{p,q}\frown D'^{q,p}$. Then by (10), $x\in C_p\otimes Z_q(G\otimes C')$. Hence $\delta x\in B_{p-1}\otimes K$, where K is the subgroup of $G\otimes B'_{q-1}$ which is the image under ∂' of $Z_q(G\otimes C')$. Thus K may be identified with $\operatorname{Tor}(G, H_{q-1}(C'))$. From (10) again clearly $\delta x\in \operatorname{Im}\partial(Z_p(C\otimes K))$ in $B_{p-1}\otimes K$, i.e. $\operatorname{Im}\delta_*$ may be identified with a subgroup of $\operatorname{Tor}\big(H_{p-1}(C), \operatorname{Tor}(G, H_{q-1}(C'))\big)$. Now let $y=\sum_i \partial f_p^{(i)}\otimes \sum_j g^{(i,j)}\otimes \partial' f_q'^{(i,j)}$ $\in \operatorname{Im}\Big(\operatorname{Tor}\big(H_{p-1}(C), \operatorname{Tor}(G, H_{q-1}(C'))\big)\Big)$ in $B_{p-1}\otimes G\otimes B'_{q-1}$. Let $x=\sum_i f_p^{(i)}\otimes \sum_j g^{(i,j)}\otimes f_q'^{(i,j)}$.

From the classical properties of torsion product ([4], p. 226), $\partial f_p^{(i)} = kz_{p-1}^{(i)}$, $k\sum_j g^{(i,j)}\otimes \partial' f_q'^{(i,j)}=0$ for some integer $k\geqslant 0$ and cycle $z_{p-1}\in Z_{p-1}$. But B' is free, so $kg^{(i,j)}=0$ for all i, j, and hence $k\sum_j g^{(i,j)}\otimes f_q'^{(i,j)}=0$. Thus $\sum_i \partial f_p^{(i)}\otimes\sum_j g^{(i,j)}\otimes f_q'^{(i,j)}=0$. And since $\sum_j g^{(i,j)}\otimes \partial' f_q'^{(i,j)}\in \operatorname{Tor}(G, H_{q-1}(C'))$ clearly $\sum_i f_p^{(i)}\otimes\sum_j g^{(i,j)}\otimes \partial' f_q'^{(i,j)}=0$. Hence x satisfies relations (10), i.e. $\{x\}\in D^{p,q}\frown D'^{q,p}$ and $y=\delta_*\{x\}$. $\operatorname{Im}\delta_*$ may therefore be identified with $\operatorname{Tor}\big(H_{p-1}(C), \operatorname{Tor}(G, H_{q-1}(C'))\big)$, and letting $\xi_\frown$ be the epimorphism induced by δ_* completes the proof of exactness.

Finally, it is easy to see that if we first consider $x\in Z_p(C, G)\otimes C'_q$ and proceed to $\delta x\in \operatorname{Tor}(H_{p-1}(C), G)\otimes B'_{q-1}$ we may identify $\operatorname{Im}\delta_*$ with $\operatorname{Tor}\big(\operatorname{Tor}(H_{p-1}(C), G), H_{q-1}(C')\big)$.

COROLLARY 7.1. *If* $\operatorname{Tor}(G, H_{q-1}(C'))=0$, *then* $D^{p,q}\frown D'^{q,p} = \eta'(H_p(C, G)\otimes H_q(C'))$.

Proof. This is immediate from the above lemma and the universal coefficient theorem.

In general $D^{p,q}\frown D'^{q-1,p+1}$ and $D^{p-1,q+1}\frown D'^{q,p}$ are not zero, but Lemma 7 enables us to establish conditions under which these groups are zero.

LEMMA 8. *When* $\operatorname{Tor}(G, H_{q-1}(C'))=0=\operatorname{Tor}(G, H_q(C'))$, *then* $D^{p-1,q+1}\frown D'^{q,p}=0$.

Proof. Let $x\in D^{p-1,q+1}\frown D'^{q,p}$. Then $x\in D^{p,q}\frown D'^{q,p}=\eta'(H_p(C, G)\otimes H_q(C'))$, and $x\in D^{p-1,q+1}\frown D'^{q+1,p-1}=\eta'(H_{p-1}(C, G)\otimes H_{q+1}(C'))$. Hence $D^{p-1,q+1}\frown D'^{q,p}=0$.

Clearly all the homomorphisms defined in the sequences of this section, together with l, λ of Section 2, are natural with respect to chain mappings $\sigma\colon C\to K$, $\sigma''\colon C''\to K''$, where K, K'' are assumed free whenever it is necessary to assume this of C or C'' respectively, i.e. these homomorphisms commute with the homomorphisms of $H_n(C\otimes C'')$, $D^{p,q}$, $\sum H_r(C)\otimes H_s(C'')$, etc. induced by σ and σ''. In the case $C'' = G\otimes C'$ a chain map σ'' is of course induced by a chain map $\sigma'\colon C' \to K'$ and a homomorphism $f\colon G \to F$, where F is an abelian group.

We complete the discussion of the groups D, D', Δ and Δ' by showing that all the exact sequences of this section split, but not necessarily naturally.

LEMMA 9. *Let (C, ∂), (C'', ∂'') be chain complexes, not necessarily free, and let $D^{p,q} \subseteq H_{p+q}(C\otimes C'')$ be defined as usual. Then whenever the following exact sequence exists it splits:*

$$\sum_{r\leqslant p} H_r(C)\otimes H_s(C'') \xrightarrow{\eta_p} D^{p,q} \xrightarrow{\xi_p} \sum_{r\leqslant p} \mathrm{Tor}\,\big(H_{r-1}(C), H_s(C'')\big).$$

Proof. This lemma is already known for the Künneth sequence ([5], p. 168) and it is necessary only to adapt the argument there given.

