COLLOQUIUM MATHEMATICUM

VOL. LVI 1988 FASC. 1

TATE RESOLUTION AND THE STRUCTURE OF THE EXACT LOCAL RING

BY

MIROSŁAW USCKI (TORUŃ)

Let R be a commutative Noetherian ring and let M be an R-module. We say that a pair (u, v), $u, v \in R$, is *M-exact* if the sequence

$$M \xrightarrow{u} M \xrightarrow{v} M \xrightarrow{u} M$$

is exact. A sequence of pairs

$$(\boldsymbol{u}, \boldsymbol{v}) = \big((u_1, v_1), (u_2, v_2), \ldots, (u_n, v_n)\big)$$

is *M-exact* if the pair (u_i, v_i) is $\big(M/(u_1, u_2, \ldots, u_{i-1}) M\big)$-exact for $i = 2, 3, \ldots, n$. Certain examples of R-exact sequence of pairs are given in Theorem 1 below. Other examples can be found in [2].

DEFINITION 1 ([2]). A commutative local ring is called *exact* if there exists an R-exact sequence of pairs

$$(\boldsymbol{u}, \boldsymbol{v}) = \big((u_1, v_1), (u_2, v_2), \ldots, (u_n, v_n)\big)$$

such that its unique maximal ideal $\mathfrak{M}$ is generated by elements $u_1, u_2, \ldots, u_n$.

In the paper we prove that each exact local Noetherian ring R of equal characteristic is of the form described in Theorem 1 below. Moreover, a construction of the Tate resolution of all exact local Noetherian rings is given. Also, making use of this resolution one proves that each exact local Noetherian ring is Artinian. All rings in this paper are assumed to be commutative and Noetherian.

The paper is a part of the author's doctoral thesis [4] written under the supervision of Professor D. Simson. The author would like to thank him and Professor A. Tyc for many suggestions and discussions.

THEOREM 1. *Let $R_1 = K[[X_1, X_2, \ldots, X_n]]$ be the formal power series ring over a field K and F_i, G_{kj} $(1 \leqslant i \leqslant n,\ j < k,\ 1 \leqslant j \leqslant n)$ be elements of the maximal ideal of R_1 such that*

$$\mathcal{U} = (X_1 F_1, X_2 F_2 - G_{21} X_1, \ldots, X_n F_n - G_{n1} X_1 - \ldots - G_{n,n-1} X_{n-1})$$

is a regular sequence in R_1 *(e.g.,* $n = 2$, $F_1 = X_2$, $F_2 = X_2$, $G_{21} = X_1$*). Let* $R = R_1/\mathscr{U}$*. Then* $(x_1, f_1), (x_2, f_2), \ldots, (x_n, f_n)$*, where* $x_i = X_i + \mathscr{U}$ *and* $f_i = F_i + \mathscr{U}$*, is an R-exact sequence of pairs. In particular, R is a local exact ring.*

Proof. By induction on n. For $n = 1$ we have $F_1 = aX_1^{h-1}$, where a is a unit, $h \geqslant 2$ and $R \cong K[[X_1]]/(X_1^h)$. Suppose that the theorem is valid for the factor ring of the formal power series ring of $n-1$ variables. Let

$$(X_1 F_1,\ X_2 F_2 - G_{21} X_1, \ldots, X_n F_n - G_{n1} X_1 - \ldots - G_{n,n-1} X_{n-1}) = \mathscr{U}$$

be a regular sequence in R_1. Then

$$(1)\quad (X_1,\ X_2 F_2 - G_{21} X_1, \ldots, X_n F_n - G_{n1} X_1 - \ldots - G_{n,n-1} X_{n-1}) = \mathscr{U}_1,$$

$$(2)\quad (F_1,\ X_2 F_2 - G_{21} X_1, \ldots, X_n F_n - G_{n1} X_1 - \ldots - G_{n,n-1} X_{n-1}) = \mathscr{U}_2$$

are regular sequences in R_1.

