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Fréchet's equation and Hyers theorem on noncommutative semigroups

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Abstract. In this paper the Fréchet's functional equation on noncommutative semigroups is dealt with it is shown that under some conditions the general solution can be described by multiadditive symmetric functions, just as in the commutative case. The study of higher order differences in the noncommutative case leads to a generalization of the theorem of Hyers on the stability of the linear functional equation, in the case of amenable semigroups.

1. Introduction. Let G be a semigroup with identity e and let H be a linear space over the rationals. For any y in G we define the left and right translation operators ${}_yT$ and T_y, and the left and right difference operators ${}_y\Delta$ and Δ_y, as follows: for a function $f\colon G \to H$ let

$${}_yTf(x) = f(yx), \qquad T_y f(x) = f(xy)$$

and

$${}_y\Delta = {}_yT - I, \qquad \Delta_y = T_y - I,$$

where x ranges over G and I denotes the identity operator: $I = {}_eT = T_e$. For products of difference operators we use the following notation: ${}_{y_1}\Delta \cdot \ldots \cdot {}_{y_n}\Delta$ is denoted by ${}_{y_1,\ldots,y_n}^{\;n}\Delta$, and $\Delta_{y_n} \cdot \ldots \cdot \Delta_{y_1}$ is denoted by $\Delta^n_{y_1,\ldots,y_n}$.

Now we recall the notion of invariant mean and amenability. If $B(G)$ denotes the Banach space of all bounded complex valued functions on a semigroup G (with sup-norm), then a positive linear functional $M\colon B(G) \to C$ is called and *invariant mean on* G if it is translation invariant (i.e., $M({}_yTf) = M(T_y f) = M(f)$ for all f in $B(G)$ and y in G) and normalized ($M(1) = 1$). We write M_x instead of M when we wish to indicate the variable. If an invariant mean exists, we call G amenable. For example, every commutative semigroup is amenable. For more information about invariant means and amenability see e.g. [7].

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2. Fréchet's functional equation. In this section G denotes a fixed semigroup with identity e, H denotes a fixed linear space over the rationals and $n \geqslant 0$ is a fixed integer.

Fréchet's functional equation

$$\Delta^{n+1}_{y_1,\dots,y_{n+1}} f(x) = 0 \tag{1}$$

for all $x, y_1, \dots, y_{n+1}$ in G, where $f\colon G \to H$ is an unknown function, has been dealt with by many authors (see e.g. [1]–[3], [6], [8], [11], [12], [14], [15]). In the case when G is Abelian, it has been proved that f is a solution of (1) if and only if it can be expressed as the sum of the diagonalizations of multiadditive symmetric functions of at most n-th degree. (For an integer $k \geqslant 1$, a function $A\colon G^k \to H$ is called *multiadditive*, or more precisely, *k-additive*, if it is a homomorphism of G into the additive group of H in each of its variables. Here the number k is the degree of A. The diagonalization $D(A)$ of A is defined by $D(A)(x) = A(x, \dots, x)$. In this connection, the constant functions are called *0-additive*.) The aim of this section is to show that this result remains valid also if G is noncommutative under the additional assumption

$$f(txy) = f(tyx) \tag{2}$$

for all x, y in G.

LEMMA 2.1. *Let $f\colon G \to H$ be arbitrary; then*

$$_{y_1,\dots,y_{n+1}}^{\quad n+1}\Delta f(x) = \Delta^{n+1}_{y_1,\dots,y_{n+1}} (T_x f)(e)$$

for all $x, y_1, \dots, y_{n+1}$ in G.

Proof. This is verified by induction on n. For $n = 0$ we have

$$\begin{aligned} {}_{y_1}\Delta f(x) &= ({}_{y_1}T - I) f(x) = f(y_1 x) - f(x), \\ \Delta_{y_1} (T_x f)(e) &= (T_{y_1} - I)\, T_x f(e) = T_x f(y_1) - T_x f(e) = f(y_1 x) - f(x). \end{aligned}$$