Let K, K'' be free chain complexes. Then the following exact sequence (whose existence is assured) splits:

$$\sum_{r\leqslant p} H_r(K)\otimes H_s(K'') \xrightarrow{\eta_p^K} D_K^{p,q} \xrightarrow{\xi_p^K} \sum_{r\leqslant p} \mathrm{Tor}\,\big(H_{r-1}(K), H_s(K'')\big).$$

The notation $D_K^{p,q}$, η_p^K, ξ_p^K is clear. The splitting is obtained from inverses $\varphi_r\colon K_r \to Z_r(K)$, $\psi_s\colon K_s'' \to Z_s(K'')$ of the embeddings $\iota\colon Z_r(K) \to K_r$, $\iota''\colon Z_s(K'') \to K_s''$ respectively. These induce a map $\varphi\otimes\psi\colon \sum_{r\leqslant p} K_r\otimes K_s'' \to \sum_{r\leqslant p} Z_r(K)\otimes Z_s(K'') \to \sum_{r\leqslant p} H_r(K)\otimes H_s(K'')$ which in turn induces $(\varphi\otimes\psi)_*\colon D_K^{p,q} \to \sum_{r\leqslant p} H_r(K)\otimes H_s(K'')$ such that $(\varphi\otimes\psi)_*\eta_p^K = 1$.

But there exist free chain complexes K, K'' and chain maps $\sigma\colon K \to C$, $\sigma''\colon K'' \to C''$ such that $\sigma_*\colon H_*(K) \to H_*(C)$, $\sigma_*''\colon H_*(K'') \to H_*(C'')$ are isomorphisms in all dimensions (see [3], p. 169). Then the following diagram commutes:

$$\begin{array}{ccccc}
\sum_{r\leqslant p} H_r(K)\otimes H_s(K'') & \xrightarrow{\eta_p^K} & D_K^{p,q} & \xrightarrow{\xi_p^K} & \sum_{r\leqslant p} \mathrm{Tor}\,\big(H_{r-1}(K), H_s(K'')\big) \\
\downarrow{\scriptstyle \sigma_*\otimes\sigma_*''} & & \downarrow{\scriptstyle (\sigma\otimes\sigma'')_*} & & \downarrow{\scriptstyle \mathrm{Tor}(\sigma_*,\sigma_*'')} \\
\sum_{r\leqslant p} H_r(C)\otimes H_s(C'') & \xrightarrow[\eta_p]{} & D^{p,q} & \xrightarrow[\xi_p]{} & \sum_{r\leqslant p} \mathrm{Tor}\,\big(H_{r-1}(C), H_s(C'')\big)
\end{array}$$

The five-lemma ensures that the two sequences are isomorphic, hence since the top sequence splits the bottom sequence also splits.

The inverses φ_r, ψ_s are not in general natural, i.e. they do not commute with chain maps of K, K'' and hence the splittings are also not natural.

Exactly similar reasonings show that the sequences of Theorem 3′ and Lemma 6 split.

In order to see that sequence (7) splits it is only necessary to define $\varkappa\colon B_{p-1} \to C_p$, $\varkappa'\colon B'_{q-1} \to C'_q$ as left inverses of ∂, ∂' respectively. Then $\varkappa\otimes 1\otimes\varkappa'\colon B_{p-1}\otimes G\otimes B'_{q-1} \to C_p\otimes G\otimes C'_q$ is a right inverse of δ and it is not difficult to see that it induces a right inverse of $\delta\colon Z(C_p\otimes G\otimes C'_q) \to \operatorname{Tor}\big(H_{p-1}(C), \operatorname{Tor}(G, H_{q-1}(C'))\big)$, where $Z(C_p\otimes G\otimes C'_q)$ denotes the group of cycles in $C\otimes G\otimes C'$ of elements of $C_p\otimes G\otimes C'_q$. This gives a right inverse to $\xi_\frown$ and hence a splitting, again not natural, of the sequence.

The following corollary is an immediate consequence of Lemma 9:

COROLLARY 9.1. *Let C be free, and let $C\otimes C''$, $D^{p,q}$ be defined as usual. Then*

(a) *there is a non-natural isomorphism* $D^{p,q} \cong \sum_{r\leqslant p} H_r\big(C, H_s(C'')\big)$,

(b) *there is a non-natural projection* $\theta_q\colon H_{p+q}(C\otimes C'') \to H_p\big(C, H_q(C'')\big)$ *in each dimension q.*

5. The projection $\theta_q\colon H_{p+q}(C\otimes G\otimes C') \to H_p\big(C, H_q(G\otimes C')\big)$

We prove a preliminary lemma.

Let $(C, +)$ be an abelian group, A and B subgroups of C, and write $A \frown B = D$. Then D is a subgroup of A, B and C. There is also defined a subgroup $A+B$ of C.

Let $A\oplus B$ be the direct sum of A and B, and let E be the subgroup of $A\oplus B$ generated by elements $(d, -d)$, $d\in D$.

LEMMA 10. *There is a canonical isomorphism* $m\colon \dfrac{A\oplus B}{E} \to A+B$.

Proof. Consider the sequence $D \overset{\mu}{\rightarrowtail} A\oplus B \overset{\varepsilon}{\twoheadrightarrow} A+B$, where $\mu(d) = (d, -d)$ and $\varepsilon(a, b) = a+b$, $a\in A$, $b\in B$ and $d\in D$. Then $\varepsilon\mu(d) = d+(-d) = 0$; and if $\varepsilon(a, b) = 0$, then $b = -a$, i.e. $a\in D$, and $(a, b)\in\mu D$. Clearly μ is a monomorphism and ε an epimorphism. The sequence is therefore exact, and since $\mu D = E$ the lemma is proved.

Now consider again the complex $C\otimes C''$ where C is free. Use the notation of Section 2 for D, Δ' etc. By Lemma 10 there is a canonical isomorphism $D^{p,q}+\Delta'^{q,p} \cong \dfrac{D^{p,q}\oplus\Delta'^{q,p}}{E}$, where E is generated by elements $(d, -d)$, $d\in D^{p,q}\frown\Delta'^{q,p}$.