We shall show first that the pair (x_1, f_1) is R-exact. Let $\bar{H} f_1 = 0$, i.e., $HF_1 \in \mathscr{U}$. Then

$$(H - GX_1) F_1 \in (X_2 F_2 - G_{21} X_1, \ldots, X_n F_n - G_{n1} X_1 - \ldots - G_{n,n-1} X_{n-1}) = \mathscr{U}_3$$

for any $G \in R_1$. From the regularity of (2) we obtain $H - GX_1 \in \mathscr{U}_3$ and, consequently, $\bar{H} = \bar{G} x_1$ in R. In the same way we prove that $\mathrm{Ann}_R(x_1) = (f_1)$.

From the regularity of (1) we infer that the sequence

$$(X_2 F_2,\ X_3 F_3 - G_{32} X_2, \ldots, X_n F_n - G_{n2} X_2 - \ldots - G_{n,n-1} X_{n-1}) = \mathscr{U}_4$$

is regular in $K[[X_2, X_3, \ldots, X_n]]$. But

$$R/(x_1) = K[[X_2, X_3, \ldots, X_n]]/\mathscr{U}_4,$$

whence, by the inductive assumption, $(x_2, f_2), \ldots, (x_n, f_n)$ is $R/(x_1)$-exact. Consequently, the sequence $(x_1, f_1), (x_2, f_2), \ldots, (x_n, f_n)$ is R-exact, which completes the proof.

Let R be a fixed local ring with the maximal ideal $\mathfrak{M}$. We recall the construction of the Tate resolution of R (see [3]). By a *graded R-algebra* we mean a positively graded R-algebra

$$A = \bigoplus A_i$$

satisfying the following conditions:

$A_0 = R$;

$xy = (-1)^{pq} yx$ if $x \in A_p$, $y \in A_q$;

$x^2 = 0$ if $x \in A_p$ and p is odd.

Let A be a fixed graded differential R-algebra and let B be a differential A-algebra. For any homogeneous cycle $u \in B_{p-1}$ we put

$$B\langle T; dT = u\rangle = \begin{cases} B \otimes_A \Gamma(AT) & \text{if } p \text{ is even}, \\ B \otimes_A \Lambda(AT) & \text{if } p \text{ is odd}, \end{cases}$$

where $\Lambda(AT)$ and $\Gamma(AT)$ are graded commutative A-algebras given by the equalities

$$\Lambda(AT) = A \oplus AT, \quad T^2 = 0, \quad \deg T = p,$$

$$\Gamma(AT) = (AT)^{(0)} \oplus AT^{(1)} \oplus \ldots, \quad T^{(i)} T^{(j)} = \frac{(i+j)!}{i!j!} T^{(i+j)},$$

$$\deg T^{(i)} = pi.$$

Recall that for a symbol X and any integer $p \geq 0$, AX is a graded A-module $A \otimes_R RX$, where $(RX)_p = (RX)_j = 0$ if $j \neq p$.

In the A-algebra $B\langle T; dT = u\rangle$ one can define a differential d such that

$$d(a) = d_B(a) \text{ for } a \in B, \quad dT = u,$$

and $B\langle T; dT = u\rangle$ with d is a differential A-algebra. The variable T "kills" a given cycle $u \in B$ because we have an isomorphism

$$H_n(B\langle T; dT = u\rangle) \simeq H_n(B)/B\sigma,$$

where $\sigma = \bar{u} \in H_n(B)$.

The above construction may be generalized in a way such that one can "kill" an arbitrary set of homogeneous cycles $\{u_i; i \in I\}$. In this case the corresponding differential A-algebra is denoted by $B\langle T_j; dT_j = u_j\rangle$.

It is proved in [3] that for every cyclic R-module R/M there exists a free cyclic R-algebra X such that $H_0(X) = R/M$. In other words, there exists a free resolution of R/M which is an R-algebra. We denote it by X. The algebra X is obtained as the union of an ascending chain of differential R-algebras:

$$X^{(0)} \subset X^{(1)} \subset X^{(2)} \subset \ldots$$

We define $X^{(0)}$ to be the algebra R itself ($X_0^{(0)} = R$, $X_n^{(0)} = 0$, $n \neq 0$), $d = 0$ and

$$X^{(k+1)} = X^{(k)}\langle T_1, T_2, \ldots, T_{n_k}; dT_j = u_j\rangle,$$

where $u_1, \ldots, u_{n_k}$ are generators of the module $Z_k(X^{(k)})$ of k-cycles in $X^{(k)}$. The differential graded algebra X is called the *Tate resolution* of R.

