ZASTOSOWANIA MATEMATYKI
V (1960)

J. ŁUKASZEWICZ (Wrocław) and W. SADOWSKI (Warszawa)

ON COMPARING SEVERAL POPULATIONS WITH A CONTROL POPULATION [1]

1. The following two problems have been the starting point of this paper.

A. We consider k continuous populations with distribution functions

$$F(x-a_1),\ F(x-a_2),\ \ldots,\ F(x-a_k)$$

which are identical disregarding translation. Both the functional form of the distribution function $F(x)$ and the translations $a_1, a_2, \ldots, a_k$ are unknown. On the basis of samples—of n elements each—taken from each population it is to be decided whether, among the populations concerned, there is a population with a greater mean value than the mean values of the remaining populations, i. e. whether there is a population shifted farther to the right than the remaining populations ($a_i > a_j$ for $j \neq i$). If the answer is positive, that population should be indicated.

B. We consider k continuous populations with distribution functions

$$F\left(\frac{x-\mu}{\lambda_1}\right),\ F\left(\frac{x-\mu}{\lambda_2}\right),\ \ldots,\ F\left(\frac{x-\mu}{\lambda_k}\right)$$

which are identical disregarding extension around the common mean value μ. As before, the functional form of the distribution function $F(x)$, the mean value μ and the factors $\lambda_1, \lambda_2, \ldots, \lambda_k$ are unknown. On the

---

[1] Niniejsza praca była ogłoszona w naszym piśmie po polsku w tomie 3 (1957), str. 204-216. Obecnie ogłaszamy ją po angielsku, aby jej treść uczynić dostępniejszą obcym czytelnikom. *Redakcja.*

Данная работа была опубликована в нашем журнале на польском языке в томе 3 (1957), стр. 204-216. В настоящее время публикуем её на английском языке для того, чтобы содержание работы сделать даступпым читателям, не знакомым с польском языком. *Редакция.*

This paper appeared in Polish in our periodical in vol. 3 (1957), pp. 204-216. We are now publishing it in English in order to make it easier for foreign readers to get acquainted with its contents. *Editors.*

basis of samples—of n elements each—taken from each population it is to be decided whether among the populations concerned there is a population with a greater variance than the variances of the remaining populations ($\lambda_i > \lambda_j$ for $j \neq i$). If the answer is positive, that population should be indicated.

Very simple tests are known, permitting the solution of both problems stated above. F. Mosteller [1] has given a test for problem A and W. Sadowski [5] for problem B ([2]). In both cases each of the k samples should be ordered according to the increasing values of the observations. Mosteller's test consists in choosing a sample containing the greatest element. We count in that sample the number $R \geqslant 1$ of elements greater than any element in the remaining samples. If $R \geqslant R_\alpha$, then we adopt the hypothesis that the selected sample with the greatest element comes from the population with the greatest mean value. In the opposite case we adopt the zero hypothesis, namely that the mean values of all the populations are equal ($a_1 = a_2 = \ldots = a_k$). The critical value R_α is calculated in such a way that the probability of rejecting the zero hypothesis when it is the correct one is not greater than α. Sadowski's test consists in selecting a sample having at the same time an element greater than any element in the remaining samples and an element smaller than any element in the remaining samples. If a sample of this kind exists, then we count in it the number $r \geqslant 2$ of protruding elements, i. e. the total number of elements of that sample that are greater than any element of the remaining samples and that are smaller than any element in the remaining samples. If $r \geqslant r_\alpha$, then we adopt the hypothesis that the sample containing the greatest and the smallest element comes from the population with the greatest variance. If, however, $r < r_\alpha$, or if the greatest element and the smallest element are to be found in two different samples, then we adopt the hypothesis that the variances of all the samples are equal. The critical value r_α is calculated in such a way that the probability of rejecting the zero hypothesis when it is correct is not greater than α.