Let $l\colon D^{p,q}/D^{p-1,q+1} \to H_p\big(C, H_q(C'')\big)$ be the isomorphism of Lemma 1 (c) and $\lambda'\colon \Delta'^{q,p}/D'^{q-1,p+1} \to H_q\big(H_p(C)\otimes C''\big)$ the monomorphism of Corollary 1.2, while $t\colon D^{p,q} \to D^{p,q}/D^{p-1,q+1}$ and $\tau'\colon \Delta'^{q,p} \to \Delta'^{q,p}/D'^{q-1,p+1}$

are the canonical projections. Then lt, $\lambda'\tau'$ define a map q: $D^{p,q} \oplus \Delta'^{q,p} \to H_p(C, H_q(C''))\oplus H_q(H_p(C)\otimes C'')$, $q(x, x') = lt(x), \lambda'\tau'(x')$, $x \in D^{p,q}$, $x' \in \Delta'^{q,p}$. Then q induces a homomorphism

$$Q\colon \frac{D^{p,q}\oplus \Delta'^{q,p}}{E} \to \frac{H_p(C, H_q(C''))\oplus H_q(H_p(C)\otimes C'')}{M},$$

where M is generated by elements $(lt(d), -\lambda'\tau'(d))$, $d \in D^{p,q} \cap \Delta'^{q,p}$. Combining Q with the isomorphism m of Lemma 10, and applying Theorem 2 gives.

THEOREM 11. *If (C, ∂), (C'', ∂'') are chain complexes and C is free there is a homomorphism, natural with respect to chain maps $C \to K$, $C'' \to K''$, where K'' is free, namely*

$$\mathcal{Q}\colon H_{p+q}(C\otimes C'') \to \frac{H_p(C, H_q(C''))\oplus H_q(H_p(C)\otimes C'')}{M}.$$

Proof. Since $H_{p+q}(C\otimes C'') = D^{p,q} + \Delta'^{q,p}$, then $\mathcal{Q} = Qm^{-1}$.

This theorem is obviously applicable to $C\otimes G\otimes C'$, where C, C' are free, simply by taking $C'' = G\otimes C'$. Let ϱ_*: $H_q(H_p(C)\otimes G\otimes C') \to H_q(H_p(C, G)\otimes C')$ be induced by the canonical monomorphism ϱ: $H_p(C)\otimes G \to H_p(C, G)$. Let P be the homomorphism of

$$\frac{H_p(C, H_q(G\otimes C'))\oplus H_q(H_p(C)\otimes G\otimes C')}{M}$$

to

$$\frac{H_p(C, H_q(G\otimes C'))\oplus H_q(H_p(C, G)\otimes C')}{L}$$

induced by the identity on the first summand and by ϱ_* on the second summand. Then L is generated by elements $(lt(d), -\varrho_*\lambda'\tau'(d))$, $d \in D^{p,q} \cap \Delta'^{q,p}$. But by Corollary 2.2, $H_{p+q}(C\otimes G\otimes C') = D^{p,q} + D'^{q,p}$, and since C' is free there is defined the map $l't'$: $D'^{q,p} \to H_q(H_p(C, G)\otimes C')$, using the notation of Section 2. Then lt and $l't'$ define an epimorphism

$$\mathcal{P}\colon H_{p+q}(C\otimes G\otimes C') \to \frac{H_p(C, H_q(G\otimes C'))\oplus H_q(H_p(C, G)\otimes C')}{L'},$$

where L' is generated by elements $(lt(d'), -l't'(d'))$, $d' \in D^{p,q} \cap D'^{q,p}$.

$\mathcal{P}$ is clearly onto since l, l' are isomorphisms and t, t' are epimorphisms.

Now apply Corollary 1.3 to D' and Δ'. Then $\varrho_*\lambda'\tau' = l't'\iota'$, where ι' is the embedding of Δ' in D'. So ι' induces a map $\mathcal{I}$ of

$$\frac{H_p(C, H_q(G\otimes C'))\oplus H_q(H_p(C, G)\otimes C')}{L}$$

onto

$$\frac{H_p(C, H_q(G\otimes C'))\oplus H_q(H_p(C, G)\otimes C')}{L'}.$$

COROLLARY 11.1. $\mathscr{I}P\mathscr{Q} = \mathscr{P}$.

Proof. This is immediate from the above definitions.

The projection $\mathscr{Q}$ gives rise to the conditions for the existence of θ_q: $H_{p+q}(C\otimes G\otimes C') \to H_p(C, H_q(G\otimes C'))$.

THEOREM 12. *Let C, C' be free chain complexes, G an abelian coefficient group. Then there exists a natural projection $\theta_q: H_{p+q}(C\otimes G\otimes C') \to H_p(C, H_q(G\otimes C'))$ whenever* $\operatorname{Tor}(G, H_{q-1}(C')) = 0$ *and* $\operatorname{Tor}(H_p(C)\otimes G, H_{q-1}(C')) = 0$.

Here the term natural applies in the category of chain complexes and maps such that the two torsion groups vanish.

Proof. The vanishing of the torsion groups implies that $H_q(H_p(C)\otimes G\otimes C') = H_p(C)\otimes G\otimes H_q(C')$ and that $D^{p,q} \frown \Delta'^{q,p} = \eta(H_p(C)\otimes G\otimes H_q(C'))$ (see Lemma 4). Let $\varrho: H_p(C)\otimes G\otimes H_q(C') \to H_p(C, G\otimes H_q(C')) = H_p(C, H_q(G\otimes C'))$ be the canonical monomorphism. Then $\varrho = lt\eta$ (Corollary 3.2).