In what follows we shall give a precise description of the Tate resolution of the local ring R under the assumption that R is exact. We start with the following

PROPOSITION 1. *Suppose $(R, \mathfrak{M})$ is a local ring and $(u_1, v_1), \ldots, (u_k, v_k)$ is an R-exact sequence of pairs with $u_i \in \mathfrak{M}$. If*

$$u_i v_i = \sum_{j=1}^{i-1} t_{ij} u_j,$$

$u_1 v_1 = 0$ *for some* $t_{ij} \in R$, *and* E *is the Koszul complex associated with* $(u_1, u_2, \ldots, u_k)$, *then*

$$\bar{\sigma}_i = v_i T_i - \sum_{j=1}^{i-1} t_{ij} T_j + B_1(E), \quad i = 1, 2, \ldots, k,$$

form a minimal sequence of generators of $H_1(E)$.

Proof. We apply induction on k. The case $k = 1$ is easy. Now assume that $k > 1$ and denote by E' and E Koszul complexes associated with sequences $(u_1, \ldots, u_{k-1})$ and $(u_1, \ldots, u_k)$, respectively. If $\sigma \in Z_1(E)$, then

$$\sigma = \sum_{i=1}^{k} r_i T_i \quad \text{and} \quad d_1(\sigma) = \sum_{i=1}^{k} r_i u_i = 0. \tag{3}$$

Thus $r_k u_k \in (u_1, \ldots, u_{k-1})$. Since (u_k, v_k) is $\big(R/(u_1, \ldots, u_{k-1})\big)$-exact, there exist $x \in R$ and $b_i \in R$ such that

$$r_k = v_k x + \sum_{i=1}^{k-1} b_i u_i.$$

Combining this with (3) we obtain

$$\begin{aligned}\sigma &= \sum_{i=1}^{k-1} r_i T_i + \big(v_k x + \sum_{i=1}^{k-1} b_i u_i\big) T_k \\ &= \sum_{i=1}^{k-1} (r_i + x t_{ki} + b_i u_k) T_i + x\sigma_k + \sum_{i=1}^{k-1} b_i (u_i T_k - u_k T_i).\end{aligned}$$

It is easy to check that

$$\sigma' = \sum_{i=1}^{k-1} (r_i + x t_{ki} + b_i u_k) T_i$$

is an element of $Z_1(E')$ and

$$b = \sum_{i=1}^{k-1} b_i (u_i T_k - u_k T_i) \in B_1(E).$$

By induction we obtain

$$\bar{\sigma} = x_1 \bar{\sigma}_1 + \ldots + x_{k-1} \bar{\sigma}_{k-1} + x\bar{\sigma}_k \quad \text{for any } x_i \in R.$$

Now we shall show that $\{\bar{\sigma}_i\}_{i=1}^{k}$ form a minimal sequence of generators of $H_1(E)$. For this aim it is sufficient to show that the equality

$$\sum_{i=1}^{k} a_i \bar{\sigma}_i = 0 \quad (a_i \in R)$$

implies $a_i \in \mathfrak{M}$. Note that $E = E' \otimes_R \Lambda(RT_k)$. Since the equality

$$\sum_{i=1}^{k} a_i \bar{\sigma}_i = 0$$

is equivalent to

$$d_2(b) = \sum_{i=1}^{k} a_i \sigma_i \quad \text{for some } b \in E_2 = E'_2 \otimes E'_1 \otimes_R (RT_k),$$

we have $b = x_2 + x_1 \otimes T_k$, where x_2, $x_1 \in E'$. From the definition of the differential in E we have

$$\sum_{i=1}^{k} a_i \sigma_i = d_2 x_2 + (d_1 x_1) T_k - x_1 u_k,$$

whence

$$\sum_{i=1}^{k} a_i \big(v_i T_i - \sum_{j<i} t_{ij} T_j\big) = d_2 x_2 + (d_1 x_1) T_k - x_1 u_k. \tag{4}$$