This paper aims at finding suitable test for slightly modified problems. In applications we often come across problems in which among the populations which are being compared one population is singled out and we are particularly interested in comparing the remaining populations with that control population. On the basis of samples of size n taken from each population it is to be decided whether any of the populations other than the control population has a mean value (variance)

([2]) For the case of normal distributions E. Paulson has given a test for problem A [3], and D. R. Truax for problem B [6].

greater than the mean values (variances) of all the remaining populations. If the answer is positive, we want to indicate the population with the greatest mean value (variance), and we are particularly anxious to avoid the error of selecting any of the non-control populations when the control population has a mean value (variance) not smaller than the mean values (variances) of the remaining populations. These modified forms of problems A and B for the mean value and the variance will hence forth be termed problems A_1 and B_1. Problem A_1 has been solved by E. Paulson [4] but only for the case of a population with normal distribution.

2. Both problem A_1 and problem B_1 occur in certain applications, and the control population mentioned above plays the role of a standard with which the remaining populations are compared.

In particular those standards occur in pharmacological experiments, where the efficacy of a given medicine is compared with the efficacy of a standard medicine. If we are to call a medicine better if the mean value of some characteristic of that medicine is greater, we are confronted with problem A_1. Naturally, the error which we are particularly anxious to avoid is that of ascribing to a medicine a higher quality than that of the standard medicine when in reality the standard medicine has a mean value not smaller than the mean values of the medicines which are being compared with it. Analogical situations arise in agricultural and other experiments.

Problem B_1 does not occur so frequently in applications; however, some examples of it can be given. For instance, J. Oderfeld [2] has shown that one of the criteria of estimating the spraying system in the chamber of an aircraft engine is the distribution of the size of fuel droplets. It has been found that the greater the dispersion of that distribution the better is the system. Thus, if we conduct an experiment consisting in comparing several sprayers with a control sprayer, then the indication of the optimal sprayer is reduced to problem B_1.

3. The test for problem A_1 which we are going to propose is based on a statistics R', which will be defined as follows: $R' = 0$ if a sample of the control population contains an element greater than any element in the remaining samples. If the greatest element is contained in any of the non-control populations, then $R' \geqslant 1$ is the number of those elements of the sample containing the greatest element which are greater than any element of the sample from the control population.

The test for problem A_1 consists in the following alternative:

if $R' \geqslant R'_\alpha$ we assume that the greatest mean value is that of the population whose sample contains the greatest element, if $R' < R'_\alpha$ we assume that no population has a mean value greater than the mean value of the control population.

The critical value R'_α should be calculated in such a way that the probability of the inequality $R' \geqslant R'_\alpha$ under the assumption that none of the populations concerned has a mean value greater than the mean value of the control population is not greater than α. This condition can be written as a formula

$$(1) \qquad P\{R' \geqslant R'_\alpha | a_1 \geqslant \max(a_2, a_3, \ldots, a_k)\} \leqslant \alpha$$

where the numbers $a_1, a_2, \ldots, a_k$ denote—as at the beginning of this paper—the translations of the individual populations (they are—disregarding a constant—equal to the mean values in the individual populations) and the first population is singled out.

The probability standing on the left side of inequality (1) is a function of R'_α and of the values of the translations $a_1, a_2, \ldots, a_k$. It can easily be seen, however, that for fixed R'_α that probability has its maximum when the translations $a_1, a_2, \ldots, a_k$ are equal. We shall thus determine the quantity R'_α from the condition

$$(2) \qquad P\{R' \geqslant R'_\alpha | a_1 = a_2 = \ldots = a_k\} \leqslant \alpha,$$

which implies inequality (1).

