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# *ALMOST CONVEX FUNCTIONS*

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**1.** Let $R$ be the set of all real numbers and let $\Delta$ be an open real interval (finite or not). A function $f\colon \Delta \to R$ is called *convex* iff

$$f\left(\frac{x+y}{2}\right) \leqslant \frac{f(x)+f(y)}{2} \tag{1}$$

holds for all pairs $(x, y)$ in $\Delta\times\Delta$. A function $f\colon \Delta \to R$ is called *almost convex* iff (1) holds in $\Delta\times\Delta$ except for a set $M \subset \Delta\times\Delta$ of the planar Lebesgue measure zero.

A natural question arises whether every almost convex function is equal almost everywhere in $\Delta$ (in the sense of the linear Lebesgue measure) to a convex function. An affirmative answer is established in Theorem 5.

The corresponding problem for additive functions, i.e. for functions $f\colon R \to R$ satisfying Cauchy's functional equation

$$f(x+y) = f(x)+f(y) \tag{2}$$

had been raised by Erdös [4] and remained open for some years (cf. [1]) Recently it has been solved by de Bruijn [3] and Jurkat [6] (cf. also Hartman [5]) who have proved that every almost additive function (i.e. function satisfying (2) for almost all pairs $(x, y)$) is equal almost everywhere to an additive function.

Similarly to [3] and [6], our considerations will be based on a certain consequence of the Fubini theorem, which may be formulated as follows:

*Let $a, b, c, d$ be fixed real numbers such that $ad-bc \neq 0$, and let $M \subset R\times R$ be a set of the planar Lebesgue measure zero. Then for almost every $s$ the set*

$$\{t \mid (at+bs,\ ct+ds) \in M\}$$

*has the linear Lebesgue measure zero.*

(Geometrically this means, roughly speaking, that almost every line parallel to a given direction intersects $M$ in a set of linear measure zero).

In the sequel *measure* will always mean the Lebesgue measure.

**2.** The interval $\Delta$ being fixed, we denote by $\Delta_x$ the set

$$\Delta_x = \{h \mid x-h \in \Delta \text{ and } x+h \in \Delta\}.$$

For every $x \in \Delta$, $\Delta_x$ is an open interval centered at zero.

It is obvious that a function $f\colon \Delta \to R$ is convex if and only if

$$f(x) = \inf_{\Delta_x} \frac{f(x+h)+f(x-h)}{2}$$

(the infimum being attained at $h = 0$). Now we are going to investigate to what extent the above statement remains true when the infimum is replaced by the essential infimum.

We shall make use of the following result due to Bernstein and Doetsch [2]:

*Let $f\colon J \to R$, where $J$ is a finite closed interval, be a convex function, and let $J_0$ be the set of points which divide $J$ rationally. Then the restriction of $f$ to $J_0$ is continuous.*

Now we prove the following

THEOREM 1. *If a function $f\colon \Delta \to R$ is convex, then*

$$f(x) = \operatorname*{inf\,ess}_{\Delta_x} \frac{f(x+h)+f(x-h)}{2}. \tag{3}$$

Proof. Let us fix an $x \in \Delta$ and let us take an arbitrary set $A \subset \Delta_x$ of measure zero. The set

$$B = \bigcup_{n=0}^{\infty} 2^n\big(A \cup (-A)\big)$$

also has measure zero. For any $h \in \Delta_x \setminus B$ we have $\pm 2^{-n} h \in \Delta_x \setminus A$, $n = 0, 1, 2, \ldots$ By the theorem of Bernstein and Doetsch (applied to $J = \langle x-h, x+h \rangle$) we have

$$\lim_{n\to\infty} f(x \pm 2^{-n} h) = f(x),$$

which, together with the inequality $f(x) \leqslant \frac{1}{2}[f(x+h)+f(x-h)]$ resulting from the convexity of $f$, proves that

$$\inf_{\Delta_x \setminus A} \frac{f(x+h)+f(x-h)}{2} = f(x). \tag{4}$$

(3) results from (4) in view of the assumption that $A$ is an arbitrary set of measure zero.

COROLLARY. *If two convex functions, defined on a common interval $\Delta$, are equal almost everywhere in $\Delta$, then they are identical.*

In order to derive from (3) the convexity of $f$ we must assume something more about the function $f$.