Consequently,

$$a_k v_k = d_1(x_1) \in B_0(E') = (u_1, \ldots, u_{k-1}). \tag{5}$$

Hence $a_k = ru_k + c_1$ for some $r \in R$ and $c_1 \in (u_1, \ldots, u_{k-1})$. Now let

$$c_1 = \sum_{i=1}^{k-1} r_i u_i, \quad \eta = \sum_{i=1}^{k-1} r_i T_i \quad \text{and} \quad \gamma = \sum_{i=1}^{k-1} t_{ki} T_i.$$

By (5) we have

$$\begin{aligned} d_1 x_1 &= (ru_k + c_1) v_k = ru_k v_k + c_1 v_k \\ &= r \sum_{i=1}^{k-1} t_{ki} u_i + c_1 v_k = d_1(r\gamma + v_k \eta), \end{aligned}$$

which means that $(x_1 - r\gamma - v_k \eta) \in Z_1(E')$. Using again the fact that $\bar{\sigma}_1, \ldots, \bar{\sigma}_{k-1}$ generate $H_1(E')$, we conclude that there are $s_i \in R$ such that

$$x_1 - r\gamma - v_k \eta - \sum_{i=1}^{k-1} s_i \sigma_i \in B_1(E'). \tag{6}$$

From (4) and (5) we get

$$\begin{aligned} u_k x_1 &= d_2 x_2 + d_1(x_1) T_k - \sum_{i=1}^{k} a_i \sigma_i \\ &= d_2 x_2 - \sum_{i=1}^{k-1} a_i \sigma_i - a_k (v_k T_k - \gamma) + (d_1 x_1) T_k \\ &= d_2 x_2 - \sum_{i=1}^{k-1} a_i \sigma_i + a_k \gamma. \end{aligned}$$

This equality together with (6) gives

$$u_k\Big(x_1 - r\gamma - v_k\eta - \sum_{i=1}^{k-1} s_i\sigma_i\Big)$$

$$= -\sum_{i=1}^{k-1} a_i\sigma_i + [d_2x_2 + (a_k + ru_k)\gamma - u_kv_k\eta] - \sum_{i=1}^{k-1} u_ks_i\sigma_i \in B_1(E').$$

On the other hand,

$$(a_k - ru_k)\gamma - u_kv_k\eta = c_1\gamma - u_kv_k\eta$$

$$= \Big(\sum_{i=1}^{k-1} r_iu_i\Big)\Big(\sum_{j=1}^{k-1} t_{kj}T_j\Big) - \Big(\sum_{j=1}^{k-1} t_{kj}u_j\Big)\Big(\sum_{i=1}^{k-1} r_iT_i\Big)$$

$$= \sum_{i,j=1}^{k-1} r_it_{kj}(u_iT_j - u_jT_j)$$

is an element of $B_1(E')$. Consequently,

$$\sum_{i=1}^{k-1} (s_iu_k + a_i)\bar{\sigma}_i = 0$$

and from the minimality of the set $\bar{\sigma}_1, \bar{\sigma}_2, \ldots, \bar{\sigma}_{k-1}$ it follows that $a_i + s_iu_k \in \mathfrak{M}$. Therefore, $a_i \in \mathfrak{M}$, $i = 1, 2, \ldots, k-1$. This completes the proof since $a_k = ru_k + c_1$ with $c_1 \in \mathfrak{M}$ belongs also to $\mathfrak{M}$.

Proposition 2. *Let A be a graded differential R-algebra and $H_0(A) = R/\mathfrak{A}$, where $\mathfrak{A}$ is an ideal of R. Let*

$$B = A\langle T; dT = u\rangle\langle S; dS = vT - b\rangle,$$

where $\deg T = 1$, $\deg S = 2$, $b \in A_1$, *and* (u, v) *is* $(R/\mathfrak{A})$*-exact. If* $H_i(A) = 0$ *for* $i > 0$, *then* $H_i(B) = 0$ *for* $i > 0$ *and* $H_0(B) = R/(\mathfrak{A}, u)$.