Suppose that the statement holds true for n; then we get

$$\begin{aligned} {}_{y_1,\dots,y_n,y_{n+1}}^{\qquad n+1}\Delta f(x) &= {}_{y_2,\dots,y_{n+1}}^{\quad n}\Delta f(y_1 x) - {}_{y_2,\dots,y_{n+1}}^{\quad n}\Delta f(x) \\ &= [\Delta^n_{y_2,\dots,y_{n+1}} (T_{y_1 x} f) - \Delta^n_{y_2,\dots,y_{n+1}} (T_x f)](e) \\ &= [\Delta^n_{y_2,\dots,y_{n+1}} (T_{y_1 x} - T_x) f(e)] \\ &= [\Delta^n_{y_2,\dots,y_{n+1}} (T_{y_1} - I)\, T_x f](e) = \Delta^{n+1}_{y_1,\dots,y_{n+1}} (T_x f)(e) \end{aligned}$$

and the lemma is proved.

LEMMA 2.2. *Let $f\colon G\to H$ be arbitrary. Then*

$$(3)\qquad {}_{y_1,\ldots,y_{n+1}}^{\;\;\;n+1}\Delta f(x) = \Delta^{n+1}_{x,y_1,\ldots,y_n} f(y_{n+1}) + \Delta^{n+1}_{y_1,\ldots,y_{n+1}} f(e) - \Delta^{n+1}_{x,y_1,\ldots,y_n} f(e)$$

for all $x, y_1, \ldots, y_{n+1}$ in G.

Proof. The proof is a matter of a simple calculus with use of Lemma 2.1. The right-hand side of (3) is

$$\begin{aligned}[][T_{y_{n+1}}(T_{y_n}-I)\ldots(T_{y_1}-I)(T_x-I)+(T_{y_{n+1}}-I)\ldots(T_{y_1}-I)-\\ -(T_{y_n}-I)\ldots(T_{y_1}-I)(T_x-I)]f(e)\\ =[(T_{y_{n+1}}-I)(T_{y_n}-I)\ldots(T_{y_1}-I)(T_x-I)+\\ +(T_{y_{n+1}}-I)\ldots(T_{y_1}-I)]f(e) = \Delta^{n+1}_{y_1,\ldots,y_{n+1}}(T_x f)(e),\end{aligned}$$

which is just the left-hand side, by Lemma 2.1.

LEMMA 2.3. *Let $A\colon G^n\to H$ be an n-additive symmetric function. Then*

$$\text{(i)}\qquad A(xy, \ldots, xy) = \sum_{k=0}^{n}\binom{n}{k} A(\underbrace{x, \ldots, x}_{k\text{-times}}, \underbrace{y, \ldots, y}_{n-k\text{-times}}),$$

$$\text{(ii)}\qquad \Delta^n_{y_1,\ldots,y_n} DA(x) = n!\, A(y_1, \ldots, y_n),$$

for all $x, y_1, \ldots, y_n, y$ in G.

Proof. Both statements can be proved just as in the commutative case (see e.g. [3]).

LEMMA 2.4. *Let $f\colon G\to H$ be a function satisfying* (1). *Then*

$$\delta_{y_1}\cdot\ldots\cdot\delta_{y_{n+1}} f(x) = 0$$

for all $x, y_1, \ldots, y_{n+1}$ in G, where each δ_y denotes either ${}_y\Delta$ or Δ_y, independently.

Proof. First of all we remark that the operators ${}_y\Delta$ and Δ_z commute for any y, z in G; this follows directly from their definitions. This means that it is enough to show that

$${}_{y_1,\ldots,y_k}^{\;\;\;k}\Delta\cdot\Delta^{n+1-k}_{y_{k+1},\ldots,y_{n+1}} f(x) = 0$$

for all $x, y_1, \ldots, y_{n+1}$ in G and for any integer k with $1 \leqslant k \leqslant n+1$. By Lemma 2.2 we have

$$\begin{aligned}{}_{y_1,\ldots,y_k}^{\;\;\;k}\Delta(\Delta^{n+1-k}_{y_{k+1},\ldots,y_{n+1}} f)(x) = \Delta^{n+1}_{y_{k+1},\ldots,y_{n+1},x,y_1,\ldots,y_{k-1}} f(y_k)+\\ +\Delta^{n+1}_{y_{k+1},\ldots,y_{n+1},y_1,\ldots,y_k} f(e) - \Delta^{n+1}_{y_{k+1},\ldots,y_{n+1},x,y_1,\ldots,y_{k-1}} f(e) = 0\end{aligned}$$

and this proves the lemma.