THEOREM 2. *If a function $f\colon \Delta \to R$ is almost convex and* (3) *holds, then $f$ is convex.*

Proof. It follows from (3) that for every $x \in \Delta$ there is a set $E_x \subset \Delta_x$ of measure zero such that

$$f(x) \leqslant \tfrac{1}{2}[f(x+h)+f(x-h)] \quad \text{for } h \in \Delta_x \setminus E_x. \tag{5}$$

On the other hand, since $f$ is almost convex, the set

$$M = \left\{(x, y) \in \Delta \times \Delta \mid f\left(\frac{x+y}{2}\right) > \frac{f(x)+f(y)}{2}\right\} \tag{6}$$

has the planar measure zero. Hence the set

$$V_x = \{y \in \Delta \mid (x, y) \in M\}$$

has the linear measure zero for almost every $x \in \Delta$; i.e., there exists a set $U \subset \Delta$ of measure zero such that $x \in \Delta \setminus U$ implies $|V_x| = 0$.

Let us fix an arbitrary $\varepsilon > 0$ and put

$$\Omega_x = \{h \in \Delta_x \mid \tfrac{1}{2}[f(x+h)+f(x-h)] < f(x)+\varepsilon\}. \tag{7}$$

In view of (3), for every $x \in \Delta$ the set $\Omega_x$ has a positive outer measure.

Let us take arbitrary $x, y \in \Delta$ and choose an $h'$ such that

$$h' \in \Omega_x \setminus [(x-U) \cup (U-x)]. \tag{8}$$

Then the sets $V_{x-h'}$ and $V_{x+h'}$ have the measure zero, and thus we may choose an $h''$ such that

$$h'' \in \Omega_y \setminus [(y-V_{x-h'}) \cup (V_{x+h'}-y) \cup (2E_p-h')], \tag{9}$$

where

$$p = \frac{x+y}{2}. \tag{10}$$

Condition (9) guarantees that $(x-h', y-h'') \notin M$, $(x+h', y+h'') \in M$, i.e.,

$$\begin{aligned} f\left(p-\frac{h'+h''}{2}\right) &\leqslant \tfrac{1}{2}[f(x-h')+f(y-h'')], \\ f\left(p+\frac{h'+h''}{2}\right) &\leqslant \tfrac{1}{2}[f(x+h')+f(y+h'')]. \end{aligned} \tag{11}$$

Further, it follows from (9) that $\frac{1}{2}(h'+h'') \notin E_p$, whence

$$f(p) \leqslant \tfrac{1}{2}\left[f\left(p+\frac{h'+h''}{2}\right)+f\left(p-\frac{h'+h''}{2}\right)\right]. \tag{12}$$

Finally, by (8), (9) and (7)

$$\tag{13} \begin{aligned} \tfrac{1}{2}[f(x+h')+f(x-h')] &< f(x)+\varepsilon, \\ \tfrac{1}{2}[f(y+h'')+f(y-h'')] &< f(y)+\varepsilon. \end{aligned}$$

It follows from (10), (12), (11) and (13) that

$$f\left(\frac{x+y}{2}\right) < \frac{f(x)+f(y)}{2}+\varepsilon.$$

Letting $\varepsilon$ tend to zero, we obtain (1).

Let us note that for measurable $f$ condition (5) implies that $f$ is almost convex. Thus we have the following

THEOREM 3. *If a function $f\colon \Delta \to R$ is measurable and* (3) *holds, then $f$ is convex.*

Remark. It may be interesting to point out that the set (7) need not have a positive inner measure. In fact, let $\varphi(x)$ be a discontinuous additive function and put $f(x) = \exp\varphi(x)$. Then $f(x)$ is a discontinuous convex function. Taking $x = 0$, we have, for $h \in \Omega_0$,

$$f(h) < f(h)+f(-h) < 2(f(0)+\varepsilon) = 2(1+\varepsilon).$$

Thus $f(h)$ is bounded from above on $\Omega_0$. By a theorem of Ostrowski [7], $\Omega_0$ cannot have a positive inner measure. (In fact, for every $x \in R$ the set $\Omega_x$ has the inner measure zero).