Proof. Let $A' = A\langle T; dT = u\rangle$; then $B = A' \otimes_A \Gamma(AS)$. We have the following exact sequences:

$$0 \to A \xrightarrow{\alpha} A' \xrightarrow{\beta} A \to 0,$$

$$0 \to A' \xrightarrow{\sigma} B \xrightarrow{\tau} B \to 0,$$

where

$$\alpha(a) = a\otimes 1, \quad \beta(a'\otimes T) = a', \quad \sigma(a') = a'\otimes 1$$

and

$$\tau(a\otimes S^{(k)}) = a\otimes S^{(k-1)}.$$

Hence we obtain the long homology exact sequences

$$(*) \quad \ldots \to H_k(A) \xrightarrow{\delta} H_k(A) \xrightarrow{\alpha_*} H_k(A') \xrightarrow{\beta_*} H_{k-1}(A) \to$$
$$\ldots \to H_2(A) \xrightarrow{\delta} H_2(A) \xrightarrow{\alpha_*} H_2(A') \xrightarrow{\beta_*} H_1(A) \xrightarrow{\delta} H_1(A) \xrightarrow{\alpha_*} H_1(A')$$
$$\xrightarrow{\beta_*} H_0(A) \xrightarrow{\delta} H_0(A) \to 0,$$

$$(**) \quad \ldots \to H_k(A') \xrightarrow{\alpha_*} H_k(B) \xrightarrow{\tau_*} H_{k-2}(B) \xrightarrow{\Delta} H_{k-1}(A') \xrightarrow{\alpha_*}$$
$$\ldots \xrightarrow{\alpha_*} H_1(B) \xrightarrow{\Delta} H_2(A') \xrightarrow{\alpha_*} H_2(B) \xrightarrow{\tau_*} H_0(B) \xrightarrow{\Delta} H_1(A') \to H_1(B) \to 0.$$

From $(*)$ and the assumption it follows that $H_i(A') = 0$ for $i > 1$ and $H_1(A') = \operatorname{Ker}\delta$. Hence by $(**)$ we get $H_k(B) \simeq H_{k-2}(B)$ for $k > 2$; clearly, $H_0(B) = R/(\mathfrak{A}, u)$. It is sufficient to prove that $H_1(B) = H_2(B) = 0$. Consider the following sequence:

$$0 \to H_2(B) \xrightarrow{\tau_*} H_0(B) \xrightarrow{\Delta} H_1(A') \xrightarrow{\alpha_*} H_1(B) \to 0.$$

As a part of the exact sequence $(**)$ it is exact. Since

$$\Delta(\bar{\xi}) = \xi(vT-b) + B_1(A') \quad \text{for } \bar{\xi} \in H_0(B)$$

and δ is the multiplication by u we have

$$H_1(A') \simeq \operatorname{Ker}\delta = \operatorname{Im}\beta_* \simeq R\bar{v} \subset R/\mathfrak{A}.$$

Consider the following diagram:

$$\begin{array}{ccccc} H_0(B) = R/(\mathfrak{A}, u) & \xrightarrow{v} & R/\mathfrak{A} & \xrightarrow{u} & R/\mathfrak{A} \\ \Delta\downarrow & & \| & & \| \\ 0 \to H_1(A') & \xrightarrow{\beta_*} & H_0(A) & \xrightarrow{u} & H_0(A) \end{array}$$

The lower sequence is exact as a part of the exact sequence $(*)$ $(H_1(A) = 0)$. This diagram is commutative since

$$\beta_* \Delta(\bar{\xi}) = \beta_*[\xi(vT-b) + B_1(A')] = v\xi + B_0(A) = v\xi + \mathfrak{A} = v\bar{\xi}.$$

By the assumption ((u, v) is $R/\mathfrak{A}$-exact) the multiplication by v is an isomorphism of the R-modules $R/(\mathfrak{A}, u)$ and $R\bar{v}$. Consequently, Δ is an isomorphism and

$$H_1(B) \simeq \operatorname{Coker}\Delta = 0 \quad \text{and} \quad H_2(B) \simeq \operatorname{Ker}\Delta = 0.$$

This completes the proof of the proposition.