THEOREM 2.5. *Let $f\colon G \to H$ be a function. Then the system of functional equations* (1), (2) *is satisfied if and only if there exist k-additive symmetric functions $A_k\colon G^k \to H$ $(k = 0, \ldots, n)$ such that*

$$f(x) = DA_n(x) + \ldots + DA_1(x) + A_0$$

for all x in G.

Proof. The *if* part is trivial by Lemma 2.3. Now suppose that (1) holds. Then obviously $x \to \Delta^n_{y_1,\ldots,y_n} f(x)$ is constant for all fixed $y_1, \ldots, y_n$ in G. Let

$$A_n(y_1, \ldots, y_n) = \frac{1}{n!}\Delta^n_{y_1,\ldots,y_n} f(e)$$

for all $y_1, \ldots, y_n$ in G. By (2) it follows that this function is symmetric. We claim that it is also n-additive. Indeed, we have

$$\begin{aligned}
&A_n(y_1\hat{y}_1, y_2, \ldots, y_n) - A_n(y_1, \ldots, y_n) - A_n(\hat{y}_1, \ldots, y_n)\\
&\quad = \frac{1}{n!}[\Delta^{n-1}_{y_2,\ldots,y_n}(T_{y_1\hat{y}_1} - I) - \Delta^{n-1}_{y_2,\ldots,y_n}(T_{y_1} - I) - \Delta^{n-1}_{y_2,\ldots,y_n}(T_{\hat{y}_1} - I)]\, f(e)\\
&\quad = \frac{1}{n!}[\Delta^{n-1}_{y_2,\ldots,y_n}(T_{y_1\hat{y}_1} - T_{y_1} - T_{\hat{y}_1} + I)]\, f(e)\\
&\quad = \frac{1}{n!}[\Delta^{n-1}_{y_2,\ldots,y_n}(T_{y_1} - I)(T_{\hat{y}_1} - I)]\, f(e) = \frac{1}{n!}\Delta^{n+1}_{y_1,\hat{y}_1,y_2,\ldots,y_n} f(e) = 0
\end{aligned}$$

for all $y_1, \hat{y}_1, y_2, \ldots, y_n$ in G. Now we define $q = f - DA_n$, and we get by Lemma 2.3

$$\begin{aligned}
\Delta^n_{y_1,\ldots,y_n} g(x) &= \Delta^n_{y_1,\ldots,y_n} f(x) - \Delta^n_{y_1,\ldots,y_n} DA_n(x)\\
&= \Delta^n_{y_1,\ldots,y_n} f(x) - n!\, A_n(y_1, \ldots, y_n) = 0,
\end{aligned}$$

which yields by induction on n the statement of the theorem.

3. The theorem of Hyers. Hyers' theorem in its original form [9] states that any approximately additive function can be approximated by an additive function. More precisely, if f is a real function for which the expression $f(x+y) - f(x) - f(y)$ is bounded, then there exists an additive function A such that $f - A$ is bounded. This result has been generalized by several authors in several directions (see e.g. [2], [4], [10], [13], [16]–[20]). The extension of this result to Abelian groups, and even to Abelian semigroups, is obvious. The question whether the result remains valid in the noncommutative case was answered by Forti [5], who has shown that on the free group on two generators the statement of Hyers' theorem is false. On the other hand, we showed at the 22nd International Symposium on Functional Equations in Oberwolfach, West Germany, 1984, that the theorem remains valid, if the group (or semigroup) in question admits an invariant mean. By using the same technique, in [16] we proved the following theorem, which is also due to Hyers [10]: Let G be an Abelian semigroup with identity, f a complex-valued function on G, for which the function $(x, y) \to \Delta_{y,\dots,y}^{n+1} f(x)$ is bounded. Then $f-P$ is bounded for some complex-valued function P on G satisfying (1). The aim of this section is to prove the analogue of this result in the more general case, when G is merely an amenable semigroup with identity. We denote by C the set of complex numbers.