The probability of the inequality $R' \geqslant R'_\alpha$ under the assumption of identical distributions in all the populations ($a_1 = a_2 = \ldots = a_k$) can easily be expressed as a function of R'_α, k and n. We are of course interested only in the natural values of R'_α. Observe first that for $R'_\alpha = 1$ we have

$$P\{R' \geqslant 1 | a_1 = a_2 = \ldots = a_k\} = (k-1)/k,$$

since the event $R' \geqslant 1$ is equivalent to an event consisting in the greatest element being found in one of the $k-1$ non-control samples. For $R'_\alpha \geqslant 2$ the probability

$$P\{R' \geqslant R'_\alpha | a_1 = a_2 = \ldots = a_k\}$$

is equal to the product of the probability of one of the $k-1$ non-control samples containing the greatest element (i. e. of the probability, already found, of the inequality $R'_\alpha \geqslant 1$) by the probability that $R'_\alpha - 1$ (or more) elements of the sample containing the greatest element are greater than any element of the control sample. That probability is equal to

$$\frac{n-1}{2n-1} \cdot \frac{n-2}{2n-2} \cdot \ldots \cdot \frac{n-R'_\alpha+1}{2n-R'_\alpha+1} = \frac{(n-1)!}{(n-R'_\alpha)!} \cdot \frac{(2n-R'_\alpha)!}{(2n-1)!}$$
$$= \binom{n-1}{R'_\alpha-1} \Big/ \binom{2n-1}{R'_\alpha-1}.$$

Thus

$$\text{(3)}\qquad P\{R' \geqslant R'_a | a_1 = a_2 = \ldots = a_k\} = \frac{k-1}{k}\binom{n-1}{R'_a-1}\Big/\binom{2n-1}{R'_a-1},$$

and, according to the generally accepted convention that $\binom{m}{0} = 1$, formula (3) is valid also for $R'_a = 1$.

With n tending to infinity the right side of equality (3) tends to the limit

$$\frac{k-1}{k}\cdot\frac{1}{2^{R'_a-1}}.$$

Thus under the assumption of $a_1 = a_2 = \ldots = a_k$ the statistics R' has the limit distribution

$$\lim_{n\to\infty} P\{R' \geqslant R'_a | a_1 = a_2 = \ldots = a_k\} = \frac{k-1}{k}\cdot 2^{1-R'_a}.$$

For several values of k and different R'_a and n we have found the following table of probabilities $P\{R' \geqslant R'_a | a_1 = a_2 = \ldots = a_k\}$ (pp. 314-315).

4. Just as in the preceding section we adapted Mosteller's test to problem A_1, we shall now modify Sadowski's test to suit it to problem B_1. For this purpose we shall define statistics r' in the following way:

If among $k-1$ samples from the non-control populations there is a sample containing an element greater than the elements in all the remaining samples and an element smaller than the elements in all the remaining samples, then $r' \geqslant 2$ is the number of protruding elements of that sample as compared with the control sample (the *protruding element* as compared with the control sample is the name which we give to elements greater than any element of the control sample and to elements smaller than any element of the control sample). In the opposite case, i. e. when the greatest element and the smallest element are to be found in different samples or when both the extremal elements are in the control sample, we assume $r' = 0$.

The test for problem B_1 consists in the following alternative:

if $r' \geqslant r'_a$, we assume that the greatest variance is that of the population whose sample contains the extremal elements,

if $r' < r'_a$, we assume that none of the populations has a variance greater than the variance of the control population.

The critical value r'_a should be found from the condition

$$\text{(4)}\qquad P\{r' \geqslant r'_a | \lambda_1 = \lambda_2 = \ldots = \lambda_k\} \leqslant a$$

where the numbers $\lambda_1, \lambda_2, \ldots, \lambda_k$, as at the beginning of the paper, denote the extension coefficients of the individual populations (they are equal,

