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Arithmetic mean function of the real part of entire Dirichlet series I

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Abstract. Let E be the set of mappings $f\colon \boldsymbol{C}\to\boldsymbol{C}$ ($\boldsymbol{C}$ is the complex plane) such that the image under f of a point $s\in\boldsymbol{C}$ is

$$f(s) = \sum_{n\in N} a_n e^{s\lambda_n} \quad \text{with} \quad \limsup_{n\to+\infty} \frac{\log n}{\lambda_n} = D\in R_+\cup\{0\}, \text{ and } \sigma_c^f = +\infty.$$

Then f is an entire function and is bounded on each vertical line $\mathrm{Re}(s) = \sigma_0$.

In this paper we have defined the arithmetic mean function A and the generalized arithmetic mean function J_r, $r\in\boldsymbol{R}$, of $\mathrm{Re}(f)$, respectively, as

$$A(\sigma,f) = \lim_{T\to+\infty}\frac{1}{2T}\int_{-T}^{T}|\mathrm{Re}(f(\sigma+it))|\,dt, \quad \forall\sigma<\sigma_c^f,$$

and

$$J_r(\sigma,f) = \lim_{T\to+\infty}\frac{1}{2Te^{r\sigma}}\int_0^{\sigma}\int_{-T}^{T}|\mathrm{Re}(f(x+it))|\,e^{rx}\,dx\,dt, \quad \forall\sigma<\sigma_c^f,$$

and have studied a few properties of the functions A and J_r. Beside establishing the convexity of $\log J_r$, we have shown that if f is of Ritt order ϱ and lower order λ, then

$$\frac{\varrho}{\lambda} = \lim_{\sigma\to+\infty}\frac{\sup}{\inf}\frac{\log_2 F(\sigma,f)}{\sigma} = \lim_{\sigma\to+\infty}\frac{\sup}{\inf}\frac{\log_2 A(\sigma,f)}{\sigma} = \lim_{\sigma\to+\infty}\frac{\sup}{\inf}\frac{\log_2 J_r(\sigma,f)}{\sigma},$$

where $\log_2 x = \log\log x$, and $F(\sigma,f) = \sup_{t\in\boldsymbol{R}}\{|\mathrm{Re}(f(\sigma+it))|\}$, and if $\varrho\in R_+$ and f is of type T and lower type t, then

$$\frac{T}{t} = \lim_{\sigma\to+\infty}\frac{\sup}{\inf}\frac{\log F(\sigma,f)}{e^{\varrho\sigma}} = \lim_{\sigma\to+\infty}\frac{\sup}{\inf}\frac{\log A(\sigma,f)}{e^{\varrho\sigma}} = \lim_{\sigma\to+\infty}\frac{\sup}{\inf}\frac{\log J_r(\sigma,f)}{e^{\varrho\sigma}}.$$

Finally, we have proved that

$$\lim_{\sigma\to+\infty}\sup[\inf]\frac{1}{\sigma}\log(A(\sigma,f)/J_r(\sigma,f)) = \varrho[\lambda].$$

1. Let E be the set of mappings $f\colon C\to C$ (C is the complex plane) such that the image under f of a point $s\in C$ is

$$f(s) = \sum_{n\in N} a_n e^{s\lambda_n} \quad \text{with} \quad \limsup_{n\to+\infty} \frac{\log n}{\lambda_n} = D\in R_+\cup\{0\}$$

(R_+ is the set of positive reals), and $\sigma_c^f = +\infty$ (σ_c^f is the abscissa of convergence of the Dirichlet series defining f); N is the set of natural numbers $0, 1, 2, \ldots$, $\langle a_n \mid n\in N\rangle$ is a sequence in C, $s = \sigma+it$, σ, $t\in R$ (R is the field of reals), and $\langle \lambda_n \mid n\in N\rangle$ is a strictly increasing unbounded sequence of non-negative reals. Since the Dirichlet series defining f converges for each $s\in C$, f is an entire function. Also, since $D\in R_+\cup\{0\}$, we have ([1], p. 168), $\sigma_a^f = +\infty$ (σ_a^f is the abscissa of absolute convergence of the Dirichlet series defining f) and we conclude that f is bounded on each vertical line $\mathrm{Re}(s) = \sigma_0$.