Define $R: H_p(C, H_q(G\otimes C'))\oplus H_p(C)\otimes G\otimes H_q(C') \to H_p(C, H_q(G\otimes C'))$ by $R(x, y) = x + \varrho y$, $x \in H_p(C, H_q(G\otimes C'))$, $y \in H_p(C)\otimes G\otimes H_q(C')$. Let $d = \eta(e)$, $e \in H_p(C)\otimes G\otimes H_q(C')$. Then M is generated by elements $(lt\eta(e), -\lambda'\tau'\eta(e))$. But $lt\eta = \varrho$ and $\lambda'\tau'\eta = \lambda'\tau'\eta' = 1$. Hence R and M satisfy the conditions of Lemma 10 so that R induces the isomorphism

$$\mathscr{R}: \frac{H_p(C, H_q(G\otimes C'))\oplus H_q(H_p(C)\otimes G\otimes C')}{M} \to H_p(C, H_q(G\otimes C')).$$

Combining $\mathscr{R}$ with $\mathscr{Q}$ gives the required projection

$$\theta_q: H_{p+q}(C\otimes G\otimes C') \to H_p(C, H_q(G\otimes C')).$$

Clearly θ_q is an epimorphism and is natural with respect to chain maps of free chain complexes $C_1 \to C_2$, $C_1' \to C_2'$ and homomorphisms $G_1 \to G_2$ such that $\operatorname{Tor}(G_i, H_{q-1}(C_i')) = 0$ and $\operatorname{Tor}(H_p(C_i)\otimes G_i, H_{q-1}(C_i')) = 0$, $i = 1, 2$.

An attempt on similar lines to obtain conditions for the existence of θ_q from the projection $\mathscr{P}$ in fact produces the same conditions. In order to be able to project the second summand $H_q(H_p(C, G)\otimes C')$ to $H_p(C, H_q(G\otimes C'))$ we need $\operatorname{Tor}(H_p(C, G), H_{q-1}(C')) = 0$; and in order that L' should be mapped to zero by the induced homomorphism of the direct sum it is necessary that $D^{p,q} \frown D'^{q,p} = \eta'(H_p(C, G)\otimes H_q(C'))$, which requires that $\operatorname{Tor}(G, H_{q-1}(C')) = 0$ (see Lemma 7). These two conditions are equivalent to those of Theorem 12. They are the conditions for $D^{p,q}$ to be a natural direct summand of $H_{p+q}(C\otimes G\otimes C')$. Then θ_q may be defined by projecting to $D^{p,q}$ followed by lt: $D^{p,q} \to H_p(C, H_q(G\otimes C'))$. The projection χ_q: $H_{p+q}(C\otimes G\otimes C') \to D^{p,q}$ is given by

$$H_{p+q}(C\otimes G\otimes C') \cong \frac{D^{p,q}\oplus \Delta'^{q,p}}{E} \xrightarrow{(1,\lambda'\tau')} \frac{D^{p,q}\oplus H_p(C)\otimes G\otimes H_q(C')}{N} \xrightarrow{T} D^{p,q}.$$

Here let $t(x, y) = x + \eta y$; N is generated by elements $(\eta(d), -d)$, $d \in H_p(C) \otimes G \otimes H_q(C')$; so $t(N) = 0$ and the induced map T is an isomorphism.

$$\begin{array}{ccc} H_{p+q}(C\otimes G\otimes C') \to & \dfrac{D^{p,q} \oplus H_p(C)\otimes G\otimes H_q(C')}{N} & \xrightarrow{T} D^{p,q} \\ \downarrow \varrho & \downarrow (u,1) & \downarrow u \\ & \dfrac{H_p(C, H_q(G\otimes C')) \oplus H_p(C)\otimes G\otimes H_q(C')}{M} & \to H_p(C, H_q(G\otimes C')) \end{array}$$

The diagram commutes.

COROLLARY 12.1. *When* Tor $(G, H_{q-1}(C')) = 0$ *there is a natural projection* θ_q: $H_{p+q}(C\otimes G\otimes C') \to H_p(C, H_q(G\otimes C'))$ *if the following universal coefficient exact sequence splits naturally*

$$H_p(C)\otimes G\otimes H_q(C') \overset{\varrho'}{\rightarrowtail} H_q(H_p(C)\otimes G\otimes C') \twoheadrightarrow \operatorname{Tor}(H_p(C)\otimes G, H_{q-1}(C')).$$

Proof. Again $D^{p,q} \cap \Delta'^{q,p} = \eta(H_p(C)\otimes G\otimes H_q(C'))$. Let σ': $H_q(H_p(C)\otimes G\otimes C') \to H_p(C)\otimes G\otimes H_q(C')$ be a splitting homomorphism, i.e. $\sigma'\varrho' = 1$. Then with the notation of the theorem define R by $R(x, y) = x + \varrho\sigma' y$. In order that M should be mapped by R to zero we need $\varrho = \varrho\sigma'\lambda'\tau'\eta' = \varrho\sigma'\varrho'$. This is satisfied, so θ_q is defined.

COROLLARY 12.2. *Under the conditions of Theorem* 12 *the projection* θ_q *satisfies the relation* $\theta_q\eta = \varrho$, *where* η: $H_p(C)\otimes G\otimes H_q(C') \to H_{p+q}(C\otimes G\otimes C')$ *and* ϱ: $H_p(C)\otimes G\otimes H_q(C') \to H_p(C, G\otimes H_q(C'))$ *are the canonical monomorphisms.*

Proof. This follows at once from the definition of θ_q and the relation $\varrho = lt\eta$ (Corollary 3.2).

If X, Y are topological spaces with a singular homology theory as in Corollary 2.3, then Theorem 12 and Corollary 12.2 imply

COROLLARY 12.3. *Let* Tor $(H_{q-1}(Y), G) = 0 =$ Tor $(H_p(X)\otimes G, H_{q-1}(Y))$. *Then there exists a natural projection* θ_q: $H_{p+q}(X\times Y, G) \to H_p(X, H_q(Y, G))$ *such that* $\theta_q\eta = \varrho$, *where* ϱ: $H_p(X)\otimes H_q(Y, G) \to H_p(X, H_q(Y, G))$ *and* η: $H_p(X)\otimes H_q(Y, G) \to H_{p+q}(X\times Y, G)$ *are the canonical monomorphisms.*

Naturality here is in the category of topological spaces X_i, Y_i and continuous maps $X_1 \to X_2$, $Y_1 \to Y_2$ such that

$$\operatorname{Tor}(H_{q-1}(Y_i), G) = 0 = \operatorname{Tor}(H_p(X_i)\otimes G, H_{q-1}(Y_i)), \quad i = 1, 2.$$

The following counter-example serves to show that there does not always exist a natural projection θ_q: $H_{p+q}(C\otimes G\otimes C') \to H_p(C, H_q(G\otimes C'))$ such that $\theta_q\eta = \varrho$.