TABLE 1

Values of probabilities $P(R' \geqslant R'_a | a_1 = a_2 = \ldots = a_k)$ depending on the number of populations k and the sizes of samples n

n \ R'_a	1	2	3	4	5	6	7	8	9	10
					$k = 2$					
2	0,5000	0,1667								
3	,5000	,2000	0,0500							
4	,5000	,2143	,0714	0,0143						
5	,5000	,2222	,0833	,0238	0,0040					
6	,5000	,2273	,0909	,0303	,0076	0,0011				
7	,5000	,2308	,0962	,0350	,0105	,0023	0,0003			
8	,5000	,2333	,1000	,0385	,0128	,0035	,0007	0,0001		
9	,5000	,2353	,1029	,0412	,0147	,0045	,0011	,0002	0,0000	
10	,5000	,2368	,1053	,0433	,0163	,0054	,0016	,0004	,0001	0,0000
12	0,5000	0,2391	0,1087	0,0466	0,0186	0,0069	0,0023	0,0007	0,0002	0,0000
14	,5000	,2407	,1111	,0489	,0204	,0080	,0029	,0010	,0003	,0001
16	,5000	,2419	,1129	,0506	,0217	,0088	,0034	,0012	,0004	,0001
18	,5000	,2429	,1143	,0520	,0228	,0096	,0038	,0014	,0005	,0002
20	,5000	,2436	,1154	,0530	,0236	,0101	,0042	,0016	,0006	,0002
25	0,5000	0,2449	0,1173	0,0549	0,0251	0,0111	0,0048	0,0020	0,0008	0,0003
30	0,5000	0,2458	0,1186	0,0562	0,0261	0,0119	0,0053	0,0023	0,0010	0,0004
∞	0,5000	0,2500	0,1250	0,0625	0,0313	0,0156	0,0078	0,0039	0,0020	0,0010
					$k = 3$					
2	0,6667	0,2222								
3	,6667	,2667	0,0667							
4	,6667	,2857	,0952	0,0190						
5	,6667	,2963	,1111	,0317	0,0053					
6	,6667	,3030	,1212	,0404	,0101	0,0014				
7	,6667	,3077	,1282	,0466	,0140	,0031	0,0004			
8	,6667	,3111	,1333	,0513	,0171	,0047	,0009	0,0001		
9	,6667	,3137	,1373	,0549	,0196	,0060	,0015	,0003	0,0000	
10	,6667	,3158	,1404	,0578	,0217	,0072	,0021	,0005	,0001	0,0000
12	0,6667	0,3188	0,1449	0,0621	0,0248	0,0092	0,0031	0,0009	0,0002	0,0000
14	,6667	,3210	,1481	,0652	,0272	,0106	,0039	,0013	,0004	,0001
16	,6667	,3226	,1505	,0675	,0289	,0118	,0045	,0016	,0005	,0002
18	,6667	,3238	,1524	,0693	,0304	,0127	,0051	,0019	,0007	,0002
20	,6667	,3248	,1538	,0707	,0314	,0135	,0055	,0022	,0008	,0003
25	0,6667	0,3265	0,1565	0,0732	0,0334	0,0149	0,0064	0,0027	0,0011	0,0004
30	0,6667	0,3277	0,1582	0,0749	0,0348	0,0158	0,0070	0,0030	0,0013	0,0005
∞	0,6667	0,3333	0,1667	0,0833	0,0417	0,0208	0,0104	0,0052	0,0026	0,0013

TABLE 1 (continued)