Let

$$M(\sigma, f) = \sup_{t\in R}\{|f(\sigma+it)|\}, \qquad \forall \sigma < \sigma_c^f, \tag{1.1}$$

be the maximum modulus of an entire function $f\in E$ on any vertical line $\mathrm{Re}(s) = \sigma$,

$$\mu(\sigma, f) = \max_{n\in N}\{|a_n| e^{\sigma\lambda_n}\}, \qquad \forall \sigma < \sigma_c^f, \tag{1.2}$$

be the maximum term, for $\mathrm{Re}(s) = \sigma$, in the Dirichlet series defining f,

$$\nu(\sigma, f) = \max_{n\in N}\{n \mid \mu(\sigma, f) = |a_n| e^{\sigma\lambda_n}\}, \qquad \forall \sigma < \sigma_c^f, \tag{1.3}$$

be the rank of the maximum term, and

$$F(\sigma, f) = \sup_{t\in R}\{|\mathrm{Re}(f(\sigma+it))|\}, \qquad \forall \sigma < \sigma_c^f, \tag{1.4}$$

be the maximum modulus of $\mathrm{Re}(f)$ on the vertical line $\mathrm{Re}(s) = \sigma$.

We define the arithmetic mean function A of $\mathrm{Re}(f)$, for any $f\in E$, as

$$A(\sigma, f) = \lim_{T\to+\infty} \frac{1}{2T} \int_{-T}^{T} |\mathrm{Re}(f(\sigma+it))|\, dt, \qquad \forall \sigma < \sigma_c^f, \tag{1.5}$$

and the generalized arithmetic mean function J_r, for any $r\in R$, of $\mathrm{Re}(f)$ as

$$J_r(\sigma, f) = \lim_{T\to+\infty} \frac{1}{2Te^{r\sigma}} \int_0^{\sigma} \int_{-T}^{T} |\mathrm{Re}(f(x+it))| e^{rx}\, dx\, dt, \qquad \forall \sigma < \sigma_c^f, \tag{1.6}$$

and investigate a few properties of the functions A and J_r in this paper. Most of the time, however, we shall study J_r since it is a generalization of A.

2. We first study the convexity of J_r in

THEOREM 1. *J_r is a steadily increasing function, and $\log J_r$ is a convex function of σ.*

Proof. We adopt the method of Titchmarsh ([5], p. 174) to prove this theorem. Let $\sigma_1, \sigma_2, \sigma_3 \in \boldsymbol{R}$ be such that $0 < \sigma_1 < \sigma_2 < \sigma_3$. Also let $g: \boldsymbol{R} \to \boldsymbol{C}$ and $h: \boldsymbol{C} \to \boldsymbol{C}$ be two functions defined, respectively, as

$$g(t_2) = \frac{|\mathrm{Re}(f(\sigma_2+it_2))|}{\log|f(\sigma_2+it_2)|}, \qquad \forall t_2 \in \boldsymbol{R},$$

and

$$h(s) = \lim_{T\to+\infty} \frac{1}{2Te^{r\sigma}} \int_0^{\sigma} \int_{-T}^{T} \log|f(s+it_2)|\, g(t_2) e^{rx}\, dx\, dt_2, \qquad \forall s \in C.$$

It is clear from the definition of h that it is analytic in the half-plane $\mathrm{Re}(s) \leqslant \sigma_3$, and that $|h|$ attains its supremum on the boundary $\mathrm{Re}(s) = \sigma_3$, say at $s = \sigma_3 + it_3$. Hence

$$J_r(\sigma_2, f) = h(\sigma_2) \leqslant h(\sigma_3+it_3) \leqslant J_r(\sigma_3, f),$$

which shows that J_r increases steadily with σ.

We now choose β so that

$$e^{\beta\sigma_1} J_r(\sigma_1, f) = e^{\beta\sigma_3} J_r(\sigma_3, f). \tag{2.1}$$

Then

$$e^{\beta\sigma_2} J_r(\sigma_2, f) = e^{\beta\sigma_2} h(\sigma_2) \leqslant \sup_{\sigma_1 \leqslant \mathrm{Re}(s) \leqslant \sigma_3} |e^{\beta s} h(s)| \leqslant e^{\beta\sigma_1} h(\sigma_1) \leqslant e^{\beta\sigma_1} J_r(\sigma_1, f),$$

whence

$$e^{\beta\sigma_2} J_r(\sigma_2, f) \leqslant e^{\beta\sigma_1} J_r(\sigma_1, f). \tag{2.2}$$

Putting the value of β from (2.1) in (2.2), we get

$$\log J_r(\sigma_2, f) \leqslant \frac{\sigma_3-\sigma_2}{\sigma_3-\sigma_1} \log J_r(\sigma_1, f) + \frac{\sigma_2-\sigma_1}{\sigma_3-\sigma_1} \log J_r(\sigma_3, f),$$

which proves the convexity of $\log J_r$.