THEOREM 3.1. *Let G be an amenable semigroup with identity and $f\colon G \to C$ a function for which the function $(x, y_1, \dots, y_{n+1}) \to \Delta_{y_1,\dots,y_{n+1}}^{n+1} f(x)$ is bounded. Then there exists a function $P\colon G \to C$ satisfying* (1) *for which $f-P$ is bounded.*

Proof. Let M denote any invariant mean on G. From our assumption on f and from (3) it follows that also the function $(x, y_1, \dots, y_{n+1}) \to {}_{y_1,\dots,y_{n+1}}^{\;\;n+1}\Delta f(x)$ is bounded. Further, we have for all $y_1, \dots, y_{n+1}$ in G:

$$\begin{aligned} M_x\left[{}_{y_1,\dots,y_{n+1}}^{\;\;n+1}\Delta f(x)\right] &= M_x\left[{}_{y_2,\dots,y_{n+1}}^{\;\;n}\Delta f(y_1 x) - {}_{y_2,\dots,y_{n+1}}^{\;\;n}\Delta f(x)\right] \\ &= M_x\left[{}_{y_1 x, y_2,\dots,y_{n+1}}^{\;\;n+1}\Delta f(e) - {}_{x, y_2,\dots,y_{n+1}}^{\;\;n+1}\Delta f(e)\right] = 0, \end{aligned}$$

because M is invariant. From this we obtain the identity

$$M_{y_1} \dots M_{y_k}\left({}_{u_1,\dots,u_{n+1}}^{\;\;n+1}\Delta \cdot \Delta_{y_1,\dots,y_k}^{k} f\right) = (-1)^k\, {}_{u_1,\dots,u_{n+1}}^{\;\;n+1}\Delta f \tag{4}$$

for all $y_1, \dots, y_k, u_1, \dots, u_{n+1}$ in G, by using (2) and induction on n. Now we define

$$f_0(x) = (-1)^n M_{y_1} \dots M_{y_n}\left(\Delta_{y_1,\dots,y_n,x}^{n+1} f(e)\right)$$

for any x in G. Then obviously f_0 is bounded. Further

$$\begin{aligned} &{}_{u_1,\dots,u_{n+1}}^{\;\;n+1}\Delta (f-f_0)(x) = {}_{u_1,\dots,u_{n+1}}^{\;\;n+1}\Delta f(x) + \\ &\quad + (-1)^{n+1}\, {}_{u_1,\dots,u_{n+1}}^{\;\;n+1}\Delta M_{y_1} \dots M_{y_n}\left[\Delta_{y_1,\dots,y_n}^{n} f(x) - \Delta_{y_1,\dots,y_n}^{n} f(e)\right] \\ &= {}_{u_1,\dots,u_{n+1}}^{\;\;n+1}\Delta f(x) + (-1)^{n+1} M_{y_1} \dots M_{y_n}\left[{}_{u_1,\dots,u_{n+1}}^{\;\;n+1}\Delta (\Delta_{y_1,\dots,y_n}^{n} f)(x)\right] = 0 \end{aligned}$$

by (4), and if we let $P = f - f_0$, then the results of the previous section – which obviously remain valid when we interchange the roles of the left and right difference operators – imply our statement.

Remark 3.2. The proof of the theorem and the identity

$$\Delta^{n+1}_{y_1,\ldots,y_{n+1}} f(x) = \Delta^{n+1}_{xy_{n+1},y_1,\ldots,y_n} f(e) - \Delta^{n+1}_{x,y_1,\ldots,y_n} f(e)$$

show that, for the validity of 3.1, it is enough to assume that the function $(y_1, \ldots, y_{n+1}) \to \Delta^{n+1}_{y_1,\ldots,y_{n+1}} f(e)$ is bounded.

Remark 3.3. Combining the assertions of 2.5 and 3.1 we have the following statement: if G is any amenable semigroup with identity and $f\colon G \to C$ is a function satisfying (2), then the function $(x, y_1, \ldots, y_{n+1}) \to \Delta^{n+1}_{y_1,\ldots,y_{n+1}} f(x)$ is bounded if and only if there exist k-additive symmetric functions $A_k\colon G^k \to H$ $(k = 1, \ldots, n)$ and a bounded function $f_0\colon G \to C$ such that

$$f(x) = DA_n(x) + \ldots + DA_1(x) + f_0(x)$$

for all x in G. Here the functions A_k $(k = 1, \ldots, n)$ are unique.

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