$n \backslash R'_a$	1	2	3	4	5	6	7	8	9	10
					$k = 4$					
2	0,7500	0,2500								
3	,7500	,3000	0,0750							
4	,7500	,3214	,1071	0,0214						
5	,7500	,3333	,1250	,0357	0,0060					
6	,7500	,3409	,1364	,0455	,0114	0,0016				
7	,7500	,3462	,1442	,0524	,0157	,0035	0,0004			
8	,7500	,3500	,1500	,0577	,0192	,0052	,0011	0,0001		
9	,7500	,3529	,1544	,0618	,0221	,0068	,0017	,0003	0,0000	
10	,7500	,3553	,1579	,0650	,0244	,0081	,0023	,0005	,0001	0,0000
12	0,7500	0,3587	0,1630	0,0699	0,0280	0,0103	0,0034	0,0010	0,0003	0,0001
14	,7500	,3611	,1667	,0733	,0306	,0120	,0044	,0014	,0004	,0001
16	,7500	,3629	,1694	,0759	,0325	,0133	,0051	,0018	,0006	,0002
18	,7500	,3643	,1714	,0779	,0342	,0143	,0057	,0022	,0008	,0003
20	,7500	,3654	,1731	,0795	,0353	,0152	,0062	,0025	,0009	,0003
25	0,7500	0,3674	0,1760	0,0824	0,0376	0,0167	0,0072	0,0030	0,0012	0,0005
30	0,7500	0,3686	0,1780	0,0843	0,0391	0,0178	0,0079	0,0034	0,0014	0,0006
∞	0,7500	0,3750	0,1875	0,0938	0,0469	0,0234	0,0117	0,0059	0,0029	0,0015
					$k = 5$					
2	0,8000	0,2667								
3	,8000	,3200	0,0800							
4	,8000	,3429	,1143	0,0229						
5	,8000	,3556	,1333	,0381	0,0064					
6	,8000	,3636	,1455	,0485	,0121	0,0017				
7	,8000	,3692	,1538	,0559	,0168	,0037	0,0005			
8	,8000	,3733	,1600	,0615	,0205	,0056	,0011	0,0001		
9	,8000	,3765	,1647	,0659	,0235	,0072	,0018	,0003	0,0000	
10	,8000	,3789	,1684	,0694	,0260	,0087	,0025	,0006	,0001	0,0000
12	0,8000	0,3826	0,1739	0,0745	0,0298	0,0110	0,0037	0,0011	0,0003	0,0001
14	,8000	,3852	,1778	,0782	,0326	,0128	,0046	,0015	,0005	,0001
16	,8000	,3871	,1806	,0810	,0347	,0141	,0054	,0020	,0007	,0002
18	,8000	,3886	,1829	,0831	,0364	,0153	,0061	,0023	,0008	,0003
20	,8000	,3897	,1846	,0848	,0377	,0162	,0067	,0026	,0010	,0004
25	0,8000	0,3918	0,1878	0,0879	0,0401	0,0178	0,0077	0,0032	0,0013	0,0005
30	0,8000	0,3932	0,1898	0,0899	0,0418	0,0190	0,0084	0,0037	0,0015	0,0006
∞	0,8000	0,4000	0,2000	0,1000	0,0500	0,0250	0,0125	0,0062	0,0031	0,0016

disregarding a constant factor, to the mean square deviations of the individual populations). If the first population is singled out, then inequality (4) implies the inequality

$$\mathrm{P}\{r' \geqslant r'_\alpha \mid \lambda_1 \geqslant \max(\lambda_2, \lambda_3, \ldots, \lambda_k)\} \leqslant \alpha. \tag{5}$$

Statistics r' can assume zero value or natural values from 2 to n. In the first place it will be observed that

$$\mathrm{P}\{r' \geqslant 2 \mid \lambda_1 = \lambda_2 = \ldots = \lambda_k\} = \frac{k-1}{k}\cdot\frac{n-1}{kn-1},$$

since the event $r' \geqslant 2$ is equivalent to the event of the greatest elemens being found in a non-control sample (the probability of that event it $(k-1)/k$) and of the smallest element being found in the same sample (the probability of that being $(n-1)/(kn-1)$).

Let us now observe the fact that for $\varrho \geqslant 2$ the probability $\mathrm{P}\{r' = \varrho \mid \lambda_1 = \lambda_2 = \ldots = \lambda_k\}$ is equal to the product of the probability that one of the $k-1$ non-control samples contains both extremal elements (it is the probability—already found—of the inequality $r' \geqslant 2$) by the probability that exactly $\varrho-2$, other elements of the sample with the extremal elements protrude beyond the elements of the control sample. The last event can be realized in $\varrho-1$ disjoint ways, consisting in there being among the $\varrho-2$ protruding elements (excluding the extremal elements):

$\varrho-2$ elements greater than any elements of the control sample,

$\varrho-3$ elements greater and 1 element smaller than any element of the control sample,

$\varrho-4$ elements greater and 2 elements smaller than any element of the control sample,

. . . .