Then we establish

THEOREM 2. *For every entire function $f \in E$ of Ritt order $\varrho \in R_+^* \cup \{0\}$ (R_+^* is the set of extended positive reals) and lower order $\lambda \in R_+^* \cup \{0\}$,*

(2.3)

$$\begin{matrix}\varrho\\ \lambda\end{matrix} = \lim_{\sigma\to+\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log_2 F(\sigma, f)}{\sigma} = \lim_{\sigma\to+\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log_2 A(\sigma, f)}{\sigma} = \lim_{\sigma\to+\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log_2 J_r(\sigma, f)}{\sigma},$$

where $\log_2 x = \log\log x$. *Moreover, if* $\varrho \in R_+$ *and* f *is of type* $T \in R_+^* \cup \{0\}$ *and lower type* $t \in R_+^* \cup \{0\}$, *then*

(2.4)
$$\frac{T}{t} = \lim_{\sigma\to+\infty} \frac{\sup}{\inf} \frac{\log F(\sigma,f)}{e^{\varrho\sigma}} = \lim_{\sigma\to+\infty} \frac{\sup}{\inf} \frac{\log A(\sigma,f)}{e^{\varrho\sigma}} = \lim_{\sigma\to+\infty} \frac{\sup}{\inf} \frac{\log J_r(\sigma,f)}{e^{\varrho\sigma}}.$$

Proof. We have, from ([1], p. 170)

$$(2.5)\qquad a_n e^{\sigma\lambda_n} = \lim_{T\to+\infty} \frac{1}{T} \int_{t_0}^{T} e^{-it\lambda_n} \overline{f(s)}\, dt, \qquad \forall n \in N \text{ and } \sigma < \sigma_c^f.$$

But

$$(2.6)\qquad \lim_{T\to+\infty} \frac{1}{T} \int_{t_0}^{T} e^{-it\lambda_n} \overline{f(s)}\, dt = \lim_{T\to+\infty} \frac{1}{T} \int_{t_0}^{T} \sum_{m\in N} \bar{a}_m e^{(\sigma-it)\lambda_m} e^{-it\lambda_n} dt$$
$$= \lim_{T\to+\infty} \frac{1}{T} \sum_{m\in N} \int_{t_0}^{T} \bar{a}_m e^{-it(\lambda_m+\lambda_n)} e^{\sigma\lambda_m} dt = 0,$$

the term by term integration being valid since the series $\sum_{m\in N} \bar{a}_m e^{(\sigma-it)\lambda_m}$ converges uniformly. By the addition of (2.5) and (2.6), it follows that

$$a_n e^{\sigma\lambda_n} = \lim_{T\to+\infty} \frac{1}{T} \int_{t_0}^{T} 2\mathrm{Re}\big(f(s)\big) e^{-it\lambda_n} dt.$$

Therefore

$$|a_n| e^{\sigma\lambda_n} \leqslant \lim_{T\to+\infty} \frac{1}{T} \int_{-T}^{T} 2\big|\mathrm{Re}\big(f(s)\big)\big| dt.$$

Hence

$$(2.7)\qquad \mu(\sigma,f) \leqslant 4A(\sigma,f) \leqslant 4F(\sigma,f) \leqslant 4\, M(\sigma,f).$$

Since ([2], Theorems 2.7 and 2.8), for every entire function $f \in E$ of Ritt order $\varrho \in R_+^* \cup \{0\}$ and lower order $\lambda \in R_+^* \cup \{0\}$,

(2.8)
$$\frac{\varrho}{\lambda} = \lim_{\sigma\to+\infty} \frac{\sup}{\inf} \frac{\log_2 M(\sigma,f)}{\sigma} = \lim_{\sigma\to+\infty} \frac{\sup}{\inf} \frac{\log_2 \mu(\sigma,f)}{\sigma} = \lim_{\sigma\to+\infty} \frac{\sup}{\inf} \frac{\log \lambda_{\nu(\sigma,f)}}{\sigma},$$

and ([4], Theorem 5) for any entire function $f \in E$ of Ritt order $\varrho \in R_+ \cup \{0\}$, as $\sigma \to +\infty$,

$$(2.9)\qquad \log M(\sigma,f) \sim \log \mu(\sigma,f),$$

the first two equalities in (2.3) and (2.4) now follow from (2.7) in view of (2.8) and (2.9), respectively.