COUNTER-EXAMPLE. Denote by P^r real projective r-space. Let C be the chain group (for some homology theory) of P^4 and let $C' = C(P^2)\otimes C(P^2)$. Let $G = T$, the integers, and let T_2 denote the integers modulo 2.

The homology groups of P^4 are

$$H_0(P^4) = T, \quad H_1(P^4) = H_3(P^4) = T_2, \quad H_r(P^4) = 0, \quad r \neq 0, 1, 3.$$

The homology groups of $P^2\times P^2$ may be calculated from the Künneth formula

$$H_0(P^2\times P^2) = T, \quad H_1(P^2\times P^2) = T_2\oplus T_2, \quad H_2(P^2\times P^2) = T_2,$$
$$H_3(P^2\times P^2) = T_2, \quad H_r(P^2\times P^2) = 0, \quad r > 3.$$

Consider $H_6(C\otimes C') \cong H_6(P^4\times(P^2\times P^2))$ filtered in the two ways described in Section 2. The groups D, D' may be calculated from Theorem 3. Since C' is free $D' = \Delta'$. Then

$$D^{0,6} = D^{1,5} = D^{2,4} = 0, \quad D^{3,3} = \eta\big(H_3(C)\otimes H_3(C')\big),$$
$$D'^{0,6} = D'^{1,5} = D'^{2,4} = 0, \quad D'^{3,3} = l'^{-1}\big(H_3\big(H_3(C)\otimes C'\big)\big).$$

Here η is the canonical monomorphism $D^{3,3} \subseteq H_6(C\otimes C')$ and l' is the isomorphism of Lemma 1. Let ϱ': $H_3(C)\otimes H_3(C') \to H_3\big(H_3(C)\otimes C'\big)$ be the canonical monomorphism. Note that $H_3(C)\otimes H_3(C') = H_3\big(C, H_3(C')\big)$. Apply Theorem 2 to $H_6(C\otimes C')$ with $p = q = 3$. Since $D^{3,3} \subseteq D'^{3,3}$ ($l'\eta = \varrho'$, see Corollary 3.2), then $H_6(C\otimes C') = D'^{3,3} = l'^{-1}\big(H_3\big(H_3(C)\otimes C'\big)\big)$. So the required natural projection θ_3: $H_6(C\otimes C') \to H_3\big(C, H_3(C')\big)$ exists if and only if there is a natural projection

$$\Theta\colon l'^{-1}\big(H_3\big(H_3(C)\otimes C'\big)\big) \to H_3(C)\otimes H_3(C')$$

such that $\Theta l'^{-1}\varrho' = 1$ on $H_3(C)\otimes H_3(C')$, i.e. we require a natural left inverse of ϱ': $H_3(P^2\times P^2)\otimes H_3(P^4) \to H_3\big(P^2\times P^2, H_3(P^4)\big)$. We show that no such map exists. This serves also to demonstrate that the universal coefficient exact sequence has in general no natural splitting, for a natural left inverse of ϱ' provides a natural splitting of the sequence

$$H_3(P^2\times P^2)\otimes H_3(P^4) \overset{\varrho'}{\rightarrowtail} H_3\big(P^2\times P^2, H_3(P^4)\big) \twoheadrightarrow \operatorname{Tor}\big(H_2(P^2\times P^2), H_3(P^4)\big).$$

Let ζ_1 be a cycle whose class generates $H_1(P^2)$. Then $\partial\zeta_1 = 0$. Let $\zeta_2 \in C_2(P^2)$ be a chain such that $\partial\zeta_2 = 2\zeta_1$ (see [4], p. 215). Consider the chain $\zeta_2\otimes\zeta_1 + \zeta_1\otimes\zeta_2 \in C(P^2)\otimes C(P^2)$. Then $\partial^{\otimes}(\zeta_2\otimes\zeta_1 + \zeta_1\otimes\zeta_2) = 2(\zeta_1\otimes\zeta_1 - \zeta_1\otimes\zeta_1) = 0$, and $\partial^{\otimes}(\zeta_2\otimes\zeta_2) = 2(\zeta_1\otimes\zeta_2 + \zeta_2\otimes\zeta_1)$. Thus $\{\zeta_2\otimes\zeta_1 + \zeta_1\otimes\zeta_2\}$ generates $H_3\big(C(P^2)\otimes C(P^2)\big) \cong H_3(P^2\times P^2)$. Now $H_3(P^4) = T_2$, and it is clear that with this coefficient group $\partial^{\otimes}(\zeta_2\otimes\zeta_1) = 2(\zeta_1\otimes\zeta_1) = 0 = \partial^{\otimes}(\zeta_1\otimes\zeta_2)$. So $\{\zeta_2\otimes\zeta_1\}$ and $\{\zeta_1\otimes\zeta_2\}$ generate $H_3\big(C(P^2)\otimes C(P^2), T_2\big) = T_2\oplus T_2$.

In terms of these generators

$$\varrho'\colon\ H_3\big(C(P^2)\otimes C(P^2)\big)\otimes T_2 \to H_3\big(C(P^2)\otimes C(P^2),\, T_2\big)$$

is defined by

$$\varrho'\{\zeta_2\otimes\zeta_1+\zeta_1\otimes\zeta_2\}\otimes j = \{\zeta_2\otimes\zeta_1\otimes j\}+\{\zeta_1\otimes\zeta_2\otimes j\}, \quad j\in T_2,$$

i.e. it is the diagonal map $T_2 \to T_2\oplus T_2$

Let Φ: $C(P^2)\otimes C(P^2) \to C(P^2)\otimes C(P^2)$ be the chain equivalence which interchanges the factors $C(P^2)$. Obviously the induced homology automorphism Φ_*: $H_3(C') \to H_3(C')$ is the identity, while Φ_*: $H_3(C', T_2) \to H_3(C', T_2)$ interchanges the generators of $H_3(C', T_2) = T_2\oplus T_2$. The situation is illustrated by the diagram

$$\begin{tikzcd} T_2 \arrow[r, "\varrho'"] \arrow[d, "1={}_*"'] & T_2\oplus T_2 \arrow[d, "\Phi_*"] \\ T_2 \arrow[r, dashed, "\varrho'"] & T_2\oplus T_2 \end{tikzcd} \qquad \text{interchanges generators}$$

Any left inverse of ϱ' must be trivial on one factor of $T_2\oplus T_2$ and non-trivial on the other, and therefore cannot commute with Φ_*. Thus ϱ' has no natural left inverse.