0 elements greater and $\varrho-2$ elements smaller than any element of the control sample.

Each of those alternatives has a conditional probability, on condition that $r' \geqslant 2$ and $\lambda_1 = \lambda_2 = \ldots = \lambda_k$, equal to

$$\frac{n-2}{2n-2}\cdot\frac{n-3}{2n-3}\cdot\ldots\cdot\frac{n-\varrho+1}{2n-\varrho+1}\cdot\frac{n}{2n-\varrho}\cdot\frac{n-1}{2n-\varrho-1}$$

$$= \frac{n!}{(n-\varrho)!}\cdot\frac{(2n-\varrho-2)!}{(2n-2)!} = \binom{n}{\varrho}\Big/\binom{2n-2}{\varrho}.$$

The initial $\varrho-2$ factors of the above product express the probability that $\varrho-2$ protruding elements (excluding the extremal elements) belong to the sample with the extremal elements and the last two factors are the probabilities that the next successive elements—according to magnitude from above and from below—belong to the control sample (since otherwise we should have $r' > \varrho$).

We thus finally obtain

$$P\{r' = \varrho | \lambda_1 = \lambda_2 = \ldots = \lambda_k\} = \frac{k-1}{k} \cdot \frac{n-1}{kn-1} (\varrho - 1) \binom{n}{\varrho} \Big/ \binom{2n-2}{\varrho} \quad \text{for} \quad \varrho \geqslant 2,$$

and

$$P\{r' = 0 | \lambda_1 = \lambda_2 = \ldots = \lambda_k\} = 1 - \frac{k-1}{k} \cdot \frac{n-1}{kn-1}.$$

With n tending to infinity the statistics r' has a limit distribution

$$\lim_{n\to\infty} P\{r' = \varrho | \lambda_1 = \lambda_2 = \ldots = \lambda_k\} = \frac{k-1}{k^2} \cdot \frac{\varrho - 1}{2^\varrho} \quad \text{for} \quad \varrho \geqslant 2,$$

and

$$\lim_{n\to\infty} P\{r' = 0 | \lambda_1 = \lambda_2 = \ldots = \lambda_k\} = 1 - \frac{k-1}{k^2}.$$

The probability of the inequality $r' \geqslant r'_a$ needed for finding the critical values r'_a is found from the formula

$$P\{r' \geqslant r'_a | \lambda_1 = \lambda_2 = \ldots = \lambda_k\} = \sum_{\varrho = r'_a}^{n} P\{r' = \varrho | \lambda_1 = \lambda_2 = \ldots = \lambda_k\}$$

$$= 1 - \sum_{\varrho=0}^{r'_a - 1} P\{r' = \varrho | \lambda_1 = \lambda_2 = \ldots = \lambda_k\}.$$

For several values of k and different r'_a and n we have calculated tables of probabilities $P\{r' \geqslant r'_a | \lambda_1 = \lambda_2 = \ldots = \lambda_k\}$ (table 2 on pp. 318-319).

5. In problems A and A_1 we have sought populations with the greatest mean value. The test of Mosteller [1] for problem A and ours for problem A_1 can of course be also used for seeking a population with the least mean value. For that purpose it suffices to replace everywhere in the definition of statistics R or R' the word "greatest" and "greater" by the words "smallest" and "smaller" respectively. It will be the same if, without changing the tests, we change the signs of all the observations, owing to which smaller elements will become greater elements and the population with the least mean value will become the population with the greatest mean value.