COROLLARY 1. *If* $(u_1, v_1), \ldots, (u_n, v_n)$ *is an R-exact sequence of pairs, then the algebra*

$$X = R\langle T_1, \ldots, T_k; dT_i = u_i\rangle\langle S_1, \ldots, S_k; dS_i = \sigma_i\rangle$$

is a free resolution of the R-module $R/(u_1, \ldots, u_k)$.

The corollary can be proved by an easy induction on n using Proposition 2.

THEOREM 2. *If* $(R, \mathfrak{M})$ *is an exact local Noetherian ring and* $(u_1, v_1), \ldots, (u_n, v_n)$ *is an R-exact sequence of pairs with* $(u_1, \ldots, u_n) = \mathfrak{M}$, *then R is Artinian.*

Proof. By [2], Corollary 1.7, the completion $\hat{R}$ of the ring R is also an exact local ring. Now from [1], Theorem 2.7, and our Corollary 1 it follows that R is a local complete intersection; in particular, it is a Cohen–Macaulay ring. On the other hand,

$$v = v_1 v_2 \ldots v_n \neq 0 \quad \text{and} \quad v\hat{\mathfrak{M}} = 0$$

by [2], Lemma 3.1. This means that $\mathrm{dh}(\hat{\mathfrak{M}}) = 0$, where $\mathrm{dh}(\mathfrak{A})$ denotes the depth of the ideal $\mathfrak{A}$. Hence $\dim R = \dim \hat{R} = 0$ since in a local Cohen–Macaulay ring $(A, \mathfrak{M})$ $\dim A = \mathrm{dh}(\mathfrak{M})$. The theorem now follows because any Noetherian ring of the Krull dimension zero is Artinian.

COROLLARY 2. *If R is an exact local ring, then R is complete.*

Now assume that $(R, \mathfrak{M})$ is an exact local ring of equal characteristic and $(u_1, v_1), \ldots, (u_n, v_n)$ is an R-exact sequence of pairs with

$$(u_1, \ldots, u_n) = \mathfrak{M}.$$

By [6] there exists an epimorphism ring

$$g\colon K[[X_1, \ldots, X_n]] \to R, \quad K = R/\mathfrak{M},$$

such that $g(X_i) = u_i$ $(i = 1, 2, \ldots, n)$.

Let $G_{ij}, F_i \in K[[X_1, \ldots, X_n]]$ be such that

$$g(F_i) = v_i \quad \text{and} \quad g(G_{ij}) = t_{ij}, \quad \text{where } u_i v_i = \sum_{j=1}^{i-1} t_{ij} u_j.$$

Then

THEOREM 3. *We have*

$$\mathrm{Ker}\, g = (X_1 F_1, X_2 F_2 - G_{21} X_1, \ldots, X_n F_n - G_{n1} X_1 - \ldots - G_{n,n-1} X_{n-1}).$$

Proof. Let us denote the right hand ideal by $\mathfrak{A}$. We apply induction on n.

If $n = 1$, then $v_1 = u_1^{h-1}$, where

$$h = \min\{m;\ u_1^m = 0\} \quad \text{and} \quad R \simeq K[[X_1]]/(X_1^h).$$

Suppose that the theorem is true for all exact local rings with $(u_1, v_1), \ldots, (u_k, v_k)$ being an R-exact sequence of pairs ($k < n$) and consider the following commutative diagram:

$$\begin{array}{ccccc} K[[X_1, \ldots, X_n]] & \xrightarrow{g} & R & \to & 0 \\ \downarrow & & \downarrow & & \\ K[[X_1, \ldots, X_n]]/(X_1) & \xrightarrow{g'} & R/(u_1) & \to & 0 \end{array}$$

where

$$g'(X_i) = \bar{u}_i, \qquad \bar{u}_i = u_i + (u_1),$$

$$g'(\bar{F}_i) = \bar{v}_i, \qquad g'(\bar{G}_{ij}) = \bar{t}_{ij}$$

for $2 \leqslant i \leqslant n$, $1 \leqslant j \leqslant n-1$. From the definition of the R-exact sequence of pairs it follows that $(\bar{u}_2, \bar{v}_2), \ldots, (\bar{u}_n, \bar{v}_n)$ is an $R/(u_1)$-exact sequence of pairs. Thus, by the induction hypothesis,

$$\operatorname{Ker} g' = (\bar{X}_2 \bar{F}_2, \bar{X}_3 \bar{F}_3 - \bar{G}_{32} \bar{X}_2, \ldots, \bar{X}_n \bar{F}_n - \bar{G}_{n2} \bar{X}_2 - \ldots - \bar{G}_{n,n-1} \bar{X}_{n-1}).$$