In order to establish the last equalities in (2.3) and (2.4), we observe, by (1.5) and (1.6), that

$$J_r(\sigma,f) = \frac{1}{e^{r\sigma}}\int_0^{\sigma} A(x,f)e^{rx}dx \tag{2.10}$$

$$\leqslant A(\sigma,f)\frac{1}{r}(1-e^{-r\sigma}). \tag{2.11}$$

Hence

$$\lim_{\sigma\to+\infty}\begin{matrix}\sup\\ \inf\end{matrix}\frac{\log\log J_r(\sigma,f)}{\sigma} \leqslant \lim_{\sigma\to+\infty}\begin{matrix}\sup\\ \inf\end{matrix}\frac{\log\log A(\sigma,f)}{\sigma}, \tag{2.12}$$

and

$$\lim_{\sigma\to+\infty}\begin{matrix}\sup\\ \inf\end{matrix}\frac{\log J_r(\sigma,f)}{e^{\varrho\sigma}} \leqslant \lim_{\sigma\to+\infty}\begin{matrix}\sup\\ \inf\end{matrix}\frac{\log A(\sigma,f)}{e^{\varrho\sigma}}. \tag{2.13}$$

Again, from (2.10), for any $\varepsilon\epsilon R_+$,

$$J_r(\sigma+\varepsilon,f) \geqslant \frac{1}{e^{r(\sigma+\varepsilon)}}\int_{\sigma}^{\sigma+\varepsilon} A(x,f)e^{rx}dx \geqslant A(\sigma,f)\frac{1}{r}(1-e^{-r\varepsilon}). \tag{2.14}$$

Hence

$$\lim_{\sigma\to+\infty}\begin{matrix}\sup\\ \inf\end{matrix}\frac{\log\log J_r(\sigma,f)}{\sigma} \geqslant \lim_{\sigma\to+\infty}\begin{matrix}\sup\\ \inf\end{matrix}\frac{\log\log A(\sigma,f)}{\sigma} \tag{2.15}$$

and

$$\lim_{\sigma\to+\infty}\begin{matrix}\sup\\ \inf\end{matrix}\frac{\log J_r(\sigma,f)}{e^{\varrho\sigma}} \geqslant \frac{1}{e^{\varrho\varepsilon}}\lim_{\sigma\to+\infty}\begin{matrix}\sup\\ \inf\end{matrix}\frac{\log A(\sigma,f)}{e^{\varrho\sigma}}. \tag{2.16}$$

Since the left-hand side of (2.16) is independent of ε, putting $\varepsilon\to 0$, we, therefore, get

$$\lim_{\sigma\to+\infty}\begin{matrix}\sup\\ \inf\end{matrix}\frac{\log J_r(\sigma,f)}{e^{\varrho\sigma}} \geqslant \lim_{\sigma\to+\infty}\begin{matrix}\sup\\ \inf\end{matrix}\frac{\log A(\sigma,f)}{e^{\varrho\sigma}}. \tag{2.17}$$

Combining (2.12) with (2.15), and (2.13) with (2.17) we get the desired result.

Remark. The result in the first equality of (2.3) is known ([3], Formula (2.8)). But we have given here an alternative and shorter proof of it.

COROLLARY 1. *For every entire function $f\epsilon E$ of Ritt order $\varrho\epsilon R_+\cup\{0\}$, as $\sigma\to+\infty$,*

$$\log\mu(\sigma,f) \sim \log A(\sigma,f) \sim \log F(\sigma,f) \sim \log M(\sigma,f). \tag{2.18}$$

The result in (2.18) is obvious from (2.7) and (2.9).

Finally we show that

THEOREM 3. *For every entire function* $f \in E$ *of Ritt order* $\varrho \in R_+^* \cup \{0\}$ *and lower order* $\lambda \in R_+^* \cup \{0\}$,

(2.19) $$\lim_{\sigma\to+\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log\big(A(\sigma,f)/J_r(\sigma,f)\big)}{\sigma} = \begin{matrix}\varrho\\ \lambda\end{matrix}.$$

In order to prove this theorem we need the following lemma:

LEMMA. *For every entire function* $f \in E$, $e^{r\sigma}A(\sigma,f)$ *is an increasing convex function of* $e^{r\sigma}J_r(\sigma,f)$.