Similarly in the conditions of problems B and B_1 we can seek the population with the least variance. Such problems have even greater

TABLE 2

Values of probabilities $P\{r' \geqslant r_a | \lambda_1 = \lambda_2 = \ldots = \lambda_k\}$ depending on the number of populations k and the sizes of samples n

n \ r'_a	2	3	4	5	6	7	8	9	10
					$k = 2$				
2	0,1667								
3	,2000	0,1000							
4	,2143	,1286	0,0429						
5	,2222	,1429	,0635	0,0159					
6	,2273	,1515	,0758	,0271	0,0054				
7	,2308	,1573	,0839	,0350	,0105	,0017			
8	,2333	,1615	,0897	,0408	,0147	,0038	0,0005		
9	,2353	,1647	,0941	,0452	,0181	,0058	,0013	0,0002	
10	,2368	,1672	,0975	,0488	,0209	,0075	,0021	,0004	0,0000
12	0,2391	0,1708	0,1025	0,0539	0,0252	0,0104	0,0037	0,0011	0,0003
14	,2407	,1733	,1059	,0576	,0283	,0126	,0050	,0018	,0005
16	,2419	,1752	,1085	,0603	,0306	,0143	,0061	,0024	,0008
18	,2429	,1766	,1104	,0623	,0324	,0156	,0070	,0029	,0011
20	,2436	,1776	,1119	,0640	,0339	,0168	,0078	,0034	,0014
25	0,2449	0,1798	0,1146	0,0669	0,0365	0,0188	0,0092	0,0043	0,0019
30	0,2458	0,1811	0,1164	0,0688	0,0382	0,0202	0,0102	0,0050	0,0023
∞	0,2500	0,1875	0,1250	0,0781	0,0469	0,0273	0,0156	0,0088	0,0049
					$k = 3$				
2	0,1250								
3	,1600	0,0800							
4	,1765	,1059	0,0353						
5	,1860	,1196	,0532	0,0133					
6	,1923	,1282	,0641	,0229	0,0046				
7	,1967	,1341	,0715	,0298	,0089	0,0015			
8	,2000	,1385	,0769	,0350	,0126	,0033	0,0005		
9	,2025	,1418	,0810	,0389	,0156	,0050	,0011	0,0001	
10	,2046	,1444	,0842	,0421	0,180	,0065	,0019	,0004	0,0000
12	0,2076	0,1483	0,0890	0,0468	0,0218	0,0090	0,0032	0,0010	0,0002
14	,2097	,1510	,0923	,0501	,0246	,0109	,0044	,0016	,0005
16	,2113	,1530	,0947	,0526	,0267	,0125	,0053	,0021	,0007
18	,2125	,1545	,0966	,0545	,0284	,0137	,0062	,0026	,0010
20	,2135	,1558	,0981	,0558	,0297	,0147	,0068	,0030	,0012
25	0,2152	0,1580	0,1008	0,0588	0,0321	0,0165	0,0079	0,0038	0,0017
30	0,2164	0,1595	0,1025	0,0606	0,0337	0,0178	0,0090	0,0044	0,0020
∞	0,2222	0,1667	0,1111	0,0694	0,0417	0,0243	0,0139	0,0078	0,0043

TABLE 2 (continued)