Using this equality one can show that $\mathfrak{A} = \operatorname{Ker} g$. It is obvious that $\mathfrak{A} \subset \operatorname{Ker} g$. Let $F \in \operatorname{Ker} g$. Then $g'(F + (X_1)) = 0$, and therefore

$$F = r_1 X_1 + r_2 F_2 X_2 + \ldots + r_n (X_n F_n - G_{n2} X_2 - \ldots - G_{n,n-1} X_{n-1})$$

for some $r_1, \ldots, r_n \in K[[X_1, \ldots, X_n]]$. Since $g(F) = 0$, we have

$$0 = g(r_1) u_1 + g(r_2) u_2 v_2 + \ldots + g(r_n)(u_n v_n - t_{n2} u_2 - \ldots - t_{n,n-1} u_n)$$

$$= u_1 \big(g(r_1) + g(r_2) t_{21} + \ldots + g(r_n) t_{n1}\big).$$

From the exactness of the pair (u_1, v_1) we get

$$g(r_1) + g(r_2) t_{21} + \ldots + g(r_n) t_{n1} = v_1 x \quad \text{for some } x \in R.$$

Let us denote by G an element of $K[[X_1, \ldots, X_n]]$ such that $g(G) = x$. Then

$$g(r_1 + r_2 G_{21} + \ldots + r_n G_{n1} - GF_1) = 0$$

and

$$w = r_1 + r_2 G_{21} + \ldots + r_n G_{n1} - GF_1 \in \operatorname{Ker} g.$$

Hence

$$F = r_1 X_1 + r_2 X_2 F_2 + \ldots + r_n (X_n F_n - G_{n2} X_2 - \ldots - G_{n,n-1} X_{n-1})$$

$$= \Big(w - \sum_{i=2}^{n} r_i G_{i1} + GF_1\Big) X_1 + r_2 X_2 F_2 + \ldots$$

$$+ r_n (X_n F_n - G_{n2} X_2 - \ldots - G_{n,n-1} X_{n-1})$$

$$= wX_1 + GF_1 X_1 + r_2 (X_2 F_2 - G_{21} X_1) + \ldots$$

$$+ r_n (X_n F_n - G_{n1} X_1 - \ldots - G_{n,n-1} X_{n-1}).$$

Therefore

$$F \in \mathfrak{A} + (X_1) \operatorname{Ker} g \quad \text{and} \quad \operatorname{Ker} g = \mathfrak{A} + (X_1) \operatorname{Ker} g.$$

By the Nakayama lemma, $\operatorname{Ker} g = \mathfrak{A}$ and the theorem is proved.

Note that the ideal $\mathfrak{A}$ is generated by a regular sequence (Corollary 1 and [1], Theorem 2.7).

Remark. The description of exact local rings of nonequal characteristics remains open.

REFERENCES

[1] E. F. Assmus, *On the homology of local rings*, Illinois J. Math. 3 (1959), pp. 187–199.
[2] R. Kiełpiński, D. Simson and A. Tyc, *Exact sequences of pairs in commutative rings*, Fund. Math. 99 (1978), pp. 113–121.
[3] J. Tate, *Homology of Noetherian rings and local rings*, Illinois J. Math. 1 (1957), pp. 14–27.
[4] M. Uscki, *Rings with exact sequence of pairs* (in Polish), Doctoral Thesis, preprint No. 1/77, Institute of Mathematics, N. Copernicus University, Toruń.
[5] — *On exact modules over a commutative exact ring*, Colloq. Math. 40 (1979), pp. 197–202.
[6] C. Zariski and P. Samuel, *Commutative Algebra*, Vol. II, New York 1960.

INSTITUTE OF MATHEMATICS
N. COPERNICUS UNIVERSITY, TORUŃ

Reçu par la Rédaction le 5.8.1980