Proof. We have, from the definitions of A and J_r,

$$\frac{d\big(e^{r\sigma}A(\sigma,f)\big)}{d\big(e^{r}J_r(\sigma,f)\big)} = r + \frac{d}{d\sigma}\big(\log A(\sigma,f)\big),$$

whence follows the lemma, since by Theorem 1 it follows that $\log A$ is an increasing convex function of σ.

Proof of the theorem. We have

$$\frac{d}{d\sigma}\big(r\sigma+\log J_r(\sigma,f)\big) = \frac{A(\sigma,f)}{J_r(\sigma,f)}.$$

Therefore, for arbitrary $\sigma, \sigma_0, \sigma > \sigma_0$,

$$r(\sigma-\sigma_0)+\log J_r(\sigma,f)-\log J_r(\sigma_0,f) = \int_{\sigma_0}^{\sigma} \frac{A(x,f)}{J_r(x,f)}\,dx,$$

or

(2.20) $$\log J_r(\sigma,f) = \log J_r(\sigma_0,f)+ \int_{\sigma_0}^{\sigma} m_r(x,f)\,dx,$$

where

(2.21) $$m_r(x,f) = \frac{A(x,f)}{J_r(x,f)} - r,$$

increases with σ, by virtue of the lemma. Thus, for $\sigma > \sigma_0$, (2.20) gives

$$\log J_r(\sigma,f)-\log J_r(\sigma_0,f) \leqslant (\sigma-\sigma_0)\,m_r(\sigma,f).$$

Therefore

$$\lim_{\sigma\to+\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log\log J_r(\sigma,f)}{\sigma} \leqslant \lim_{\sigma\to+\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log m_r(\sigma,f)}{\sigma},$$

or, using (2.3),

(2.22) $$\begin{matrix}\varrho\\ \lambda\end{matrix} \leqslant \lim_{\sigma\to+\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log m_r(\sigma,f)}{\sigma}.$$

Again, from (2.20), we get, for any $h \in R_+$,

$$\log J_r(\sigma+h, f) - \log J_r(\sigma_0, f) \geqslant \int_{\sigma}^{\sigma+h} m_r(x, f)\, dx \geqslant h m_r(\sigma, f),$$

which gives, in view of (2.3),

$$\frac{\varrho}{\lambda} \geqslant \lim_{\sigma\to+\infty} {\sup \atop \inf} \frac{\log m_r(\sigma, f)}{\sigma}. \tag{2.23}$$

Combining (2.22) and (2.23), we get

$$\lim_{\sigma\to+\infty} {\sup \atop \inf} \frac{\log m_r(\sigma, f)}{\sigma} = \frac{\varrho}{\lambda}. \tag{2.24}$$

The theorem now follows from (2.21) and (2.24).

The following corollary is immediate from Theorem 3.

COROLLARY 2. *For every entire function $f \in E$ of Ritt order $\varrho \in R_+ \cup \{0\}$, as $\sigma \to +\infty$,*

$$\log J_r(\sigma, f) \sim \log A(\sigma, f). \tag{2.25}$$

Remark. Combining Corollaries 1 and 2, we find that for every entire function $f \in E$ of Ritt order $\varrho \in R_+ \cup \{0\}$, as $\sigma \to +\infty$,

$$\log \mu(\sigma, f) \sim \log A(\sigma, f) \sim \log J_r(\sigma, f). \tag{2.26}$$

References

[1] S. Mandelbrojt, The Rice Institute Pamphlet (Dirichlet series), Vol. 31, No. 4, Houston 1944.

[2] Q. I. Rahman, *On the maximum modulus and the coefficients of entire Dirichlet series*, Tôhoku Math. J. (2), 8 (1956), p. 108–113.

[3] S. N. Srivastava, *On the maximum real part of an entire function defined by Dirichlet series*, Rev. Math. Hisp. Amer. 32 (1972), p. 119–127.

[4] K. Sugimura, *Übertragung einiger Sätze aus der Theorie der ganzen Funktionen auf Dirichletsche Reihen*, Math. Z. 29 (1929), p. 264–277.

[5] E. C. Titchmarsh, *The theory of functions*, Sec. Ed., Oxford 1939.

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