n \ r'_a	2	3	4	5	6	7	8	9	10
					$k = 4$				
2	0,1071								
3	,1364	0,0682							
4	,1500	,0900	0,0300						
5	,1579	,1015	,0451	0,0113					
6	,1630	,1087	,0543	,0194	0,0039				
7	,1667	,1136	,0606	,0253	,0076	0,0013			
8	,1694	,1172	,0651	,0296	,0107	,0028	0,0004		
9	,1714	,1200	,0686	,0330	,0132	,0042	,0010	0,0001	
10	,1731	,1222	,0713	,0356	,0153	,0055	,0016	,0003	0,0000
12	0,1755	0,1254	0,0752	0,0396	0,0185	0,0076	0,0027	0,0008	0,0002
14	,1773	,1276	,0780	,0424	,0208	,0092	,0037	,0013	,0004
16	,1786	,1293	,0800	,0445	,0226	,0105	,0045	,0018	,0006
18	,1796	,1306	,0816	,0461	,0240	,0116	,0052	,0022	,0008
20	,1804	,1316	,0829	,0474	,0251	,0124	,0058	,0025	,0010
25	0,1818	0,1335	0,0851	0,0496	0,0271	0,0140	0,0068	0,0032	0,0014
30	0,1828	0,1347	0,0866	0,0512	0,0284	0,0150	0,0076	0,0037	0,0017
∞	0,1875	0,1406	0,0938	0,0586	0,0352	0,0205	0,0117	0,0066	0,0037
					$k = 5$				
2	0,0889								
3	,1143	0,0571							
4	,1263	,0758	0,0253						
5	,1333	,0857	,0381	0,0095					
6	,1379	,0920	,0460	,0164	0,0033				
7	,1412	,0963	,0513	,0214	,0064	0,0011			
8	,1436	,0994	,0552	,0251	,0090	,0023	0,0003		
9	,1454	,1018	,0582	,0280	,0112	,0036	,0008	0,0001	
10	,1469	,1037	,0605	,0303	,0130	,0047	,0013	,0003	0,0000
12	0,1492	0,1065	0,0639	0,0336	0,0157	0,0065	0,0023	0,0007	0,0002
14	,1507	,1085	,0663	,0360	,0177	,0079	,0031	,0011	,0003
16	,1519	,1100	,0681	,0378	,0192	,0090	,0038	,0015	,0005
18	,1528	,1111	,0695	,0392	,0204	,0098	,0044	,0018	,0007
20	,1535	,1120	,0705	,0403	,0213	,0106	,0049	,0021	,0009
25	0,1548	0,1137	0,0725	0,0423	0,0231	0,0119	0,0058	0,0027	0,0012
30	0,1557	0,1147	0,0738	0,0436	0,0242	0,0128	0,0065	0,0031	0,0015
∞	0,1600	0,1200	0,0800	0,0500	0,0300	0,0175	0,0100	0,0056	0,0031

practical significance, since on the whole a great variance is an undesirable property of a population and we attempt to find conditions minimizing the variance. Sadowski's test [2] for problem B and ours for problem B_1 can also be applied in seeking a population with the least variance. For that purpose it is sufficient—before working through the experimental material—to map all the observations by means of a projection transformation

$$y = 1/(x-\mu), \tag{6}$$

where μ is the common mean of all the populations. Such mapping changes the population with the least variance into the population with the greatest variance. Instead of transforming all observations in this way, we can only change the definition of the statistics r and r'. Instead of counting the protruding elements it is sufficient to count the internal elements, i. e. to count on both sides of the mean μ the elements nearer to that mean then the elements of the samples from the remaining populations. This procedure requires the knowledge of the mean value μ. In the case of the mean value μ not being known, we can replace in formula (6) or in the procedure described above the mean μ by its approximation found on the basis of the elements of all the samples. It should be remembered, however, that the latter procedure somewhat changes the distribution of the statistics r and r'.

Bibliography

[1] F. Mosteller, *A k-sample slippage test for an extreme population*, Annals of Math. Statistics 19 (1948), pp. 58-65.

[2] J. Oderfeld, *O wielkości kropelek w rozpylonym paliwie* (*On the size of droplets in sprayed fuel*), Archiwum Budowy Maszyn 1 (1954), pp. 363-369.

[3] E. Paulson, *An optimum solution to the k-sample problem for the normal distribution*, Annals of Math. Stat. 23 (1952), pp. 610-616.

[4] — *On the comparison of several experimental categories with a control*, Annals of Math. Stat. 23 (1952), pp. 669-674.

[5] W. Sadowski, *O nieparametrycznym teście na porównywanie rozsiewów* (*On a non-parametric test for comparing dispersions*), Zast. Mat. 2 (1955), pp. 161-171.

[6] D. R. Truax, *An optimum slippage test for the variance of k normal distributions*, Annals of Math. Stat. 24 (1953), pp. 669-674.

INSTYTUT MATEMATYCZNY POLSKIEJ AKADEMII NAUK
MATHEMATICAL INSTITUTE OF THE POLISH ACADEMY OF SCIENCES

(*The Polish version of this paper has been received on 10. 4. 1956*)