6. Products

The maps $\mathscr{P}, \mathscr{Q}$ of Section 5 are used in this section to define a product $H^q(\mathrm{Hom}\,(C', F))\otimes H_{p+q}(C\otimes G\otimes C') \to H_p(C, G\otimes F)$, where F is an abelian coefficient group and Hom (C', F) has the standard coboundary operator δ' induced by ∂'.

Let f_q: $C'_q \to F$ be a cocycle of Hom (C', F); i.e. $f'_q\partial'(C'_{q+1}) = 0$. Define a chain complex $(\mathscr{F}, \partial_f)$ by $\mathscr{F}_q = F$, $\mathscr{F}_i = 0$, $i \neq q$, and $\partial_f = 0$ in all dimensions. Then there is defined a chain map f: $C' \to \mathscr{F}$ such that $f_i(C'_i) = 0$, $i \neq q$, and f_q is the cocycle defined above.

Now consider the chain complex $A_f = C\otimes G\otimes\mathscr{F}$ with standard tensor product differential $\partial_f^\otimes$. Filtering A_f as usual we define D_f, Δ'_f, and since C is free, $H_{p+q}(A_f) = D_f^{p,q}+\Delta_f'^{q,p}$. But clearly $D_f^{p-1,q+1} = 0 = D_f'^{q-1,p+1}$, and $D_f^{p,q} = H_{p+q}(A_f)$, $\Delta_f'^{q,p} = \eta\big(H_p(C)\otimes G\otimes F\big) \subseteq D_f^{p,q}$. Then there is a canonical isomorphism l_f: $H_{p+q}(A_f) \xrightarrow{\cong} H_p(C, G\otimes F)$, where l_f is defined as in Lemma 1.

A chain map $1\otimes f$: $C\otimes G\otimes C' \to C\otimes G\otimes\mathscr{F}$ is induced by f and hence a homomorphism from $H_{p+q}(C\otimes G\otimes C')$ to $H_{p+q}(A_f)$. Combining this homomorphism with l_f gives

$$F_*\colon\ H_{p+q}(C\otimes G\otimes C') \to H_p(C, G\otimes F).$$

Then

$$F_*\Big\{\sum_i \sum_{r,s} c_r^{(i)}\otimes g^{(i)}\otimes c_s'^{(i)}\Big\} = \Big\{\sum_i c_p^{(i)}\otimes g^{(i)}\otimes f_q c_q'^{(i)}\Big\}.$$

LEMMA 13. *The homomorphism F_* depends only on the cohomology class of f_q.*

Proof. Let f_q be a coboundary, i.e. there exists $f'_{q-1}\colon C'_{q-1}\to F$ such that $f_q = f'_{q-1}\partial'$. Let $x = \sum_i \sum_{r,s} c_r^{(i)}\otimes g^{(i)}\otimes c_s'^{(i)}$ be a cycle of $C\otimes G\otimes C'$. Then

$$\varepsilon \sum_i c_p^{(i)}\otimes g^{(i)}\otimes \partial' c_q'^{(i)} + \sum_i \partial c_{p+1}^{(i)}\otimes g^{(i)}\otimes c_q'^{(i)} = 0.$$

Hence

$$\begin{aligned} F_*\{x\} &= \Big\{\sum_i c_p^{(i)}\otimes g^{(i)}\otimes f_q c_q'^{(i)}\Big\} = \Big\{\sum_i c_p^{(i)}\otimes g^{(i)}\otimes f'_{q-1}\partial' c_q'^{(i)}\Big\} \\ &= \Big\{-\varepsilon \sum_i \partial c_{p+1}^{(i)}\otimes g^{(i)}\otimes f'_{q-1} c_{q-1}'^{(i)}\Big\} = 0. \end{aligned}$$

The lemma follows.

Using the above notation for f_q, F_* and $\{x\}$, and writing $\{f_q\}$ for the cohomology class of f_q, we have the following corollary:

COROLLARY 13.1. *There is a pairing of $H^q(C', F)$ and $H_{p+q}(C\otimes G\otimes C')$ to $H_p(C, G\otimes F)$ defined by $\{f_q\}\otimes\{x\} = F_*\{x\}$.*

Denote this pairing by $\mathscr{C}\colon H^q(C', F)\otimes H_{p+q}(C\otimes G\otimes C') \to H_p(C, G\otimes F)$.

The main theorem of this section is then

THEOREM 14. *The product pairing $\mathscr{C}$ may also be defined by mapping $H_{p+q}(C\otimes G\otimes C')$ by $\mathscr{Q}$ to*

$$\frac{H_p(C, H_q(G\otimes C'))\oplus H_q(H_p(C)\otimes G\otimes C')}{M},$$

then taking the Kronecker index of $\{f_q\}$ with the coefficient group $H_q(G\otimes C')$ of the first summand, and of $\{f_q\}$ with $H_q(H_p(C)\otimes G\otimes C')$. The resulting group maps canonically to $H_p(C, G\otimes F)$.