**3.** It follows from Theorem 1 that if a function $f\colon \Delta \to R$ is equal almost everywhere in $\Delta$ to a convex function $g\colon \Delta \to R$, then

$$\tag{14} g(x) = \operatorname*{inf\,ess}_{\Delta_x} \frac{f(x+h)+f(x-h)}{2}, \quad x \in \Delta.$$

Now, we have the following

THEOREM 4. *If $f\colon \Delta \to R$ is almost convex and $g(x)$ is defined by* (14), *then*

$$\tag{15} f(x) = g(x) \textit{ almost everywhere in } \Delta.$$

Proof. Let $M$ denote the set (6) and let us write

$$T_x = \{h \in \Delta_x \,|\, (x, x+h) \in M\}, \qquad S_x = \{h \in \Delta_x \,|\, (x-h, x+h) \in M\}.$$

For almost every $x \in \Delta$ the sets $T_x$ and $S_x$ have measure zero, i.e., there exists a set $U \subset \Delta$ of measure zero such that $x \in \Delta \setminus U$ implies $|T_x| = |S_x| = 0$.

Let us fix an arbitrary $x \in \Delta \setminus U$ and an arbitrary set $A \subset \Delta_x$ of measure zero. The set

$$B = \bigcup_{n=0}^{\infty} 2^n\big(A \cup (-A) \cup T_x \cup (-T_x) \cup S_x\big)$$

also has measure zero. For any $h \in \Delta_x \setminus B$ we have

$$(16) \qquad \pm 2^{-n} h \notin A, \quad \pm 2^{-n} h \notin T_x, \quad \pm 2^{-n} h \notin S_x, \quad n = 0, 1, 2, \ldots$$

In particular, $h \notin S_x$, whence $(x-h, x+h) \notin M$ and

$$f(x) \leqslant \tfrac{1}{2}[f(x+h)+f(x-h)].$$

Consequently,

(17)
$$f(x) \leqslant \inf_{\Delta_x \setminus B} \tfrac{1}{2}[f(x+h)+f(x-h)] \leqslant \operatorname*{inf\,ess}_{\Delta_x} \tfrac{1}{2}[f(x+h)+f(x-h)] = g(x).$$

On the other hand, we have by (16), for $h \in \Delta_x \setminus B$,

$$f\left(\frac{2x \pm 2^{-n}h}{2}\right) \leqslant \frac{f(x)+f(x \pm 2^{-n}h)}{2}, \quad n = 0, 1, 2, \ldots,$$

i.e.,

$$f(x \pm 2^{-(n+1)}h) - f(x) \leqslant \tfrac{1}{2}[f(x \pm 2^{-n}h) - f(x)], \quad n = 0, 1, 2, \ldots$$

This implies

$$\limsup_{n\to\infty} f(x \pm 2^{-n}h) \leqslant f(x),$$

whence

$$\limsup_{n\to\infty} \tfrac{1}{2}[f(x+2^{-n}h)+f(x-2^{-n}h)] \leqslant f(x).$$

Consequently (cf. (16)),

$$(18) \qquad \inf_{\Delta_x \setminus A} \tfrac{1}{2}[f(x+h)+f(x-h)] \leqslant f(x).$$

Since $A$ has been an arbitrary set of measure zero, (18) implies

$$(19) \qquad g(x) = \operatorname*{inf\,ess}_{\Delta_x} \tfrac{1}{2}[f(x+h)+f(x-h)] \leqslant f(x).$$

(17) and (19) yield the equality $f(x) = g(x)$ for every $x \in \Delta \setminus U$, i.e. almost everywhere in $\Delta$.

As an immediate consequence of Theorems 1, 2 and 4 we obtain

THEOREM 5. *If $f$: $\Delta \to R$ is almost convex, then there exists a unique convex function $g$: $\Delta \to R$ such that* (15) *holds.*

Proof. We define $g$ by (14). In virtue of Theorem 4 relation (15) holds and, consequently,

$$\inf_{\Delta_x}\operatorname{ess}\tfrac{1}{2}[g(x+h)+g(x-h)] = \inf_{\Delta_x}\operatorname{ess}\tfrac{1}{2}[f(x+h)+f(x-h)] = g(x).$$

It follows from (15) that $g$ is almost convex. Hence, by Theorem 2, it is convex. Uniqueness results from the corollary to Theorem 1.

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