Proof. Since $\mathscr{Q}$ is natural the following diagram commutes, where $\mathfrak{F}_*$ is induced by $f\colon C'\to\mathscr{F}$

$$\begin{array}{ccc} H_{p+q}(C\otimes G\otimes C') & \xrightarrow{\mathscr{Q}} & \dfrac{H_p(C, H_q(G\otimes C'))\oplus H_q(H_p(C)\otimes G\otimes C')}{M} \\ \Big\downarrow{\scriptstyle (1\otimes f)_*} & & \Big\downarrow{\scriptstyle \mathfrak{F}_*} \\ H_{p+q}(C\otimes G\otimes \mathscr{F}) & \xrightarrow[\mathscr{Q}_f]{} & \underline{H_p(C, G\otimes F)\oplus H_p(C)\otimes G\otimes F} \end{array}$$

The notation Q_f, M_f is clear. M_f is generated by elements $(l_f\eta_f(x), -\lambda'_f\eta'_f(x))$, $x\in H_p(C)\otimes G\otimes F$, where clearly $\lambda'_f\eta'_f = 1$ and $l_f\eta_f = \varrho\colon H_p(C)\otimes$

$G \otimes F \to H_p(C, G \otimes F)$. Then by Lemma 10 there is an isomorphism

$$\mathscr{R}: \frac{H_p(C, G\otimes F) \oplus H_p(C) \otimes G \otimes F}{M_f} \to H_p(C, G\otimes F)$$

defined by $\mathscr{R}(a, b) = a + \varrho b$. $\mathscr{Q}_f$ is obviously an isomorphism induced by l_f, and $\mathscr{R}\mathscr{Q}_f = l_f$: $H_{p+q}(A_f) \to H_p(C, G\otimes F)$.

Then $\mathscr{C}$ is given by $F_* = l_f(1\otimes f)_* = \mathscr{R}\mathscr{Q}_f(1\otimes f)_* = \mathscr{R}\mathfrak{F}_*\mathscr{Q}$. It remains only to comment that $\mathfrak{F}_*$ is obtained by taking the Kronecker indices described, and that $\mathscr{R}$ is the canonical map to $H_p(C, G\otimes F)$.

COROLLARY 14.1. *The pairing $\mathscr{C}$ may be defined by projecting $H_{p+q}(C\otimes G\otimes C')$ by $\mathscr{P}$ to*

$$\frac{H_p(C, H_q(G\otimes C')) \oplus H_q(H_p(C, G)\otimes C')}{L'}$$

followed by taking the Kronecker index of $\{f_q\} \in H^q(C', F)$ with the coefficient group $H_q(G\otimes C')$ and with $H_q(H_p(C, G)\otimes C')$. The resulting group maps canonically to $H_p(C, G\otimes F)$.

Proof.

$$\begin{array}{ccc}
\dfrac{H_p(C, H_q(G\otimes C'))\oplus H_q(H_p(C)\otimes G\otimes C')}{M} & \xrightarrow{\mathscr{I}P} & \dfrac{H_p(C, H_q(G\otimes C'))\oplus H_q(H_p(C, G)\otimes C')}{L'} \\
\Big\downarrow \mathfrak{F}_* & & \Big\downarrow \mathfrak{F}_* \\
\dfrac{H_p(C, G\otimes F)\oplus H_p(C)\otimes G\otimes F}{M_f} & \xrightarrow{\mathscr{I}_f P_f} & \dfrac{H_p(C, G\otimes F)\oplus H_p(C, G)\otimes F}{L'_f}
\end{array}$$

The above diagram commutes. The subscript f is used to denote groups and homomorphisms associated with the complex A_f. $\mathscr{I}$ and P are defined in Corollary 11.1. Let $\mathscr{R}$ be as in Theorem 14.

Then $\mathscr{R} = \mathscr{R}'\mathscr{I}_f P_f$, where

$$\mathscr{R}': \frac{H_p(C, G\otimes F)\oplus H_p(C, G)\otimes F}{L'_f} \to H_p(C, G\otimes F)$$

is the canonical isomorphism (see Lemma 10).

Hence $\mathscr{R}\mathfrak{F}_*\mathscr{Q} = \mathscr{R}'\mathscr{I}_f P_f \mathfrak{F}_*\mathscr{Q} = \mathscr{R}'\mathfrak{F}_*\mathscr{I}P\mathscr{Q} = \mathscr{R}'\mathfrak{F}_*\mathscr{P}$ by Corollary 11.1. Then $\mathscr{R}'$ is the required canonical mapping to $H_p(C, G\otimes F)$ and the corollary is proved.

Note that the product $\mathscr{C}$ can be defined by any chain map f of C' to a complex $\mathscr{F}$ whose q-th chain group is F and whose q-th and $(q+1)$-st differentials vanish, where f is f_q in dimension q and satisfies the conditions for a chain map in other dimensions. Then the pairing is defined by mapping $H_{p+q}(C\otimes G\otimes C')$ to $H_{p+q}(C\otimes G\otimes \mathscr{F})$ followed by projecting to $\dfrac{H_{p+q}(C\otimes G\otimes\mathscr{F})}{D_f^{p-1,q+1} + D_f'^{q-1,p+1}}$ which is isomorphic to $H_p(C, G\otimes F)$. Theorem 14 and its corollary remain unaltered.

7. Cap product

Now let X, Y be topological spaces with a singular simplicial homology theory. Let $C = C(X)$, $C' = C(Y)$. Then C, C' are free so Theorem 14 and its Corollary are applicable. Combining these with the isomorphism ψ_*^{-1}: $H_{p+q}(X\times Y, G) \to H_{p+q}\big(C(X)\otimes G\otimes C(Y)\big)$ whose inverse is induced by the chain equivalence ψ: $C(X)\otimes C(Y) \to C(X\times Y)$ of Section 3 gives a product pairing $H^q(Y, F)\otimes H_{p+q}(X\times Y, G) \to H_p(X, F\otimes G)$.

Let X_0, Y_0 be subspaces of X, Y respectively (X_0, Y_0 are such that $(X\times Y_0, Y\times X_0)$ is an excisive couple in $X\times Y$) and then take $C = C(X, X_0)$, $C' = C(Y, Y_0)$. C and C' are again free, so $\mathscr{P}$, $\mathscr{Q}$ are defined. Also ψ induces a chain equivalence ψ': $C(X, X_0)\otimes C(Y, Y_0) \to C(X\times Y, X\times Y_0 \cup X_0\times Y)$. There is therefore a product pairing

$$H^q(Y, Y_0; F)\otimes H_{p+q}(X\times Y, X\times Y_0 \cup X_0\times Y; G) \to H_p(X, X_0; F\otimes G).$$

Now let $X = Y$, i.e. $C = C' = C(X)$. There is a chain map D: $C(X) \to C(X)\otimes C(X)$ (see [4], p. 365) which induces the homology homomorphism D_*: $H_n(X, G) \to H_n\big(C(X)\otimes G\otimes C(X)\big)$. Let d: $X \to X\times X$ be the diagonal map $d(x) = (x, x)$, $x\in X$. Then $\psi_* D_* = d_*$ ([4], p. 365). We do not mention explicitly the inversions $G\otimes C(X) \to C(X)\otimes G$, $G\otimes F \to F\otimes G$, e.g. the map D induces $C(X)\otimes G \to C(X)\otimes C(X)\otimes G$, and inverting $C(X)$ and G, maps to $C(X)\otimes G\otimes C(X)$, hence giving rise to D_*.

Combining D_* with F_* of Lemma 13 gives a product $H^q(X, F)\otimes H_{p+q}(X, G) \to H_p(X, F\otimes G)$.

LEMMA 15. *The above product* $H^q(X, F)\otimes H_{p+q}(X, G) \to H_p(X, F\otimes G)$ *is the classical cap product.*

Proof. The homomorphism D: $C(X) \to C(X)\otimes C(X)$ maps a singular $(p+q)$-simplex u_{p+q} to $\sum_{r+s=p+q} v_r\otimes v_s'\in C(X)\otimes C(X)$, where v_r is the first r-face of u_{p+q} and v_s' is its last s-face (see [4], p. 365). Then the product defined by D_* and F_* is by definition the cap product for singular simplicial homology ([4], p. 364).

When X is a polyhedron the cap product structure of its singular simplicial homology theory is isomorphic to the cap product defined simplicially ([4], p. 153, 154).

We have thus produced the following definition of cap product formulated solely in terms of natural homomorphisms of homology groups. Compare it with the classical definition of cap product, viz. $H^p(X, R)\otimes H^q(X, R) \xrightarrow{\pi^*} H^{p+q}\big(C(X)\otimes C(X), R\big) \xrightarrow{D^*} H^{p+q}(X, R)$, where π^* is induced by π: $\mathrm{Hom}\,\big(C(X), R\big)\otimes \mathrm{Hom}\,\big(C(X), R\big) \to \mathrm{Hom}\,\big(C(X)\otimes C(X), R\big)$ defined $\pi(d\otimes e)(f\otimes g) = d(f)\cdot g(e)$, $\big(d, e\in \mathrm{Hom}\,\big(C(X), R\big), f, g\in C(X)\big)$, D^*

is induced by D, and R is a coefficient ring. The definition is stated as a theorem.

THEOREM 15. *Let X be a topological space with the standard singular simplicial homology theory. Then the classical cap product $H^q(X, F)\otimes H_{p+q}(X, G) \to H_p(X, F\otimes G)$ may be defined by the following sequence of homomorphisms*

$$H_{p+q}(X, G) \xrightarrow{d_*} H_{p+q}(X\times X, G) \xrightarrow{\psi_*^{-1}} H_{p+q}\big(C(X)\otimes G\otimes C(X)\big) \xrightarrow{\mathscr{P}} \frac{H_p\big(X, H_q(X,G)\big)\oplus H_q\big(X, H_p(X, G)\big)}{L'} \xrightarrow{\mathscr{R}'\mathfrak{F}_*} H_p(X, F\otimes G).$$

The homomorphisms, d_*, ψ_* are defined above, while $\mathscr{P}$, $\mathfrak{F}_*$ and $\mathscr{R}'$ are as in Corollary 14.1.

This statement has been given in terms of $\mathscr{P}$ because of the symmetry of Im $\mathscr{P}$ but it could equally well be given in terms of $\mathscr{Q}$.

The relative cap product is defined in the same way. The diagonal map d: $X\to X\times X$ applied to $(X, X_0 \cup Y_0)$, where X_0, $Y_0 \subseteq X$, (X_0, Y_0) is an excisive couple in X induces $d'\colon X, X_0\cup Y_0 \to X\times X, (X_0\cup Y_0)\times (X_0\cup Y_0)$, and $X\times X$, $(X_0\cup Y_0)\times(X_0\cup Y_0)$, may in turn be mapped by inclusion to $X\times X$, $X\times(X_0\cup Y_0) = X\times X$, $X\times X_0\cup X\times Y_0$. Denote by d^X the composite of d' and the above inclusion. Then the cap product for relative homology is given by

COROLLARY 16.1. *Let X be a topological space with subspaces X_0, Y_0. Then in the above singular simplicial homology theory the cap product $H^q(X, Y_0; F)\otimes H_{p+q}(X, X_0\cup Y_0; G) \to H_p(X, X_0; F\otimes G)$ may be defined by the sequence of homomorphisms*

$$\begin{aligned} H_{p+q}(X, X_0\cup Y_0; G) &\xrightarrow{d_*^X} H_{p+q}(X\times X, X\times X_0\cup X\times Y_0; G) \\ &\xrightarrow{\psi_*'^{-1}} H_{p+q}\big(C(X, X_0)\otimes G\otimes C(X, Y_0)\big) \\ &\xrightarrow{\mathscr{P}} \frac{H_p\big(X, X_0, H_q(X, Y_0; G)\big)\oplus\big(H_q(X, Y_0; H_p(X, X_0; G)\big)}{L'} \\ &\xrightarrow{\mathscr{R}'\mathfrak{F}_*} H_p(X, X_0; F\otimes G)\colon \end{aligned}$$

The homomorphisms d_*^X and ψ_*' are defined above, while $\mathscr{P}$, $\mathfrak{F}_*$ and $\mathscr{R}'$ are as in Corollary 14.1 *with* $C = C(X, X_0)$, $C' = C(X, Y_0)$.

There is a corresponding situation in cohomology which will be described separately.

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